Algebraic Number Theory

Exercise Sheet 3

PD Dr. Maksim Zhykhovich Winter Semester 2026-27, 29.10.2026

Exercise 1. Let $\alpha \in \mathbb{C}$ be a root of the polynomial $X^3 + 2X + 1 \in \mathbb{Z}[X]$ and let $L = \mathbb{Q}(\alpha)$.
(1) Show that $[L : \mathbb{Q}] = 3$.
(2) Compute the discriminant $D^L_{\mathbb{Q}}(1, \alpha, \alpha^2)$ by definition.
(3) Deduce from (2) that $1, \alpha, \alpha^2$ is a $\mathbb{Z}$-basis of $\mathcal{O}_L$.

Exercise 2. (1) Let K be a field and let $L = K(\alpha)$ be a separable extension of finite degree n of K, and let $F(X)$ be the minimal polynomial of α over K. Prove the following formula

$$D^L_K(1, \alpha, ..., \alpha^{n-1}) = (-1)^{\frac{1}{2}n(n-1)} N(F'(\alpha)) .$$

Hint: Use Proposition 3.6(1) from the lecture.
(2) Using the formula from (1) compute the discriminant in Exercise 1.2.

Exercise 3 (**Lemma of Dedekind**)**.** Let G be a group, K a field, and let $\sigma_1, ..., \sigma_n$ be distinct homomorphisms of G into the multiplicative group K^*. Show that the homomorphisms $\sigma_1, ..., \sigma_n$ are linearly independent over K (i.e. $\sum_{i=1}^n \lambda_i \sigma_i(g) = 0$ for all $g \in G$ implies that $\lambda_i = 0$ for all i).

Exercise 4. Let A be an integral domain and K the field of fractions of A. Let L and L' be two finite field extension of K. Let B (respectively, B') be the integral closure of A in L (respectively, in L'). Assume that there is a K-morphism of fields $\sigma : L \to L'$. Show that $\sigma(B) \subset B'$.
Deduce that, if L/K is Galois, then B is stable under the action of the Galois group of L/K.