Algebraic Number Theory

Exercise Sheet 2

PD Dr. Maksim Zhykhovich Winter Semester 2026-27, 22.10.2026

Exercise 1. Let $\alpha \in \mathbb{C}$ be a root of the polynomial $X^3 - 2 \in \mathbb{Z}[X]$ and let $L = \mathbb{Q}(\alpha)$.
(1) Show that $[L : \mathbb{Q}] = 3$.
(2) Compute $\mathrm{Tr}_{\mathbb{Q}}^{L}(\alpha + 1)$ and $N_{\mathbb{Q}}^{L}(\alpha + 1)$.

Exercise 2. Let A be an integral domain and let F be the field of fractions of A. Let B be a ring, such that $A \subseteq B \subseteq F$. Show that the canonical morphism of fields $f : F \to K$ is an isomorphism, where K is the field of fractions of B.

Exercise 3. Let B be an integral domain and A a subring of B such that B is integral over A. Show that B is a field if and only if A is a field.

Exercise 4. Let A be an integrally closed ring. Let K be the field of fractions of A. Let $K \subset L$ be a finite separable field extension of K, B the integral closure of A in L and let $\alpha \in B$. Show that α is a unit in B if and only if $N_K^L(x)$ is a unit in A.