Algebraic Number Theory

Exercise Sheet 6

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Exercise 1. Recall that for any prime number p the ring $\mathbb{Z}_{(p)}$ is a dvr and $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}_{(p)}$. Let $L = \mathbb{Q}(i)$ be the quadratic field with $i^2 = -1$. Find all primes p for which the integral closure of $\mathbb{Z}_{(p)}$ in L is a dvr.
Hint: Use first Exercise 4, Sheet 4.

Exercise 2. Let A be a noetherian integral domain. Let $f_1, ..., f_n$ be elements of A, such that the ideal generated by $f_1, ..., f_n$ is A. Let K denote the field of fractions of A. For $f \in A$ we denote by A_f the localization $S^{-1}A$, where $S = \{f^k \mid k \geq 0\}$.
(1) Show that $A = \cap_{i=1}^n A_{f_i}$ in K.
(2) Show that A is a Dedekind ring if and only if A_{f_i} is a Dedekind ring for any $i = 1, ..., n$.

Exercise 3. Let $L = \mathbb{Q}(\alpha)$ be a quadratic field extension of $\mathbb{Q}$, where $\alpha^2 = -5$. Let $\mathcal{O}_L$ be the ring of integers of L. Recall that $\mathcal{O}_L = \mathbb{Z}[\alpha]$.

(1) Show that the ideal $\rho = (2, 1 + \alpha)$ is prime and not principle in $\mathcal{O}_L$.
Hint: Use Exercise 1 and 2 from Sheet 5 to show that ρ is prime.

From the lecture we know that the localization $(\mathcal{O}_L)_\rho$ is a discrete valuation ring and hence, by Theorem 4.23, is a PID.

(2) Find a generator of the principal ideal $\rho(\mathcal{O}_L)_\rho$.
Hint: Recall that $2 \cdot 3 = (1 + \alpha)(1 - \alpha)$ in $\mathcal{O}_L$.

(3) Compute $v(2)$, $v(3)$, $v(1 + \alpha)$, $v(1 - \alpha)$, where v is the discrete valuation in the dvr $(\mathcal{O}_L)_\rho$. Explain why the two factorizations

$$6 = 2 \cdot 3 = (1 + \alpha)(1 - \alpha)$$

in $(\mathcal{O}_L)_\rho$ do not contradict the fact that $(\mathcal{O}_L)_\rho$ is a PID and hence a factorial ring.
Remark: Compare with Exercise 3, Sheet 4.