Algebraic Number Theory

Exercise Sheet 4

PD Dr. Maksim Zhykhovich Winter Semester 2026-27, 05.11.2026

Exercise 1. Let p be an odd prime number, let $\zeta_p \in \mathbb{C}$ be a primitive p-th root of unity and let $L = \mathbb{Q}(\zeta_p)$ be the corresponding cyclotomic field. Show that $d_L = (-1)^{\frac{p-1}{2}} p^{p-2}$.
Hint: Use the formula from Exercise 2.1, Sheet 3.

Exercise 2. For positive integer n let $\zeta_n \in \mathbb{C}$ denote a primitive n-th root of unity. Let p be an odd prime number, let $\mathbb{Q}(\zeta_p)$ be the corresponding cyclotomic field.
Recall that $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ is Galois field extension and $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) = (\mathbb{Z}/p\mathbb{Z})^*$.
(1) Deduce that there is a unique intermediate field K of $\mathbb{Q}(\zeta_p)/\mathbb{Q}$, which is of degree 2 over $\mathbb{Q}$.
(2) Using Proposition 3.6 from the lecture and Exercise 1, find explicitly the unique quadratic extension $K/\mathbb{Q}$ contained in $\mathbb{Q}(\zeta_p)$.
(3) Deduce that any quadratic extension of $\mathbb{Q}$ is contained inside some cyclotomic field.
Hint: Show that $\sqrt{2} \in \mathbb{Q}(\zeta_8)$ and use that $\mathbb{Q}(\zeta_n) \subset \mathbb{Q}(\zeta_{nm})$ for any positive integers n and m.

Exercise 3. Let $L = \mathbb{Q}(\alpha)$ be a quadratic field, where $\alpha^2 = -5$. Let $\mathcal{O}_L$ be the ring of integers of L. Show that $\mathcal{O}_L$ is not a principal ideal domain.
Hint: Observe that $(1+\alpha)(1-\alpha) = 2 \cdot 3$. Are the elements $(1-\alpha), (1+\alpha), 2, 3$ irreducible in $\mathcal{O}_L$?

Exercise 4. Let A be an integral domain and $S \subset A$ a multiplicative subset. Let K denote the field of fractions of A and let L be a finite field extension of K. Let B denote the integral closure of A in L. Show that the integral closure of $S^{-1}A$ in L is $S^{-1}B$, where the localizations $S^{-1}A$ and $S^{-1}B$ are considered as subrings of K and L respectively.
Remark: We will recall localization in Monday's lecture. For this exercise, it is enough only to know the definition.