Stein Manifolds

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I prepared these notes for the participants of my lecture. The course took place at the mathematical department of LMU (Ludwig-Maximilians-Universität) at Munich during the winter semester 2021/2022. Main parts of the texts rely on notes of a lecture of O. Forster [6].

Compared to the oral lecture in class these written notes contain some additional material. I thank all participants of the course, in particular J. Bartenschlager and L. Schönlinner, who have pointed out errors and typos in the notes.

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any errors or typos, adding the release of the lecture notes.

Release notes:

- **Release 4.3**: Chapter 4, Section 4.1 Minor additions.
- **Release 4.2**: Chapter 4: Minor revisions.
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Contents

Introduction  ................................................................. 2

1 Holomorphic functions of several variables  ........................................ 5
  1.1 Convergent power series  .................................................. 5
  1.2 First applications of Cauchy’s integral formula  .......................... 11
  1.3 Dolbeault’s lemma  .......................................................... 20

2 Complex manifolds and sheaves  .......................................... 37
  2.1 Complex manifolds  ......................................................... 37
  2.2 Sheaves and their stalks  .................................................. 46
  2.3 The Cousin problems  ...................................................... 62

3 Sheaf cohomology  .......................................................... 75
  3.1 Čech cohomology groups  .................................................. 76
  3.2 The long exact cohomology sequence  .................................. 86
  3.3 Acyclic sheaves and applications  ...................................... 92

4 Local theory and coherence of sheaves  .................................. 109
  4.1 The ring of convergent power series and finitely generated modules .. 109
  4.2 Oka’s coherence theorem for the structure sheaf  ....................... 129
  4.3 Coherent \( \mathcal{O} \)-modules  ................................................. 138

5 Cartan’s lemma for holomorphic matrices  ................................ 147
  5.1 Proof of Cartan’s lemma  .................................................. 147
  5.2 Hilbert’s syzygy theorem for coherent \( \mathcal{O} \)-modules  .................. 163
  5.3 Fréchet topology in the context of cohomology  ......................... 172

6 Theorem B and Theorem A on Stein manifolds  ............................ 177
  6.1 Holomorphic convexity and analytic polyhedra  ....................... 178
  6.2 Stein manifolds and Theorem B .......................................... 187
  6.3 Theorem A and further applications  ................................... 208
Introduction

Fig. 0.1 The Stein garden (Aussichtspunkt = viewpoint)
Introduction

The theory of Stein manifolds originates as generalization of the theory of open, i.e. non-compact Riemann surfaces: Each Riemann surface is either compact or a Stein manifold.

A Stein manifold \( X \) is characterized by its wealth of holomorphic functions: There exist enough holomorphic functions \( f_j \) on \( X \) to define a closed embedding

\[
f = (f_1, \ldots, f_n) : X \hookrightarrow \mathbb{C}^n
\]

for a suitable finite \( n \in \mathbb{N} \): Hence each Stein manifold \( X \) is a closed submanifold of an affine space \( \mathbb{C}^n \).

Of course the theory develops the other way round: How to find intrinsic properties of a complex manifold \( X \) which allow a closed affine embedding of \( X \)? The proper means to answer this question is sheaf cohomology.

In somewhat more detail the content of these lecture notes is as follows:

Chapter 1 carries over concepts and results from 1-dimensional complex analysis to complex analysis of several variables on open subdomains of \( \mathbb{C}^n \). We emphasize those results which carry over literally as well as those which are different in several variables.

Chapter 2 generalizes the concepts and results from Chapter 1 to complex manifolds. We introduce the language of sheaves on a complex manifold \( X \). It serves to glue local results to global results on \( X \). As a first example we state the Cousin problems from complex analysis and their solutions as questions to the sheaves of holomorphic resp. meromorphic functions on a complex manifold. Then we solve the Cousin problems for polydiscs.

Cohomology theory in Chapter 3 uses the language of sheaves to measure the obstructions against globalizing local results on manifolds. The chapter introduces Čech cohomology as an example of a cohomology theory for sheaves. We show that each sheaf has a flabby resolution which allows to compute sheaf cohomology by acyclic resolutions, thanks to the abstract de Rahm theorem. As an application we compute the Dolbeault groups for the holomorphic cohomology and show that each Cousin problem is solvable on the polydisc.

Chapter 4 introduces the concept of coherence for module sheaves over the structure sheaf of a complex manifold \( X \). This property is fundamental for the whole theory of complex manifolds. The chapter starts with the necessary results from Local Analytic Theory: The Weierstrass preparation theorem and the Weierstrass division theorem imply a series of algebraic properties of the ring \( R \) of convergent power series. From cohomological algebra we recall Hilbert’s syzygy theorem for finitely generated modules over \( R \). Locally a coherent sheaf is the cokernel of a morphism of finitely generated \( \mathcal{O} \)-modules. Hence coherence extends the commutative algebra of finitely generated modules over \( R \) to the theory of module sheaves over the structure sheaf of \( X \).
Chapter 5 introduces and proves Cartan’s lemma about the splitting of matrices of invertible holomorphic functions defined on the intersection of two adjacent product domains. Using Cartan’s lemma on holomorphic matrix functions we carry over the syzygy theorem to coherent sheaves on a polydisc. As an application, coherent sheaves are acyclic on a shrinked polydisc.

The class of Stein manifolds is introduced in Chapter 6. We prove the main results of the theory: Theorem A and B about the cohomology of coherent sheaves on Stein manifolds. The proof of Theorem B starts with the local version for polydiscs. It advances via Runge approximation for coherent sheaves up to the global result. The signpost on this path is the choice of an exhaustion of the Stein manifold by analytic polyhedra. The applications show in which sense the complex analysis of Stein manifolds generalizes the theory of open Riemann surfaces to the higher-dimensional case.

The final Chapter 7 gives an outlook to some more advanced topics from the theory of Stein manifolds, e.g. to the embedding theorem, the spectrum of a Stein algebra and duality for coherent sheaves on Stein manifolds. In addition, we make some remarks about the case of singularities, i.e. about generalizing the results to Stein spaces. A last section emphasizes the parallel between Stein spaces from complex analysis and affine schemes from algebraic geometry.

The theory of Stein manifolds is named after Karl Stein (1913-2000). Joint work together with his teacher H. Behnke on Riemann surfaces laid the foundations around the midth of the 20th century. For a historical scetch see [14, Einltg.] and [27].

All smooth and complex manifolds in these lecture notes are assumed to satisfy the second axiom of countability.
Chapter 1
Holomorphic functions of several variables

1.1 Convergent power series

Complex analysis of several variables starts with holomorphic functions defined on open subsets of the \( n \)-dimensional complex space \( \mathbb{C}^n \). Locally these functions are given by a convergent power series in several variables. Each point \( z \in \mathbb{C}^n \) is an \( n \)-tuple

\[ z = (z_1, \ldots, z_n) \]

of complex numbers \( z_1, \ldots, z_n \). As a consequence, one has often to consider polyindices

\[ i = (i_1, \ldots, i_n) \in \mathbb{N}^n. \]

As experience shows, it takes some time to become familiar with the polyindex notation. Hence one should practice to read formulas about functions of several variables alike to the familiar formulas from one variable. Many results from the 1-dimensional theory carry over to the \( n \)-dimensional theory. But there are some remarkable exceptions where one discovers surprisingly different results. We will emphasize these differences during the course of these notes.

Notation 1.1 (Polyindex and polydisc). Consider \( n \in \mathbb{N} \).

- Factorial: For a polyindex \( i = (i_1, \ldots, i_n) \in \mathbb{N}^n \) set

\[ i! := i_1! \cdot \ldots \cdot i_n! \in \mathbb{N} \]

- Modulus: For a polyindex \( i = (i_1, \ldots, i_n) \in \mathbb{N}^n \) set

\[ |i| := i_1 + \ldots + i_n \]

- Exponential: For a polyindex \( i = (i_1, \ldots, i_n) \in \mathbb{N}^n \) and a point \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) define the product

\[ z^i := z_1^{i_1} \cdot \ldots \cdot z_n^{i_n} \in \mathbb{C} \]
• Disc and polydisc: In one dimension, for \( a \in C \) and \( r \in \mathbb{R}^+ \) we denote by

\[
D(a;r) := \{ z \in C : |z - a| < r \}
\]

the open disc around \( a \) with radius \( r \). In several dimensions, for \( a = (a_1, ..., a_n) \in \mathbb{C}^n \) and a polyradius \( r = (r_1, ..., r_n) \in (\mathbb{R}^+)^n \) we denote by

\[
\Delta(a;r) := \prod_{\nu=1}^{n} D(a_\nu;r_\nu) = \{ z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_\nu - a_\nu| < r_\nu, \ \nu = 1, ..., n \} \subset \mathbb{C}^n
\]

the open polydisc around \( a \) with polyradius \( r \). The polydisc \( \Delta(a;r) \) is a product domain; it is the Cartesian product of 1-dimensional discs.

![Fig. 1.1 Compact convergence of power series](image)

**Proposition 1.2 (Convergence of power series).** Consider a complex power series in several variables \( z = (z_1, ..., z_n) \in \mathbb{C}^n \)

\[
f(z) := \sum \ \sum_{i \in \mathbb{N}^n} c_i \cdot z^i, \ c_i \in \mathbb{C} \text{ for all } i \in \mathbb{N}^n.
\]

Assume that the power series is convergent for a given point

\( w = (w_1, ..., w_n) \in \mathbb{C}^n \)

with respect to a suitable ordering of indices. If there exists a polyradius

\( r = (r_1, ..., r_n) \in \mathbb{R}^+ \) with \( r_\nu < |w_\nu|, \ \nu = 1, ..., n \),
then $f(z)$ is absolutely and uniformly convergent in the closed polydisc

$$\mathcal{A} := \{z \in \mathbb{C}^n : |z| \leq r, \, v = 1, \ldots, n\},$$

see Figure 1.1. Moreover, $f(z)$ is convergent for all $z \in \mathcal{A}$ with respect to each ordering of indices, and the value $f(z) \in \mathbb{C}$ is independent from the chosen order.

**Proof.** For $v = 1, \ldots, n$ set

$$\theta_v := \frac{r_v}{|w_v|} < 1 \quad \text{and} \quad \theta := (\theta_1, \ldots, \theta_n).$$

There exists a constant $M$ such that for all $z \in \mathcal{A}$ and all $i \in \mathbb{N}^n$

$$|c_i \cdot w^i| \leq M$$

The estimate

$$|z| \leq r_v = \frac{r_v}{|w_v|} \cdot |w_v| = \theta_v \cdot |w_v|, \quad v = 1, \ldots, n,$$

implies

$$|c_i \cdot z^i| \leq |c_i \cdot w^i| \cdot \theta \leq M \cdot \theta^i$$

The formula for the geometric series

$$\sum_{i \in \mathbb{N}^n} M \cdot \theta^i = M \cdot \frac{1}{(1 - \theta_1) \cdot \ldots \cdot (1 - \theta_n)}$$

shows that the assumptions of the rearrangement theorem are satisfied, and finishes the claim of the theorem. $\Box$

We define holomorphic functions as analytic functions, i.e. as functions which locally expand into a convergent power series.

**Definition 1.3 (Holomorphic function).** Consider an open set $U \subset \mathbb{C}^n$.

1. A function

$$f : U \to \mathbb{C}$$

is holomorphic if $f$ expands into a convergent power series locally around each point of $U$, i.e. for each point $a \in U$ exists a neighbourhood $V \subset U$ of $a$ and a power series convergent in $V$

$$\sum_{i \in \mathbb{N}^n} c_i \cdot (z-a)^i$$

such that for all $z \in V$

$$f(z) = \sum_{i \in \mathbb{N}^n} c_i \cdot (z-a)^i$$
2. A map 

\[ F = (F_1, \ldots, F_k) : U \rightarrow \mathbb{C}^k \]

is holomorphic iff each component function \( F_j, j = 1, \ldots, k \), is holomorphic.

Alike to functions of one complex variable also for functions \( f \) of several complex variables \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) the following concepts are equivalent, see [6, § 1]:

- Analyticity of \( f \), i.e. locally expandable in a convergent power series.
- Existence of continuous partial derivatives of \( f \) with respect to all real variables \( x_\nu \) and \( y_\nu \) with \( z_\nu = x_\nu + i \cdot y_\nu \), \( \nu = 1, \ldots, n \),

which satisfy the Cauchy-Riemann differential equations

\[ \frac{\partial f}{\partial z_\nu} = 0, \; \nu = 1, \ldots, n. \]

Here

\[ \frac{\partial}{\partial z_\nu} := \frac{\partial}{\partial x_\nu} + i \cdot \frac{\partial}{\partial y_\nu} \]

- Existence of a complex-linear approximation for \( f \) at each point (Total complex differentiability).

As a consequence sum and product as well as the composition of holomorphic functions are again holomorphic.

**Notation 1.4 (Ring of holomorphic functions).** For an open set \( U \subset \mathbb{C}^n \) we denote by \( \mathcal{O}(U) \) the ring of holomorphic functions on \( U \). It is a commutative ring with unit, even a \( \mathbb{C} \)-algebra.

**Remark 1.5 (Partial holomorphy).** Each holomorphic function is continuous due to the compact convergence from Proposition 1.2. Due to the same proposition each holomorphic function of several variables is holomorphic in each variable separately, i.e. if

\[ f : U \rightarrow \mathbb{C}, \; U \subset \mathbb{C}^n \text{ open}, \]

is a holomorphic function of several variables and if \( a = (a_1, \ldots, a_n) \in U \), then for each \( \nu = 1, \ldots, n \) exists a radius \( r_\nu > 0 \) such that

\[ U_\nu := \{a_1\} \times \ldots \times \{a_{\nu-1}\} \times \{z_j \in \mathbb{C} : |z - a_\nu| < r_\nu\} \times \{a_{\nu+1}\} \times \ldots \times \{a_n\} \subset U \]

and the restriction
1.1 Convergent power series

\[ f|_{U_{\nu}} : U_{\nu} \to \mathbb{C} \]

is holomorphic as a function of the distinguished single complex variable \( z_{\nu} \).

**Theorem 1.6 (Osgood’s lemma).** Consider an open set \( U \subset \mathbb{C}^n \) and a continuous function

\[ f : U \to \mathbb{C} \]

Then \( f \) is holomorphic in the sense of Definition 1.3 iff \( f \) is holomorphic in each variable separately.

**Proof.** Due to Remark 1.5 only the direction “$$\Leftarrow$$” needs a proof. Assume that \( f \) is holomorphic as a function of each variable separately.

Holomorphy is a local property. Consider a point \( a \in U \) and choose a polydisc

\[ \Delta := \Delta(a; r^n) \]

with \( \overline{\Delta} \subset U \). For each index tuple \( K = (k_1, \ldots, k_n) \in \mathbb{N}^n \) set

\[ e := (1, \ldots, 1) \in \mathbb{N}^n \]

and define

\[ c_K := \frac{1}{(2\pi i)^n} \int_{\zeta \in \partial \Delta} \frac{f(\zeta)}{(\zeta - a)^{K+e}} d\zeta \]

Note that the integral is a multiple integral

\[ d\zeta = d\zeta_1 \ldots d\zeta_n \]

along the compact set \( \partial \Delta \subset \mathbb{C}^n \).

Claim: In \( \Delta \) the function \( f \) expands in a convergent power series around \( a \)

\[ f(z) = \sum_{K \in \mathbb{N}^n} c_K \cdot (z - a)^K \]

with coefficients

\[ c_K = \frac{1}{K!} \left. \frac{\partial^{|K|}}{\partial \zeta_1^{k_1} \ldots \partial \zeta_n^{k_n}} f(a) \right|_{z=a} \]

We proof the claim by induction on the number \( n \) of variables.

Induction start \( n = 1 \): The result is one of the main results from complex analysis in one variable: Each holomorphic function \( f \) is analytic and has derivatives of each order. Due to the Cauchy integral formula the coefficients of the power series expansion derive from the derivatives of \( f \) as claimed above.

Induction step \( n-1 \mapsto n \): Split
$a = (a', a_n) \in U \cap (\mathbb{C}^{n-1} \times \mathbb{C}), K = (K', k) \in \mathbb{N}^{n-1} \times \mathbb{N}, z = (z', z_n) \in (\mathbb{C}^{n-1} \times \mathbb{C}) \cap \Delta,$

$$e = (e', 1) \text{ with } e' = (1, \ldots, 1) \in \mathbb{N}^{n-1}$$

and set

$$\Delta' := \Delta \cap \mathbb{C}^{n-1}$$

Define for each $k \in \mathbb{N}$ and $z' \in \Delta'$ the 1-dimensional integral with respect to the last variable $\zeta_n$ and depending on the first variables as the parameter $z'$$

$$c_k(z') := \frac{1}{2\pi i} \int_{|\zeta_n - a_n| = r_n} \frac{f(z', \zeta_n)}{(\zeta_n - a_n)^{k+1}} d\zeta_n.$$ 

The integral is well-defined because the integrand is continuous along the compact path of integration. By induction start

$$f(z', z_n) = \sum_{k=0}^\infty c_k(z') \cdot (z_n - a_n)^k$$

and the coefficient $c_k(z')$ depends continuously on $z'$. Because

$$\frac{\partial f}{\partial \sigma_v} = 0, \nu = 1, \ldots, n - 1,$$

the function $c_k$ is even partially holomorphic - with respect to all variables of $z'$. The induction assumption applies to each function $c_k, k \in \mathbb{N}$. It provides the convergent power series expansions with respect to $z'$

$$c_k(z') = \sum_{K' \in \mathbb{N}^{n-1}} c_{(K', k)} (z' - a')^{K'}$$

and the well-defined $n-1$-dimensional integral

$$c_{(K', k)} := \frac{1}{(2\pi i)^{n-1}} \cdot \int_{|\zeta_j - a_j| = r_j, j = 1, \ldots, n-1} \frac{c_k(\zeta')}{(\zeta' - a')^{K'+1}} d\zeta' = \frac{1}{(2\pi i)^{n-1}} \cdot \int_{|\zeta_j - a_j| = r_j, j = 1, \ldots, n} \frac{f(\zeta)}{(\zeta - a)^{K+1}} d\zeta$$

satisfies

$$c_{(K', k)} = \frac{1}{K'!} \cdot \frac{\partial^{|K'|} c_k}{\partial \zeta_1^{a_1} \cdots \partial \zeta_{n-1}^{a_{n-1}}}(a')$$

Due to the induction start

$$c_k(a') = \frac{1}{k!} \cdot \frac{\partial^k f}{\partial \zeta_n^k}(a)$$

we eventually get
\begin{align*}
\mathcal{C}_K &= c_{(K', \mathbf{k})} = \frac{1}{K!} \frac{\partial^{|K'|} f}{\partial z_{k_1}^{k_1} \cdots \partial z_{k_n}^{k_n}}(a)
\end{align*}

\hfill \square

The proof of Theorem 1.6 contains in particular the Cauchy integral formula Corollary 1.7.

**Corollary 1.7 (Cauchy integral formula).** Consider an open set \( U \subset \mathbb{C}^n \) and a holomorphic function 

\[ f : U \to \mathbb{C}. \]

Then all partial derivatives of \( f \) exist as holomorphic functions and can be represented by the following integral formula: For each point \( a \in U \) and for each polydisc 

\[ \Delta := \Delta(a; r) \] 

with closure \( \Delta \subset U \) holds for all \( z \in \Delta \) and each polyindex \( j = (j_1, \ldots, j_n) \in \mathbb{N}^n \) the Cauchy integral formula

\[ \frac{\partial^{|j|} f}{\partial z_{1}^{j_1} \cdots \partial z_{n}^{j_n}}(z) = \frac{j!}{(2\pi i)^n} \int_{\zeta \in \partial \Delta} \frac{f(\zeta)}{(\zeta - z)^{j+e}} \, d\zeta \]

with the shorthand 

\[ \zeta = (\zeta_1, \ldots, \zeta_n), \quad d\zeta = d\zeta_1 \cdots d\zeta_n, \quad e = (1, \ldots, 1) \in \mathbb{N}^n \]

**Proof.** See the inductive proof of Theorem 1.6. \hfill \square

**Remark 1.8 (Hartogs’ theorem).** It is a remarkable result due to Hartogs that in Theorem 1.6 one can dismiss the assumption that \( f \) is continuous, see [22, Chap. III, Theor. of Hartogs], [19, Theor. 2.2.8].

\[ F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} \, dx = 1 \]

\[ z = x + iy, \quad z \bar{z} = x^2 + y^2 \]

\[ \frac{1}{(2\pi i)^n} \int_{\zeta \in \partial \Delta} \frac{f(\zeta)}{(\zeta - z)^{j+e}} \, d\zeta \]

\[ |f^{(n)}(0)| \leq C_n |x|^{n-2} \]

\[ \frac{1}{(2\pi i)^n} \int_{\zeta \in \partial \Delta} \frac{f(\zeta)}{(\zeta - z)^{j+e}} \, d\zeta \]

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\[ \frac{1}{(2\pi i)^n} \int_{\zeta \in \partial \Delta} \frac{f(\zeta)}{(\zeta - z)^{j+e}} \, d\zeta \]

1.2 First applications of Cauchy’s integral formula

**Definition 1.9 (Domain).** A domain \( G \subset \mathbb{C}^n \) is a non-empty, open and connected subset of \( \mathbb{C}^n \).

**Theorem 1.10 (Identity theorem).** Consider a domain \( G \subset \mathbb{C}^n \). Two holomorphic functions 

\[ f, g : G \to \mathbb{C} \]

are equal if they coincide on a non-empty open subset of \( G \).
Proof. By definition the set of local coincidence

\[ X := \{ x \in G : \text{For a suitable neighbourhood } U \text{ of } x \text{ holds } f|U = g|U \} \]

is open and non-empty. The set \( X \) is also closed with respect to the topology of \( G \):

Consider an arbitrary point \( z_0 \in \partial X \cap G \).

We have to show \( z_0 \in X \), i.e. for a polydisc \( \Delta \) around \( z_0 \) holds

\[ f|\Delta = g|\Delta : \]

Choose a polydisc \( \Delta \subset G \) centered around \( z_0 \). Because

\[ \Delta \cap X \neq \emptyset \]

there exists a point \( a \in \Delta \cap X \).

For arbitrary \( z \in \Delta \) the function \( \Phi \) of the single complex variable \( t \)

\[ \Phi(t) := f(a + t \cdot (z - a)) - g(a + t \cdot (z - a)) \]

is holomorphic in a connected, complex neighbourhood in \( \mathbb{C} \) of the real interval \([0, 1] \), see Figure 1.2.

![Figure 1.2 Identity theorem](image)

Because both functions \( f \) and \( g \) coincide by assumption in a suitable neighbourhood \( V_a \subset \mathbb{C}^n \) of \( a \), the identity theorem of complex analysis in one variable implies
1.2 First applications of Cauchy’s integral formula

\[ \phi = 0. \]

Hence in particular

\[ \phi(1) = 0, \text{ i.e. } f(z) = g(z). \]

Because \( z \in \Delta \) has been chosen arbitrarily, there holds

\[ f|\Delta = g|\Delta. \]

It implies \( z_0 \in X \) by definition of \( X \). As a consequence \( X \subset G \) is also closed in \( G \), and the connectedness of \( G \) implies

\[ X = G. \]

\[ \square \]

**Corollary 1.11 (Integral domain).** For a domain \( G \subset \mathbb{C}^n \) the ring \( \mathcal{O}(G) \) of holomorphic functions is an integral domain.

**Proof.** We have to show for \( f, g, \in \mathcal{O}(G) \):

\[ f \cdot g = 0 \implies f = 0 \text{ or } g = 0. \]

If \( f \neq 0 \) then there exists a point \( a \in G \) with \( f(a) \neq 0 \). By continuity \( f|V \neq 0 \) for a suitable neighbourhood \( V \) of \( a \). Hence \( g|V = 0 \). Theorem 1.10 implies \( g = 0 \). \( \square \)

The conclusion of Corollary 1.11 does not hold for an open subset \( G \subset \mathbb{C}^n \) with at least two connected components.

**Corollary 1.12 (Open map).** Consider an open subset \( U \subset \mathbb{C}^n \) and a holomorphic map

\[ f : U \rightarrow \mathbb{C} \]

which is not constant on any component of \( U \). Then \( f \) is an open map.

**Proof.** Consider an arbitrary point \( a \in U \). Choose an open polydisc \( \Delta \subset U \) around \( a \). Theorem 1.10 implies the existence of a point \( b \in \Delta \) with

\[ f(a) \neq f(b) \]

If \( L \subset \mathbb{C}^n \) denotes the complex line passing through \( a \) and \( b \) then the restriction

\[ f|L \cap \Delta \]

is a non-constant holomorphic function of one variable. The open mapping theorem for holomorphic functions of one variable implies that
is an open neighbourhood of \( f(a) \) in \( \mathbb{C} \), and as a consequence also the superset
\[
 f(\Delta) \supset f(\Delta \cap L).
\]
\[ \square \]

**Corollary 1.13 (Maximum modulus theorem).** Consider a domain \( G \subset \mathbb{C}^n \) and a holomorphic function \( f \in \mathcal{O}(G) \) which attains the maximum of its modulus at a point \( a \in G \), i.e. for all \( z \in G \) holds
\[
 |f(a)| \geq |f(z)|
\]
Then \( f \) is constant in \( G \).

**Proof.** The proof is indirect. Assume that \( f \) is not constant. Then \( |f(a)| \neq 0 \) which implies
\[
 |f(a)| = \sup |f(G)| =: r > 0
\]
Corollary 1.12 implies that \( f(G) \subset \mathbb{C} \) is open. Moreover
\[
 f(G) \subset \Delta_r(0) \subset \mathbb{C},
\]
and therefore
\[
 \partial \Delta_r(0) \cap f(G) = \emptyset.
\]
But
\[
 f(a) \in \partial \Delta_r(0) \cap f(G),
\]
a contradiction. \[ \square \]

The following Theorem 1.14 shows the existence of domains
\[
 B \subset \mathbb{C}^n, \; n \geq 2,
\]
such that each holomorphic function \( f \in \mathcal{O}(B) \) extends holomorphically to a strictly larger domain in \( \mathbb{C}^n \), see Figure 1.3.
This property is in striking contrast to complex analysis of one variable: For each domain \( B \subset \mathbb{C} \) exists a holomorphic function \( f \in \mathcal{O}(B) \) which does not extend holomorphically across a certain boundary point \( z_0 \in \partial B \).
1.2 First applications of Cauchy’s integral formula

**Theorem 1.14 (Hartogs’ continuity theorem).** Let $G \subset \mathbb{C}^{n-1}, \ n \geq 2,$ be a domain. Consider the following geometric situation: A non-empty open set $U \subset G,$ and two radii $0 < r_1 < r_2$ defining the 1-dimensional complex domains

$$S := \{z \in \mathbb{C} : |z| < r_2\} \text{ (disc) and } R := \{z \in \mathbb{C} : r_1 < |z| < r_2\} \text{ (annulus),}$$

see Figure 1.3. Set

$$H := U \times S \cup G \times R$$

Then each holomorphic function

$$f \in \mathcal{O}(H)$$

extends uniquely to a holomorphic function

$$\tilde{f} \in \mathcal{O}(G \times S).$$

**Proof.** Choose an intermediate radius $r_1 < \rho < r_2,$ and define for each $z = (z', z_n) \in G \times R$

$$\tilde{f}(z) := \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(z', \zeta)}{\zeta - z_n} d\zeta$$

The resulting function $\tilde{f}$ is holomorphic on...
\[ G \times \{ z_n \in \mathbb{C} : |z_n| < \rho \} \]

On the non-empty open set
\[ \tilde{U} := U \times \{ z_n \in \mathbb{C} : |z_n| < \rho \} \subset U \times S \]
holds
\[ \tilde{f}|\tilde{U} = f|\tilde{U}. \]

Theorem 1.10 implies \( \tilde{f}|(U \times S \cup G \times R) = f. \]
\[ \square \]

The domain
\[ U \times S \cup G \times R \]
from Figure 1.3 is named a Hartogs figure in \( \mathbb{C}^n. \)

**Corollary 1.15 (Kugelsatz).** For \( n \geq 2 \) consider two polyradii
\[ r = (r_1, ..., r_n) \text{ and } \tilde{r} = (\tilde{r}_1, ..., \tilde{r}_n) \text{ with } r_j < \tilde{r}_j \text{ for all } j = 1, ..., n. \]

Then the canonical restriction
\[ \mathcal{O}(\Delta(\tilde{r})) \rightarrow \mathcal{O}(\Delta(\tilde{r}) \setminus \overline{A}(r)) \]
is surjective, see Figure 1.4

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**Fig. 1.4** Kugelsatz
1.2 First applications of Cauchy’s integral formula

**Theorem 1.16 (Weierstrass’ convergence theorem).** Consider an open set $U \subset \mathbb{C}^n$ and a compact convergent sequence $(f_\nu)_{\nu \in \mathbb{N}}$ of holomorphic functions

$$f_\nu : U \to \mathbb{C}, \ \nu \in \mathbb{N}.$$  

- There exists the limit
  $$f := \lim_{\nu \to \infty} f_\nu$$
  as a holomorphic function
  $$f : U \to \mathbb{C}$$
- For each polyindex $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$ also the sequence
  $$(f^{(j)})_{\nu \in \mathbb{N}} := \left( \frac{\partial^{\mid j\mid} f_\nu}{\partial z_1^{j_1} \cdots \partial z_n^{j_n}} \right)_{\nu \in \mathbb{N}}$$
  of partial derivatives is compact convergent in $U$ with limit $f^{(j)}$.

**Proof.** i) It is well-known that the limit of a compact convergent sequence of continuous functions exists as a continuous function. In addition, Weierstrass’ convergence theorem from complex analysis of one variable implies that the limit $f$ is holomorphic in each variable separately. Theorem 1.6 implies that $f$ is holomorphic as a function of several variables.

ii) In order to prove the compact convergence of the sequence of partial derivatives it suffices to prove the claim for the partial derivatives

$$\left( \frac{\partial f_\nu}{\partial z_k} \right)_{\nu \in \mathbb{N}}, \ k = 1, \ldots, n,$$

of order $= 1$ and to apply the result in an iterative way.

For given $k = 1, \ldots, n$ consider an arbitrary compact $K \subset U$. There exist a radius $r > 0$ and a compact $K'$ with

$$K \subset K' \subset U$$

such that for all $z = (z_1, \ldots, z_n) \in K$ and $\zeta \in \mathbb{C}$ with $|\zeta| < r$ also

$$(z_1, \ldots, z_{k-1}, z_k + \zeta, z_{k+1}, \ldots, z_n) \in K'$$

Then each holomorphic function $g \in O(U)$ satisfies according to the 1-dimensional Cauchy integral formula for $z \in K$

$$\frac{\partial g}{\partial z_k}(z) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=r} \frac{g(z_1, \ldots, z_{k-1}, z_k + \zeta, z_{k+1}, \ldots, z_n)}{\zeta^2} \, d\zeta$$

The substitution
\[ \zeta = r \cdot e^{i \phi} \]

shows: The norm with respect to the compact sets \( K \) and \( K' \) satisfies the estimate
\[
\left\| \frac{\partial g}{\partial z_k} \right\|_K \leq \frac{1}{r} \cdot \|g\|_{K'}
\]

Applying this estimate to each function
\[
g := f_\nu - f, \ \nu \in \mathbb{N},
\]
proves the compact convergence of the sequence of all derivatives of order \( \nu = 1 \) as a consequence of the compact convergence of the sequence \((f_\nu)_{\nu \in \mathbb{N}}\).

\[ \square \]

**Definition 1.17 (Fréchet space).**

1. A topological vector space is a vector space such that addition and scalar multiplication are continuous functions. In the following the base field \( \mathbb{C} \) is provided with its Euclidean topology.

2. A seminorm on a complex vector space \( V \) is a map
\[
p : V \rightarrow \mathbb{R}_+
\]
satisfying:

i) For all \( \lambda \in \mathbb{C} \) and for all \( v \in V \)
\[
p(\lambda \cdot v) = |\lambda| \cdot p(v)
\]

ii) For all \( v_1, v_2 \in V \)
\[
p(v_1 + v_2) \leq p(v_1) + p(v_2) \text{ (Triangle inequality)}. \]

A seminorm \( p \) is a norm if
\[
p(v) = 0 \implies v = 0.
\]

3. A topological complex vector space \( V \) is a Fréchet space if \( V \) is a complete Hausdorff space with the topology defined by a countable family \((p_\nu)_{\nu \in \mathbb{N}}\) of seminorms, i.e. a neighbourhood basis of \( 0 \in V \) is given by the finite intersections of sets of the form
\[
V(\nu, \varepsilon) := \{ v \in V : p_\nu(v) < \varepsilon \}, \ \varepsilon > 0, \ \nu \in \mathbb{N}.
\]

Apparently the concept of Fréchet spaces generalizes the concept of Banach spaces by replacing a fixed norm by a countable family of seminorms. A sequence
of elements of a Fréchet space \( V \) is a \textit{Cauchy sequence} if for each neighbourhood of zero \( W \subset V \) exists \( N \in \mathbb{N} \) such that for all \( \nu, \mu \geq N \)

\[ f_\nu - f_\mu \in W. \]

Each Fréchet space \( V \) is metrizable, e.g. by the metric

\[ d(f, g) := \sum_{\nu=0}^{\infty} \frac{1}{2^\nu} \frac{p_\nu(f-g)}{1 + p_\nu(f-g)}, \quad f, g \in V. \]

\textbf{Remark 1.18 (Fréchet space).}

1. \textit{Subspace and quotient:} Consider a Fréchet space \( V \): If \( W \subset V \) is a closed subspace then also \( W \) and \( V/W \) are Fréchet spaces with the induced topologies.

2. \textit{Countable product of Fréchet space:} For a countable family \( (V_i)_{i \in \mathcal{I}} \) also the product

\[ V := \prod_{i \in \mathcal{I}} V_i \]

becomes a Fréchet space when provided with the product topology.

3. \textit{Open mapping theorem:} A continuous surjective linear map \( \phi : V \to W \) between two Fréchet spaces is an open map.

A good introduction to Fréchet spaces are the first chapters of Rudin’s book [25], see also [17, App. B, Theor. 6].

\textbf{Proposition 1.19 (Topology of compact convergence).} Consider an open set \( U \subset \mathbb{C}^n \).

Provided with the topology of compact convergence both vector spaces

\[ \mathcal{C}(U) := \{ f : U \to \mathbb{C} : f \text{ continuous} \} \]

and

\[ \mathcal{O}(U) := \{ f : U \to \mathbb{C} : f \text{ holomorphic} \} \]

are Fréchet spaces. Moreover

\[ \mathcal{O}(U) \subset \mathcal{C}(U) \]

is a closed subspace. For open \( V \subset U \) the restriction map

\[ \mathcal{O}(U) \to \mathcal{O}(V), \ f \mapsto f|V, \]

is continuous.
Proof. One chooses an exhaustion \((U_\nu)_{\nu\in\mathbb{N}}\) of \(U\) by relatively compact subsets
\[U_\nu \subset \subset U_{\nu+1} \subset \subset U, \quad \nu \in \mathbb{N},\]
and defines the seminorms
\[p_\nu : \mathcal{C}(U) \to \mathbb{R}_+, \quad p_\nu(f) := \|f\|_{U_\nu} := \sup \{|f(z)| : z \in U_\nu\}\]
The finite intersections of the sets
\[V(\nu, \varepsilon) := \{f \in \mathcal{C}(U) : p_\nu(f) < \varepsilon\}, \quad \nu \in \mathbb{N}, \quad \varepsilon > 0,\]
are a neighbourhood basis of \(0 \in \mathcal{C}(U)\) of a Fréchet topology on \(\mathcal{C}(U)\).
A sequence of continuous functions on \(U\) is convergent in the Fréchet topology of \(\mathcal{C}(U)\) iff the functions are uniformly convergent on compact subsets of \(U\). Here one uses that the sequence \((U_\nu)_\nu\) is an exhaustion of \(U\) by relatively compact open subsets. The subspace
\[\mathcal{O}(U) \subset \mathcal{C}(U),\]
provided with the subspace topology, is closed as a consequence of Weierstrass’ convergence theorem, Theorem 1.16. Hence \(\mathcal{O}(U)\) is a Fréchet space with respect to the topology of compact convergence. The continuity of the restriction maps is a direct consequence of the definition of the topology, because a compact subset of \(V\) is also a compact subset of \(U\). \(\square\)

1.3 Dolbeault’s lemma

Dolbeault’s lemma deals with the exterior differential operator \(d''\) on smooth and holomorphic differential forms on polydiscs. It will be the basis for many computations of cohomology groups in later chapters.

We recall some basic results about differential forms defined in an open set
\[U \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}.\]
We split the complex coordinates \(z = (z_1, \ldots, z_n) \in \mathbb{C}^n\) in real and imaginary part
\[z_\nu = x_\nu + i \cdot y_\nu, \quad \nu = 1, \ldots, n.\]

Remark 1.20 (Differential forms and exterior derivation).

- Smooth functions: We denote by \(\mathcal{E}(U)\) the complex vector space of smooth functions
\[f : U \to \mathbb{C}\]
• **Smooth alternating differential forms:** For $1 \leq r \leq 2n$ we denote by $\mathcal{E}^r(U)$ the complex vector space of differential forms on $U$ of degree $r$. Each differential form $\omega \in \mathcal{E}^r(U)$ has with respect to real coordinates $(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$ the form

$$\omega = \sum_{1 \leq j_1 < \ldots < j_r \leq 2n} f_{j_1, \ldots, j_r} \cdot dx_{j_1} \wedge \ldots \wedge dx_{j_r}$$

with elements $f_{j_1, \ldots, j_r} \in \mathcal{E}^r(U)$. Due to the complex structure on $U \subset \mathbb{C}^n$ the vector space $\mathcal{E}^r(U)$ splits as the direct sum

$$\mathcal{E}^r(U) = \bigoplus_{1 \leq p,q \leq n \atop p+q=r} \mathcal{E}^{p,q}(U)$$

with $\mathcal{E}^{p,q}(U)$ the vector space of all $(p,q)$-forms. A differential form $\omega \in \mathcal{E}^{p,q}(U)$ has the form

$$\omega = \sum_{1 \leq j_1 < \ldots < j_p \leq n \atop 1 \leq j_{p+1} < \ldots < j_q \leq n} f_{j_1, \ldots, j_p, j_{p+1}, \ldots, j_q} \, dz_{j_1} \wedge \ldots \wedge dz_{j_p} \wedge d\bar{z}_{j_{p+1}} \wedge \ldots \wedge d\bar{z}_{j_q}$$

with $f_{j_1, \ldots, j_p, j_{p+1}, \ldots, j_q} \in \mathcal{E}^r(U)$. Here

$$d\bar{z}_v = dx_v + i \cdot dx_{n+v} \text{ (holomorphic differential)}$$

and

$$d\bar{z}_v = dx_v - i \cdot dx_{n+v} \text{ (antiholomorphic differential)}$$

We often use the shorthand

$$\omega = \sum_{IJ} f_{IJ} \, dz_I \wedge d\bar{z}_J \in \mathcal{E}^{p,q}(U)$$

• **Linear differential operators:** There are $\mathbb{C}$-linear differential operators

$$d' : \mathcal{E}(U) \to \mathcal{E}^{1,0}(U), \quad d' f := \sum_{v=1}^n \frac{\partial f}{\partial z_v} \, dz_v$$

and

$$d'' : \mathcal{E}(U) \to \mathcal{E}^{0,1}(U), \quad d'' f := \sum_{v=1}^n \frac{\partial f}{\partial \bar{z}_v} \, d\bar{z}_v$$

and

$$d := d' + d'' : \mathcal{E}(U) \to \mathcal{E}^1(U)$$

A function $f \in \mathcal{E}(U)$ is holomorphic if and only if $d'' f = 0$.

• **Exterior derivation of higher order:** For $p, q \in \mathbb{N}$ the linear differential operators from the previous step extend to $\mathbb{C}$-linear differential operators
\[ d' : \mathcal{E}^{p,q}(U) \to \mathcal{E}^{p+1,q}(U) \]

and

\[ d'' : \mathcal{E}^{p,q}(U) \to \mathcal{E}^{p,q+1}(U) \]

as follows: For

\[ \omega = f \cdot dz_I \wedge d\bar{z}_J \in \mathcal{E}^{p,q}(U) \]

define

\[ d' \omega = d' f \cdot dz_I \wedge d\bar{z}_J \in \mathcal{E}^{p+1,q}(U) \]

and

\[ d'' \omega = d'' f \cdot dz_I \wedge d\bar{z}_J \in \mathcal{E}^{p,q+1}(U) \]

and extend the definition by \( \mathbb{C} \)-linearity.

**Proposition 1.21 (Exterior derivation as a chain map).** For an open set \( U \subset \mathbb{C}^n \) the composition of exterior derivations satisfies

\[ d' \circ d' = d'' \circ d'' = 0 \]

**Proof.** Consider a single summand

\[ \omega = f \cdot dz_I \wedge d\bar{z}_J \in \mathcal{E}^{p,q}(U) \]

Compute

\[ d' \omega = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_I \wedge d\bar{z}_J \wedge d\bar{z}_K \]

and

\[ d'(d' \omega) = \sum_{k=1,j=1}^{n} \frac{\partial^2 f}{\partial z_k \partial z_j} dz_I \wedge d\bar{z}_J \wedge \left( d\bar{z}_K + d\bar{z}_j \wedge d\bar{z}_k \right) = \]

\[ = \sum_{1 \leq k < j \leq n} \left( \frac{\partial^2 f}{\partial z_k \partial z_j} - \frac{\partial^2 f}{\partial z_j \partial z_k} \right) dz_I \wedge d\bar{z}_J \wedge \left( d\bar{z}_K + d\bar{z}_j \wedge d\bar{z}_k \right) = 0 \]

and analogously for \( d'' \). The vanishing depends on the following fact: For smooth functions the higher partial derivations do not depend on the order of differentiation. \( \square \)

**Definition 1.22 (Holomorphic differential forms).** Consider an open set \( U \subset \mathbb{C}^n \).

The elements

\[ \omega = \sum_{I} f_I \cdot dz_I := \sum_{1 \leq j_1 < \ldots < j_p \leq n} f_{j_1 \ldots j_p} \cdot dz_{j_1} \wedge \ldots \wedge dz_{j_p} \in \mathcal{E}^{p,0}(U) \text{ with } f_{j_1 \ldots j_p} \in \mathcal{O}(U) \]

form the vector space \( \Omega^p(U) \) of holomorphic differential \( p \)-forms.
Note
\[ \Omega^p(U) = \ker [d'' : \mathcal{E}^{p,0}(U) \to \mathcal{E}^{p,1}(U)] \]

Remark 1.23 (Fréchet topology on the vector spaces of smooth differential forms).
Consider an open set \( U \subset \mathbb{C}^n \simeq \mathbb{R}^{2n} \).

1. **Smooth functions:** After choosing an exhaustion \((U_j)_{j \in \mathbb{N}}\) of \( U \) by relatively compact open sets one defines the family of seminorms on \( \mathcal{E}(U) \)
\[ p(j,k) : \mathcal{E}(U) \to \mathbb{R}_+ \]
with
\[ p(j,k)(f) := \left\{ \sup \left| \frac{\partial^{k_1 + \ldots + k_{2n}} f}{\partial z_{1}^{k_1} \cdots \partial z_{2n}^{k_{2n}}} (z) \right| : z \in U_j \text{ and } k_1 + \ldots + k_{2n} \leq k \right\}, \quad j, k \in \mathbb{N} \]
The family of seminorms provides \( \mathcal{E}(U) \) with the structure of a Fréchet space.

2. **Smooth differential forms:** The Fréchet structure on \( \mathcal{E}(U) \) carries over to the coefficients of the differential forms, and eventually to the vector spaces \( \mathcal{E}^{p,q}(U) \) and \( \mathcal{E}^r(U) \).

3. **Holomorphic differential forms:** Similarly, the Fréchet structure on \( \mathcal{O}(U) \) from Proposition 1.19 carries over to a Fréchet structure on \( \Omega^p(U), p \geq 1 \).

4. One checks that all Fréchet structures are independent from the choice of the exhaustion.

There are closed inclusions as closed Fréchet spaces
\[ \Omega^p(U) \subset \mathcal{E}^{p,0}(U) \text{ and } \mathcal{O}(U) \subset \mathcal{E}(U) \subset \mathcal{C}(U). \]

We now enter into Dolbeault theory. Consider an open set \( U \subset \mathbb{C}^n \). Recall that a differential form \( \omega \in \mathcal{E}^{p,q}(U) \) is named

- **closed** with respect to \( d'' \) if
  \[ d'' \omega = 0 \in \mathcal{E}^{p,q+1}(U) \]
- and **exact** with respect to \( d'' \) if
  \[ \omega = d'' \sigma \]
for a suitable \( \sigma \in \mathcal{E}^{p,q-1}(U) \).
Due to the chain condition
\[ d'' \circ d'' = 0 \]
each exact form is closed. Dolbeault’s lemma investigates the reverse direction: Which properties of \( U \) ensure that each \( d'' \)-closed \( \omega \) form is also \( d'' \)-exact? The Dolbeault groups
\[
\ker \left[ d'' : \mathcal{E}^{p,q}(U) \to \mathcal{E}^{p,q+1}(U) \right]
\]
\[ \text{im} \left[ d'' : \mathcal{E}^{p,q-1}(U) \to \mathcal{E}^{p,q}(U) \right] \]
measure the obstructions against exactness on \( U \).

From the viewpoint of the differential operator \( d'' \) one has to integrate for given \( \omega \) the differential equation on \( U \)
\[ d'' \sigma = \omega. \]

Hence the basic differential equation from Dolbeault theory is the inhomogeneous linear differential equation
\[ \frac{\partial f}{\partial z} = g \]
for a given smooth function \( g = g(z,s) \) with 1-dimensional variable \( z \) and parameter \( s \). If \( g \) has compact support with respect to the variable \( z \) then a solution \( f \) can be obtained by an integral formula with the Cauchy kernel, see Theorem 1.24.

**Theorem 1.24 (Solving the \( \partial \)-problem).** Consider an open \( U \subset \mathbb{C}^n \) and a smooth function
\[ g : \mathbb{C} \times U \to \mathbb{C}. \]
Assume that for each \( s \in U \) the restriction to the first variable
\[ g(-,s) : \mathbb{C} \to \mathbb{C} \]
has compact support. Then the function
\[ f := \mathbb{C} \times U \to \mathbb{C}, \; f(z,s) := \frac{1}{2\pi i} \int_{\mathbb{C}} g(\zeta,s) \frac{d\zeta}{\zeta - z} d\zeta \wedge d\overline{\zeta} \]
is smooth and satisfies
\[ \overline{\partial} f := \frac{\partial f}{\partial \overline{z}} = g \]

**Proof.** i) The integrand has no singularities: Fix \( z \in \mathbb{C} \) and introduce polar coordinates in \( \mathbb{C} \)
\[ \zeta = z + re^{i\phi} \]
such that
\[ d\zeta \wedge d\overline{\zeta} = -2i r \, dr \wedge d\phi \]
1.3 Dolbeault’s lemma

Then

\[ \frac{1}{\zeta - z} d\zeta \wedge d\zeta = -2ie^{-i\phi} \, dr \wedge d\phi \]

Because of the compact support of \( g(\cdot, s) \), for each \((z, s) \in \mathbb{C} \times U\) the integral

\[ \frac{1}{2\pi i} \int_{\mathbb{C}} g(\zeta, s) \frac{d\zeta}{\zeta - z} \]

is well-defined and finite.

ii) Computing the \( \overline{\partial} \)-derivative: After translating \( \zeta \) to \( \zeta + z \) to remove the parameter \( z \) from the denominator we get

\[ f(z, s) := \frac{1}{2\pi i} \int_{\mathbb{C}} g(\zeta, s) \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\mathbb{C}} g(\zeta + z, s) \frac{d\zeta}{\zeta} \]

and compute

\[ \frac{\partial f}{\partial z}(a, s) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g(z + \zeta, s)}{\partial \zeta} \frac{d\zeta}{\zeta - a} \]

We now retranslate \( \zeta + z \) to \( \zeta \), evaluate at the point \((a, s) \in \mathbb{C} \times U\), use the compact support of \( g \) with respect to the variable \( \zeta \), and apply Stokes’ theorem

\[ \frac{\partial f}{\partial z}(a, s) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g(z + \zeta, s)}{\partial \zeta} \frac{d\zeta}{\zeta - a} \]

Here the exterior derivative is computed as

\[ d \left( \frac{g(\zeta, s)}{\zeta - a} \right) = d' \left( \frac{g(\zeta, s)}{\zeta - a} \right) + d'' \left( \frac{g(\zeta, s)}{\zeta - a} \right) \]

\[ = d'' \left( \frac{g(\zeta, s)}{\zeta - a} \right) = \frac{\partial g \cdot (z - a)}{(z - a)^2} \frac{d\zeta}{\zeta - a} \]

\[ \square \]

As a consequence of the solution of the \( \overline{\partial} \)-problem in Theorem 1.24 the following Proposition 1.25 shows: For a polydisc each \( d'' \)-closed differential form is also \( d'' \)-exact - but possibly one has to shrink the domain of the solution.
Proposition 1.25 (Solving the $d''$-equation after shrinking). Consider an open polydisc $\Delta \subset \mathbb{C}^n$ and a concentric, relatively compact polydisc $\Delta' \subset \subset \Delta$.

For each pair $p, q \in \mathbb{N}$, $q \geq 1$, and for each closed differential form
\[ \omega \in E^{p,q}(\Delta) \text{ with } d''\omega = 0 \]
exists a smooth differential form
\[ \sigma \in E^{p,q-1}(\Delta') \]
satisfying
\[ d''\sigma = \omega |_{\Delta'} \]

Proof. The proof relies on Theorem 1.24. It proceeds by induction on the highest index $\nu$ of the differentials $dz_\nu$ which enter into the definition of $\omega$: For each $\nu = 0, \ldots, n$ define
\[ A_\nu(\Delta) := \left\{ \omega \in E^{p,q}(\Delta) : \omega = \sum_I \sum_{J \leq \nu} a_{IJ} dz_I \wedge d\bar{z}_J \right\} \]
Here $J \leq \nu$ indicates that each tuple $(j_1, \ldots, j_q) \in J$ satisfies
\[ j_s \leq \nu \text{ for } s = 1, \ldots, q. \]

We prove by induction on $\nu$ the statement $A_\nu(\Delta)$: The claim of the proposition holds for all polydiscs $\Delta$ and all $\omega \in A_\nu(\Delta)$.

Induction start $\nu = 0$: Then $A_0(\Delta) = \{0\}$ because $q \geq 1$. Set
\[ \sigma := 0 \in E^{p,q-1}(\Delta'). \]

Induction step $\nu - 1 \mapsto \nu$: We choose a concentric polydisc $\Delta''$ with $\Delta' \subset \subset \Delta'' \subset \subset \Delta$.

The polyradii $r'$, $r''$, $r$ of $\Delta' \subset \subset \Delta'' \subset \subset \Delta$ satisfy
\[ r' < r'' < r. \]

We decompose a given $\omega \in A_\nu(\Delta)$
\[ \omega = \sum_I \sum_{J \leq \nu} f_{IJ} \, dz_I \wedge dz_J \text{ with } d'' \omega = 0 \]

as

\[ \omega = d \bar{z} \wedge \alpha + \beta \]

with

\[ \alpha = \sum_I \sum_{J \leq \nu - 1} a_{IJ} \, dz_I \wedge dz_J \in \mathcal{E}^{p,q-1}(\Delta) \]

and

\[ \beta \in A_{\nu-1}(\Delta) \]

Then

\[ d'' \omega = \sum_I \sum_{J \leq \nu} \left( \sum_{m=1}^{\nu} \frac{\partial f_{IJ}}{\partial \bar{z}_m} \, d\bar{z}_m \wedge dz_I \wedge d\bar{z}_m \right) \]

The assumption \( d'' \omega = 0 \) implies for \( m > \nu \)

\[ \frac{\partial f_{IJ}}{\partial \bar{z}_m} = 0, \]

i.e. the coefficients \( f_{IJ} \) depend holomorphically on the final variables

\[ z_{\nu+1}, \ldots, z_n. \]

Each coefficient \( a_{IJ} \) agrees up to sign with a coefficient \( f_{I'J'} \) for a suitable pair of indices \( I'J' \). Because

\[ \Delta'' \subset \subset \Delta \]

there exists a smooth function \( \tilde{a}_{IJ} \in \mathcal{E}(\mathbb{C}^n) \) with

\[ supp \, \tilde{a}_{IJ} \subset \Delta \text{ and } \tilde{a}_{IJ}|\Delta'' = a_{IJ}|\Delta'' \]

At this point one has to shrink the domain from \( \Delta \) to \( \Delta'' \) to extend \( a_{IJ} \) to a global smooth function \( \tilde{a}_{IJ} \) with compact support. In particular \( \tilde{a}_{IJ} \) has compact support with respect to the distinguished variable \( \bar{z}_{\nu} \) and depends holomorphically on the final variables \( z_{\nu+1}, \ldots, z_n \). Hence Theorem 1.24 applies to the function

\[ \mathbb{C} \times \mathbb{C}^{n-1} \rightarrow \mathbb{C}, \ (\zeta, \bar{\zeta}') \mapsto \tilde{a}_{IJ}(z_1, \ldots, z_{\nu-1}, \bar{\zeta}, z_{\nu+1}, \ldots, z_n) \]

It provides the smooth function

\[ c_{IJ} \in \mathcal{E}(\Delta'') \]

defined as

\[ c_{IJ}(z_1, \ldots, z_n) := \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\tilde{a}_{IJ}(z_1, \ldots, z_{\nu-1}, \bar{\zeta}, z_{\nu+1}, \ldots, z_n)}{\bar{\zeta} - \bar{z}_\nu} \, d\zeta \wedge d\bar{\zeta} \]
It satisfies
\[ \frac{\partial c_{IJ}}{\partial z_\nu} = \tilde{a}_{IJ}|\Delta'' = a_{IJ}|\Delta'' \]

Like \( \tilde{a}_{IJ} \) also the function \( c_{IJ} \) depends holomorphically on the variables \( z_{v+1}, \ldots, z_n \). We define
\[
\gamma := \sum_I \sum_{J \leq v} c_{IJ} \, dz_I \wedge d\bar{z}_J \in \mathcal{E}^{p,q-1}(\Delta'')
\]

Then on \( \Delta'' \)
\[
d'' \gamma := \sum_I \sum_{J \leq v-1} \left( \sum_{m=1}^v \frac{\partial c_{IJ}}{\partial z_m} d\bar{z}_m \wedge d\bar{z}_i \wedge d\bar{z}_j \right) = d\bar{z}_v \wedge \alpha + \delta \in \mathcal{E}^{p,q}(\Delta'')
\]

with a form
\[
\delta \in A_{v-1}(\Delta'')
\]
having coefficients not depending on the final variables \( z_{v+1}, \ldots, z_n \). Then
\[
\omega - d'' \gamma = \beta - \delta \in A_{v-1}(\Delta'')
\]
The induction assumption applies to
\[
\beta - \delta \in A_{v-1}(\Delta'')
\]
because
\[
d''(\beta - \delta) = d'' \omega - (d'' \circ d'') \gamma = 0
\]
The induction assumption provides a form
\[
\eta \in \mathcal{E}^{p,q-1}(\Delta')
\]
satisfying
\[
d'' \eta = \beta - \delta = \omega - d'' \gamma
\]
Set
\[
\sigma := \gamma + \eta \in \mathcal{E}^{p,q-1}(\Delta')
\]
On \( \Delta' \)
\[
d'' \sigma = d'' \gamma + d'' \eta = \omega
\]
which finishes the induction step and terminates the proof. \( \square \)

The proof of Proposition 1.25 introduces the intermediate polydisc \( \Delta'' \) to obtain the functions \( \tilde{a}_{IJ} \) with compact support. This principle "obtaining solutions after shrinking" has its counterpart in the principle "extending solutions by exhaustion", see Proposition 1.26. We will see more examples of these two principles in subsequent passages.
Proposition 1.25 solves the $d''$-problem on relatively compact polydiscs. The standard method to extend these local solutions to a global solution on arbitrary polydiscs is to exhaust the latter by relatively compact polydiscs. The Mittag-Leffler principle of induction, Proposition 1.26, applies. It extends the local solutions after small changes during each step to a global solution. The global solution is the projective limit of a suitable sequence of compatible local solutions. The existence of the global solution of the $d''$-problem will be the Dolbeault lemma from Theorem 1.27.

**Proposition 1.26 (Mittag-Leffler principle of exhaustion).** Consider a sequence $(M_i, d_i)_{i \in \mathbb{N}}$ of complete metric spaces, $M_0 \neq \emptyset$, together with continuous maps

$$\rho_i : M_i \to M_{i-1}, \ i \geq 1$$

with dense image. Then the projective limit satisfies

$$\lim_{\leftarrow} (M_i, \rho_i) \neq \emptyset$$

i.e. there exists an infinite ascending sequence of elements

$$(f_i \in M_i)_{i \in \mathbb{N}}$$

which satisfy the compatibility condition

$$\rho(f_i) = f_{i-1}, \ i \geq 1.$$  

The idea of the proof is to construct the required sequence step by step by induction. Step $k - 1$ constructs a chain of length $k - 1$. At step $k$ each element of the chain is corrected by a difference of order at most $1/2^k$, such that the resulting chain can be continued further by one element, see Figure 1.5.
Proof. We prove by induction on \( k \in \mathbb{N} \) the statement \( \mathcal{A}(k) \): There exists a chain \((g_i^k)_{i=0,...,k}\) of length \( k \) of compatible elements \( g_i^k \in M_i, i = 0,...,k, \) satisfying
\[
d_d(g_i^k, g_{i-1}^k) < 1/2^k,
\]
i.e. for \( i = 0,...,k - 1 \) holds
\[
d_i(g_i^k, g_{i-1}^k) < 1/2^k.
\]
i) Induction start \( k = 0 \): Choose an arbitrary element \( g_0^0 \in M_0 \).

ii) Induction step \( k - 1 \rightarrow k \): Consider the chain of length \( k - 1 \) formed by the elements
\[
(g_i^{k-1})_{i=0,...,k-1}
\]
from the induction assumption. Due to the continuity of the restriction maps \( \rho_i \) and the resulting continuity of their finite composition there exists
\[
0 < \delta < 1/2^k
\]
such that each element in the \( \delta \)-neighbourhood of the final element
\[
g_{k-1}^{k-1} \in M_{k-1}
\]
maps into the \( (1/2^k) \)-neighbourhoods of the elements
\[
g_{i}^{k-1}, i = 0,...,k - 2,
\]
by the corresponding composition. Because
\[ \rho_k : M_k \to M_{k-1} \]
has dense image, one may choose as the seed of the new chain of length \( k \) an element
\[ g^k \in M_k \text{ with } \rho_k(g^k) \in M_{k-1} \]
in the \( \delta \)-neighbourhood of \( g^{k-1} \in M_{k-1} \). Defining the elements
\[ g_i^k \in M_i, \; i = 0, \ldots, k-1, \]
as the iterated images of \( g^k \) ends the induction step. \( \square \)

**Theorem 1.27 (Dolbeault’s lemma).** Consider an open polydisc \( \Delta \subset \mathbb{C}^n \). For each \( p \in \mathbb{N} \) the following sequence of complex vector spaces and morphisms is exact
\[ 0 \to \Omega^p(\Delta) \to \mathcal{E}(\Delta)^{p,0} \xrightarrow{d''} \mathcal{E}(\Delta)^{p,1} \xrightarrow{d''} \cdots \xrightarrow{d''} \mathcal{E}(\Delta)^{p,n} \to 0 \]

Exactness of the sequence means that the kernel of each morphism equals the image of the previous morphism. In Theorem 1.27 we relax the condition on the polyradius
\[ r = (r_1, \ldots, r_n) \]
and allow also the component radius \( 0 < r_j = \infty \).

We illustrate the idea of the proof of Theorem 1.27 by the special case of the exactness
\[ \mathcal{E}^{p,0}(\Delta) \xrightarrow{d''} \mathcal{E}^{p,1}(\Delta) \xrightarrow{d''} \mathcal{E}^{p,2}(\Delta) \]
Consider a given
\[ \omega \in \mathcal{E}^{p,1}(\Delta) \text{ with } d'' \omega = 0 \]
The polydisc \( \Delta \subset \mathbb{C}^n \) is not necessarily relatively compact. But there exists an exhaustion of \( \Delta \) by a sequence \( (\Delta_i)_{i \in \mathbb{N}} \) of relatively compact polydiscs \( \Delta_{i-1} \subset \subset \Delta_i \). For each \( i \in \mathbb{N} \) Proposition 1.25 provides a local solution \( \sigma_i \) on \( \Delta_i \) satisfying
\[ d'' \sigma_i = \omega|_{\Delta_i} \]
The family \( (\sigma_i)_i \) faces two problems: First, the members of the family are not necessarily compatible with respect to the restriction from \( \Delta_i \) to \( \Delta_{i-1} \). Secondly, in general a given form \( \sigma_{i-1} \) does not even extend approximatively to \( \Delta_i \), because the smooth \( \sigma_{i-1} \) has no approximation by Taylor polynomials. The local solutions \( \sigma_i \) transform the second problem into a question about holomorphic \( p \)-form on \( \Delta_i \). To solve the first problem one shows that the resulting twisted maps
have dense image due to the Taylor approximation of the holomorphic coefficient functions. Then the Mittag-Leffler principle of induction solves the problem of compatible extension.

Proof (Theorem 1.27). Recall from Remark 1.23: Each vector space $\mathcal{E}^{p,q}(P)$, $P$ polydisc carries a Fréchet topology, hence it is a complete metric space. Exterior differentiation satisfies

$$d''|_{\Omega^p} = 0 \text{ and } d'' \circ d'' = 0.$$

Hence it remains to show that each closed form is also exact, i.e. if

$$\omega \in \mathcal{E}^{p,q}(\Delta) \text{ with } d'' \omega = 0$$

then

- if $q = 0$
  $$\omega \in \Omega^p(\Delta)$$

- respectively if $q \geq 1$ exists
  $$\sigma \in \mathcal{E}^{p,q-1}(\Delta) \text{ with } \omega = d'' \sigma.$$

Choose an arbitrary but fixed $p \in \mathbb{N}$. Exactness at the index $q = 0$ is well known, hence we prove exactness at a given index $q \geq 1$.

i) Exhaustion by relatively compact concentric polydiscs: We choose an exhaustion of $\Delta$ by a sequence of relatively compact concentric polydiscs

$$\Delta_0 \subset \subset \Delta_1 \subset \subset \ldots \subset \subset \Delta_i \subset \subset \ldots$$

For given $q \geq 1$

$$\omega \in \mathcal{E}^{p,q}(\Delta) \text{ with } d'' \omega = 0$$

we consider for each $i \in \mathbb{N}$ the affine vector subspace of $\mathcal{E}^{p,q-1}(\Delta_i)$

$$M_i := M_{i,q} := \{ \sigma \in \mathcal{E}^{p,q-1}(\Delta_i) : d'' \sigma = \omega|_{\Delta_i} \}, \ i \in \mathbb{N},$$

of solutions over $\Delta_i \subset \subset \Delta$. Due to Proposition 1.25 we can choose for each $i \in \mathbb{N}$ a fixed local solution

$$\sigma_i \in M_i.$$

For $i \geq 1$ let

$$\rho_i : M_i \rightarrow M_{i-1}$$

denote the canonical restriction. The proof of the claim is by induction on $q \geq 1$.

ii) Induction start $q = 1$: We show for each $i \in \mathbb{N}$ the exactness of

$$\mathcal{E}^{p,0}(\Delta_i) \xrightarrow{d''} \mathcal{E}^{p,1}(\Delta_i) \xrightarrow{d''} \mathcal{E}^{p,2}(\Delta_i)$$
The map
\[ \psi_i : \Omega^p(\Delta_i) \to M_i, \; \eta \mapsto \sigma_i + \eta, \]
represents the space \( M_i \) of solutions on \( \Delta_i \) as the affine subspace
\[ M_i = \sigma_i + \Omega^p(\Delta_i) \subset \mathcal{E}^p(\Delta_i) : \]
Each element \( \sigma \in M_i \) has the form
\[ \sigma = \sigma_i + (\sigma - \sigma_i) \]
with
\[ \eta := \sigma - \sigma_i \in \Omega^p(\Delta_i) \text{ satisfying } d''\eta = 0. \]
Due to Remark 1.23 each vector space \( \Omega^p(\Delta_i) \) carries a Fréchet topology, hence it is a complete metric space. For \( i \geq 1 \) consider the continuous maps between complete metric spaces
\[ \tau_i : \Omega^p(\Delta_i) \to \Omega^p(\Delta_{i-1}), \; \eta \mapsto (\eta + (\sigma_i - \sigma_{i-1}))|_{\Delta_{i-1}} \]
Each map \( \tau_i \) has dense image: The holomorphic form
\[ (\sigma_i - \sigma_{i-1})|_{\Delta_{i-1}} \in \Omega^p(\Delta_{i-1}) \]
can be approximated by using the Taylor polynomials of its holomorphic coefficients. Hence each holomorphic form \( \sigma \in \Omega^p(\Delta_{i-1}) \) can be approximated by using the Taylor polynomials of the coefficients of the holomorphic form
\[ (\sigma - (\sigma_i - \sigma_{i-1}))|_{\Delta_{i-1}} \in \Omega^p(\Delta_{i-1}). \]
The following diagram commutes
\[
\begin{array}{ccc}
\Omega^p(\Delta_i) & \xrightarrow{\tau_i} & \Omega^p(\Delta_{i-1}) \\
\psi_i \downarrow & & \psi_{i-1} \downarrow \\
M_i & \xrightarrow{\rho_i} & M_{i-1}
\end{array}
\]
The Mittag-Leffler principle, Proposition 1.26, applies to the family
\[ (\Omega^p(\Delta_i), \tau_i)_{i \in \mathbb{N}} \]
It provides a compatible family
\[ (\eta_i \in \Omega^p(\Delta_i))_{i \in \mathbb{N}} \text{ with } \tau_i(\eta_i) = \eta_{i-1} \]
For each \( i \in \mathbb{N} \) define
\[ \tilde{\sigma}_i := \psi_i(\eta_i) = \sigma_i + \eta_i \in M_i \]

obtaining the compatibility of the adapted solutions \( \tilde{\sigma}_i \)

\[ \tilde{\sigma}_i|_{\Delta_{i-1}} = (\rho_i \circ \psi_i)(\eta_i) = (\psi_{i-1} \circ \tau_i)(\eta_i) = \psi_{i-1}(\eta_{i-1}) = \tilde{\sigma}_{i-1} \]

The family \((\tilde{\sigma}_i)_{i \in \mathbb{N}}\) defines the form \( \sigma \in \mathcal{E}^{p,0}(\Delta) \) satisfying \( \sigma|_{\Delta_i} = \tilde{\sigma}_i, \ i \in \mathbb{N}, \) and \( d''\sigma = \omega. \)

iii) Induction step \( q - 1 \mapsto q, \ q \geq 2: \) We show for each \( i \in \mathbb{N} \) the exactness of

\[ \mathcal{E}^{p,q-1}(\Delta_i) \overset{d''}{\rightarrow} \mathcal{E}^{p,q}(\Delta_i) \overset{d''}{\rightarrow} \mathcal{E}^{p,q+1}(\Delta_i) \]

Consider the space of local solutions

\[ M_i := M_{i,q} := \{ \sigma \in \mathcal{E}^{p,q-1}(\Delta_i) : d''\sigma = \omega|_{\Delta_i} \}, \ i \in \mathbb{N}, \]

Any pair of local solutions \( \sigma_1, \sigma_2 \in M_i \) satisfies

\[ d''(\sigma_1 - \sigma_2) = 0 \]

Hence by induction assumption there exists \( \eta \in \mathcal{E}^{p,q-2}(\Delta_i) \) with

\[ d''\eta = \sigma_1 - \sigma_2 \]

As a consequence, the space \( M_i \) is the image of the map

\[ \psi_i : \mathcal{E}^{p,q-2}(\Delta_i) \rightarrow \mathcal{E}^{p,q-1}(\Delta_i), \ \eta \mapsto \sigma_i + d''\eta, \]

namely the affine subspace

\[ M_i = \sigma_i + d''\mathcal{E}^{p,q-2}(\Delta_i) \subset \mathcal{E}^{p,q-1}(\Delta_i) \]

The map \( \psi_i \) is surjective: For a given element \( \sigma \in M_i \) the form

\[ \delta := \sigma - \sigma_i \in \mathcal{E}^{p,q-1}(\Delta_i) \]

satisfies

\[ d''\delta = 0. \]

By induction hypothesis exists

\[ \eta \in \mathcal{E}^{p,q-2}(\Delta_i) \text{ with } d''\eta = \delta. \]

Therefore

\[ \psi_i(\eta) = \sigma_i + d''\eta = \sigma_i + \delta = \sigma. \]

The element
\[ \sigma_i - \sigma_{i-1} \in \mathcal{E}^{p,q-1}(\Delta_{i-1}) \text{ satisfies } d''(\sigma_i - \sigma_{i-1}) = 0 \]

Hence the induction hypothesis provides a form
\[ \gamma_{i-1} \in \mathcal{E}^{p,q-2}(\Delta_{i-1}) \]
with
\[ d'' \gamma_{i-1} = \sigma_i - \sigma_{i-1} \in \mathcal{E}^{p,q-1}(\Delta_{i-1}) \]

For each \( i \geq 1 \) consider the "twisted" continuous map between Fréchet spaces, in particular between complete metric spaces,
\[ \tau_i : \mathcal{E}^{p,q-2}(\Delta_i) \to \mathcal{E}^{p,q-2}(\Delta_{i-1}), \, \eta \mapsto \eta|_{\Delta_{i-1}} + \gamma_{i-1}, \]

The map has dense image: For each compact set \( K \subset \Delta_{i-1} \) choose a smooth function \( s \in \mathcal{E}(\Delta_i) \) with
\[ \text{supp } s \subset \Delta_{i-1} \text{ and } s|_K = 1. \]

Multiplying by \( s \) allows to extend smooth functions on \( \Delta_{i-1} \) to smooth functions on \( \Delta_i \) without changing their values on \( K \). The following diagram commutes

\[ \begin{CD}
\mathcal{E}^{p,q-2}(\Delta_i) @> \tau_i >> \mathcal{E}^{p,q-2}(\Delta_{i-1}) \\
\downarrow \psi_i @. \downarrow \psi_{i-1} \\
M_i @> \rho_i >> M_{i-1}
\end{CD} \]

The Mittag-Leffler principle, Proposition 1.26, applies to the family
\[ (\mathcal{E}^{p,q-2}(\Delta_i), \, \tau_i)_{i \in \mathbb{N}}. \]

It provides a compatible family
\[ (\eta_i \in \mathcal{E}^{p,q-2}(\Delta_i))_{i \in \mathbb{N}} \text{ with } \tau_i(\eta_i) = \eta_{i-1} \]

Hence
\[ (\sigma_i + d'' \eta_i)|_{\Delta_{i-1}} = (\rho_i \circ \psi_i)(\eta_i) = (\psi_{i-1} \circ \tau_i)(\eta_i) = \psi_{i-1}(\eta_{i-1}) = \sigma_{i-1} + d'' \eta_{i-1} \]

and the family determines a form
\[ \sigma \in \mathcal{E}^{p,q-1}(\Delta) \text{ with } \sigma|_{\Delta_i} = \sigma_i + d'' \eta_i, \, i \in \mathbb{N}. \]

As a consequence
\[ d'' \sigma = \omega. \]
Chapter 2
Complex manifolds and sheaves

2.1 Complex manifolds

Definition 2.1 (Topological manifold, chart, complex atlas and complex structure).

1. A topological manifold $X$ of real dimension $\dim X = k$ is a topological Hausdorff space $X$ such that each point $x \in X$ has an open neighbourhood $U$ with a homeomorphism, named a chart around $x$,

$$\phi : U \cong V$$

onto an open set $V \subset \mathbb{R}^k$.

2. A complex atlas of a topological manifold $X$ of real dimension $2n$, i.e. complex dimension $\dim X = n$, is a family $\mathcal{A}$ of charts

$$\mathcal{A} = (\phi_i : U_i \to V_i)_{i \in I}$$

with open subsets

$$V_i \subset \mathbb{C}^n \cong \mathbb{R}^{2n},$$

such that

- $X = \bigcup_{i \in I} U_i$
- and for all pairs $i, j \in I$ and

$$U_{ij} := U_i \cap U_j$$

the transition map of the two charts

$$\psi_{ij} := \phi_i \circ (\phi_j|_{U_{ij}})^{-1} : \phi_j(U_{ij}) \to \phi_i(U_{ij}) \subset \mathbb{C}^n$$
is holomorphic, i.e. each component function of $\psi_{ij}$ is holomorphic on the open set $\phi_j(U_{ij}) \subset \mathbb{C}^n$, see Figure 2.1.

- Two complex atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ of $X$ are compatible if their union $\mathcal{A}_1 \cup \mathcal{A}_2$ is again a complex atlas. A maximal set of complex, biholomorphically compatible atlases of $X$ is a complex structure $\Sigma$ on $X$.

Fig. 2.1 Charts of an atlas defining a complex structure
Definition 2.2 (Complex manifold, holomorphic map).

1. An \( n \)-dimensional complex manifold is a pair \((X, \Sigma)\) with a connected, topological manifold \(X\) of real dimension \(2n\), which has a countable base of the topology, and a complex structure \(\Sigma\) on \(X\).

2. A continuous map 
\[
f : (X, \Sigma_X) \to (Y, \Sigma_Y)
\]
between two complex manifolds is holomorphic if for each point \(x \in X\) exists a chart around \(x\) from an atlas of \(\Sigma_X\)
\[
\phi : U \to V,
\]
and a chart around \(f(x)\) from an atlas of \(\Sigma_Y\)
\[
\psi : S \to T,
\]
such that \(f(U) \subset S\), and the composition
\[
\psi \circ f \circ (\phi|U \cap f^{-1}(S))^{-1} : \phi(U \cap f^{-1}(S)) \to T \subset \mathbb{C}^m, m := \dim \mathbb{C} Y,
\]
is holomorphic. Note that the definition is independent from the choice of the charts.

3. A holomorphic map \(f : X \to Y\) is biholomorphic or an isomorphism iff there exists a holomorphic map \(g : Y \to X\) such that both maps satisfy
\[
g \circ f = id_X \text{ and } f \circ g = id_Y
\]

4. A holomorphic function on \(X\) is a holomorphic map
\[
f : (X, \Sigma_X) \to \mathbb{C}.
\]
Here \(\mathbb{C}\) is considered a complex manifold in the canonical way. For each open set \(U \subset X\) the ring of all holomorphic functions on \(U\) with respect to addition and multiplication is denoted \(\mathcal{O}_X(U)\) or \(\mathcal{O}(U)\) for short.

Remark 2.3 (Fundamentals from topology).

1. An open covering \(\mathcal{U}\) of a topological space \(X\) is locally finite if every point \(x \in X\) has an open neighbourhood \(V\) which meets only finitely many sets of the covering.
2. A Hausdorff space $X$ is *paracompact* if every open covering of $X$ has a locally finite subcovering. Each locally finite covering of a topological space has a subordinate *partition of unity*, see [4, Chap. VIII, Theor. 4.2]

3. Each locally compact, second countable topological space is paracompact.

4. Consider a Hausdorff space $X$. A subset $U \subset X$ is *relatively compact* in $X$ if the closure $\overline{U} \subset X$ is compact. For two subsets $U_1, U_2 \subset X$ the notation

$$U_1 \subset\subset U_2$$

means: $U_1$ is relatively compact in $X$ and $\overline{U}_1 \subset U_2$.

5. Each locally-compact topological space $X$ has an *exhaustion* $(U_\nu)_{\nu \in \mathbb{N}}$ by relatively compact open subsets, i.e.

$$U_\nu \subset\subset U_{\nu+1}, \nu \in \mathbb{N}, \text{ and } X = \bigcup_{\nu \in \mathbb{N}} U_\nu.$$

6. Each regular, second countable space is metrizable, see [4, Chap. IX, Cor. 9.2].

7. Each subspace of a metric space is a metric space with respect to the restricted metric.

8. Each metric space is paracompact, see [4, Chap. IX, Theor. 5.3].

9. Each locally-finite covering $\mathcal{U} = (U_i)_{i \in I}$ of a paracompact topological space $X$ has a *shrinking* $\mathcal{V} = (V_i)_{i \in I}$, i.e. the open covering $\mathcal{V}$ satisfies for all $i \in I$

$$V_i \subset\subset U_i,$$

see [29, Satz 9.2.1] or [4, Theor. VII.6.1].

**Remark 2.4 (Holomorphic versus smooth).**

1. Often one uses for a chart of an $n$-dimensional complex manifold $(X, \Sigma)$ the suggestive notation

$$z : U \to V \subset \mathbb{C}^n.$$  

Then the decomposition into real part and imaginary part

$$z_\nu = x_\nu + i \cdot y_\nu, \nu = 1, \ldots, n,$$

and the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ defines a chart
2.1 Complex manifolds

\[(x_1, y_1, \ldots, x_n, y_n) : U \to V \subset \mathbb{R}^{2n}\]

of a smooth structure \(\Sigma_{\text{smooth}}\) on \(X\): When considering a holomorphic transition function \(\psi\) as a function of \(2n\) real variables then \(g\) has partial derivatives of arbitrary order. Hence the transition function is smooth, i.e. differentiable of class \(C^\infty\), and the complex structure \(\Sigma\) induces a smooth structure \(\Sigma_{\text{smooth}}\) on \(X\), such that

\[(X, \Sigma_{\text{smooth}})\]

is a \(2n\)-dimensional paracompact smooth manifold.

2. If \((X, \Sigma)\) is a complex manifold then a map

\[f : X \to \mathbb{C}\]

is smooth, if \(f\) is smooth on \((X, \Sigma_{\text{smooth}})\). For an open set \(U \subset X\) the ring of all smooth functions on \(U\) with respect to addition and multiplication is denoted \(\mathcal{E}_X(U)\), or \(\mathcal{E}(U)\) for short.

In the following we will denote a complex manifold \((X, \Sigma)\) simply by \(X\) if the details of the complex structure \(\Sigma\) are not relevant.

We now investigate holomorphic maps

\[f : X \to Y\]

between two complex manifolds from a local point of view. The basic objects at a point \(x \in X\) are

- the tangent spaces
  
  \[T_xX\text{ and } T_{f(x)}Y,\]
  
  of \(X\) at \(x \in X\) resp. \(Y\) at \(f(x) \in Y\), complex vector spaces of dimension equal to \(\dim_{\mathbb{C}} X\) resp. \(\dim_{\mathbb{C}} Y\),

- and the tangent map at \(x\)
  
  \[T_xf : T_xX \to T_{f(x)}Y,\]
  
  the complex-linear map which approximates \(f\) in a neighbourhood of \(x\).

The local properties of \(f\) at a point \(a \in X\) derive from the corresponding properties of the tangent map

\[T_af : T_aX \to T_{f(a)}Y\]

The map \(f\) is

- a local isomorphism, if it maps sufficiently small open neighbourhoods of a point isomorphically onto neighbourhoods of the image point,
2 Complex manifolds and sheaves

- an **immersion**, if it maps sufficiently small neighbourhoods of a point isomorphically onto a coordinate slice of the image point,

- and a **submersion**, if it splits sufficiently small neighbourhoods of a point in the domain of \( f \) as a product, and projects the product onto a neighbourhood of the image point.

Definition 2.5 formalizes these properties, while Proposition 2.6 relates them to the rank of the tangent map.

**Definition 2.5 (Local isomorphism, immersion, submersion).** Consider a holomorphic map

\[ f : X \rightarrow Y \]

between two complex manifolds and a point \( x \in X \). The map \( f \) is named

- **local isomorphism** at \( x \) if open neighbourhoods
  
  \[ U \text{ of } x \text{ in } X, V \text{ of } f(x) \text{ in } Y \]

  exist such that the injections
  
  \[ j_U : U \hookrightarrow X, j_V : V \hookrightarrow Y \]

  extend to the commutative diagram

  \[
  \begin{array}{ccc}
  X & \xrightarrow{f} & Y \\
  j_U \downarrow & & \downarrow j_V \\
  U & \xrightarrow{\sim} & V \\
  \end{array}
  \]

  with the biholomorphic restriction
  
  \[ f|U : U \xrightarrow{\sim} V, \]

- **immersion** at \( x \) if open neighbourhoods
  
  \[ U \text{ of } x \text{ in } X, V \text{ of } f(x) \text{ in } Y, W \text{ of } 0 \text{ in } K^m \]

  exist such that the injections
  
  \[ j_U : U \hookrightarrow X, j_V : V \hookrightarrow Y, \]

  and the embedding onto a coordinate slice

  \[ j = [U \xrightarrow{\sim} U \times \{0\} \hookrightarrow U \times W] \]
2.1 Complex manifolds

extend to a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{j_U} & & \downarrow{j_V} \\
U & \xrightarrow{f|U} & V \\
\downarrow{j} & \sim & \downarrow{pr_V} \\
U \times W & & V \times W \\
\end{array}
\]

with a biholomorphic map

\[U \times W \xrightarrow{\sim} V,\]

• submersion at \(x\) if open neighbourhoods

\[U \text{ of } x \text{ in } X, V \text{ of } f(x) \text{ in } Y, W \text{ of } 0 \text{ in } K^m\]

exist such that the injections

\[j_U : U \hookrightarrow X, j_V : V \hookrightarrow Y,\]

and the projection

\[pr_V : V \times W \rightarrow V\]

extend to a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{j_U} & & \downarrow{j_V} \\
U & \xrightarrow{f|U} & V \\
\downarrow{j} & \sim & \downarrow{pr_V} \\
U \times W & & V \times W \\
\end{array}
\]

with a biholomorphic map

\[U \xrightarrow{\sim} V \times W,\]

• respectively local isomorphism, immersion, submersion if the corresponding local property holds for each point \(x \in X\), see Figure 2.2.
Proposition 2.6 (Local isomorphism, immersion, submersion). A holomorphic map
\[ f : X \to Y \]
between two complex manifolds is at a given point \( x \in X \)
- a local isomorphism iff the tangent map \( T_x f \) is bijective.
- an immersion iff the tangent map \( T_x f \) is injective.
- respectively a submersion iff the tangent map \( T_x f \) is surjective.

For the proof see [28, Part II, Chap. III.9].

The composition of two immersions is an immersion. The composition of two submersions is a submersion.
Definition 2.7 (Embedding). Consider a holomorphic map
\[ f : X \to Y \]
between two complex manifolds and provide the set \( f(X) \subset Y \) with the subspace topology.
- The map is an embedding if \( f \) is an immersion and the induced map \( X \to f(X) \) is a homeomorphism.
- If \( f(X) \subset Y \) is also closed, then \( f \) is named a closed embedding.

Theorem 2.8 (Analytic submanifold). Consider an open set \( U \subset \mathbb{C}^n \) and a subset \( A \subset U \), which is closed with respect to the induced topology of \( U \). For a number \( k \in \mathbb{N} \) the following statements are equivalent:
1. Zero set: Each point \( a \in A \) has an open neighbourhood \( V \subset U \) and holomorphic functions \( f_1, \ldots, f_{n-k} \in \mathcal{O}(V) \) such that
\[ A \cap V = \{ z \in V : f_1(z) = \ldots = f_{n-k}(z) = 0 \} \]
and for all \( x \in A \cap V \)
\[ \text{rank} \left( \frac{\partial f_\mu}{\partial z^\nu} \right) = n - k \]
2. Parameter representation: For each point \( a \in A \) exists an open neighbourhood \( V \subset U \), and an open set \( T \subset \mathbb{C}^k \) with holomorphic functions \( \phi_1, \ldots, \phi_k \in \mathcal{O}(T) \) such that
\[ \phi := (\phi_1, \ldots, \phi_k) : T \to A \cap V \subset \mathbb{C}^n \]
is a homeomorphism satisfying for all \( t \in T \)
\[ \text{rank} \left( \frac{\partial \phi_\nu}{\partial t^j} \right) = k \]
3. Locally a \( k \)-dimensional affine plane: For each point \( a \in A \) exists an open neighbourhood \( V \subset U \), and an open neighbourhood \( W \) of the origin in \( \mathbb{C}^n \) with a biholomorphic map
\[ \Phi : V \to W \]
Complex manifolds and sheaves

$satisfying$
\[
\Phi(A \cap V) = W \cap \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : z_{k+1} = \ldots = z_n = 0\}
\]

For a proof see [9, § 14]. The proof is analogous to the case of smooth manifolds.

**Definition 2.9 (Analytic submanifold).** Consider an open set $U \subset \mathbb{C}^n$ and a subset $A \subset U$, closed with respect to the induced topology of $U$. Then $A$ is an analytic submanifold of $U$, if $A \subset U$ satisfies the equivalent properties from Theorem 2.8.

Theorem 2.8, part 3) shows that each analytic submanifold of an open set $U \subset \mathbb{C}^n$ is a complex manifold. The number $k \in \mathbb{N}$ from Theorem 2.8 is the dimension of $A$.

### 2.2 Sheaves and their stalks

We define presheaves (presheaf = Pr"agarbe) and sheaves (sheaf = Garbe) of Abelian groups first. But the definition and results transfer to other objects of Abelian categories, i.e. to commutative rings $R$ or $R$-modules and also to the category of sets.

**Definition 2.10 (Presheaf of Abelian groups).**

1. A presheaf $\mathcal{F}$ of Abelian groups on a topological space $X$ is a family of Abelian groups $\mathcal{F}(U)$, $U \subset X$ open, and for each pair $V \subset U$ of open subsets of $X$ a homomorphism of Abelian groups $\rho^V_U : \mathcal{F}(U) \to \mathcal{F}(V)$ satisfying:
   \[
   \rho^V_U = id_{\mathcal{F}(U)}
   \]
   and
   \[
   \rho^W_V \circ \rho^V_U = \rho^W_U \text{ for } W \subset V \subset U.
   \]

   The maps $\rho^V_U$ are often named restrictions and denoted
   \[
   f|_V := \rho^V_U(f)
   \]
   for $f \in \mathcal{F}(U)$, $V \subset U$ open.

   The elements of a given Abelian group
2.2 Sheaves and their stalks

\[ \mathcal{F}(U), \ U \subset X \text{ open}, \]

are named the sections of \( \mathcal{F} \) on \( U \).

2. A morphism

\[ f : \mathcal{F} \to \mathcal{G} \]

between two presheaves of Abelian groups with families of restrictions

\[ \rho = (\rho_U^{U,V})_{U,V} \text{ resp. } \sigma = (\sigma_U^{U,V})_{U,V} \]

is a family of group homomorphisms

\[ f_U : \mathcal{F}(U) \to \mathcal{G}(U), \ U \subset X \text{ open}, \]

such that for any pair \( V \subset U \) of open subsets of \( X \) the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\
\rho_U & \downarrow & \sigma_U \\
\mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V)
\end{array}
\]

**Remark 2.11 (Presheaf as a functor).** Consider a fixed topological space \( X \). Denote by \( X \) the category of open subsets of \( X \):

- Objects of \( X \) are the open sets \( U \subset X \)
- and morphisms exist only for pairs of open sets \( V \subset U \), and here the only morphism is the injection \( V \hookrightarrow U \), i.e. for open sets \( U, V \subset X \)

\[ \text{Mor}(V, U) := \begin{cases} \{V \hookrightarrow U\} & V \subset U \\ \emptyset & \text{otherwise} \end{cases} \]

Then the presheaves \( \mathcal{F} \) of Abelian groups on \( X \) are exactly the contravariant functors

\[ \mathcal{F} : X \to Ab \]

to the category \( Ab \) of Abelian groups. A morphism

\[ \mathcal{F} \to \mathcal{G} \]

between two presheaves is a functor morphism (natural transformation) from \( \mathcal{F} \) to \( \mathcal{G} \).
In general the concept of a presheaf is too weak to support any strong result on complex manifolds. The stronger concept is a sheaf. It satisfies two additional sheaf axioms. According to these conditions local sections which coincide on their common domain of definition glue to a unique global section.

**Definition 2.12 (Sheaf).** Consider a topological space $X$. A sheaf $\mathcal{F}$ of Abelian groups on $X$ is a presheaf of Abelian groups on $X$, which satisfies the following two sheaf axioms:

For each open $U \subset X$ and for each open covering $\mathcal{U} = (U_i)_{i \in I}$ of $U$:

1. If two elements $f, g \in \mathcal{F}(U)$ satisfy for all $i \in I$
   \[ f|U_i = g|U_i \]
   then
   \[ f = g, \]
   i.e. local equality implies global equality.

2. If a family
   \[ f_i \in \mathcal{F}(U_i), i \in I, \]
   satisfies for all $i, j \in I$
   \[ f_i|U_i \cap U_j = f_j|U_i \cap U_j \]
   then an element $f \in \mathcal{F}(U)$ exists satisfying for all $i \in I$
   \[ f|U_i = f_i, \]
   i.e. local sections which agree on the intersections glue to a global element.

A morphism of sheaves is a morphism of the underlying presheaves.

If one paraphrases a presheaf as a tool to handle local objects, then a sheaf is a tool to glue local objects, which fit together, in order to construct a unique global object. If it is not possible to make the parts fit, then cohomology theory is a means to measure the obstructions, see Chapter 3.

**Definition 2.13 (Subsheaf of Abelian groups).** Consider a presheaf $\mathcal{F}$ of Abelian groups on a topological space $X$.

1. A presheaf of Abelian groups $\mathcal{G}$ on $X$ is a subpresheaf of $\mathcal{F}$
   - if for all open sets $U \subset X$
     \[ \mathcal{G}(U) \subset \mathcal{F}(U) \]
     is a subgroup, and
2.2 Sheaves and their stalks

- if the restriction maps of \( G \) are induced by the restriction maps of \( F \).

2. If \( F \) is a sheaf, then a sheaf \( G \) is a subsheaf of \( F \) if \( G \) is a subpresheaf of \( F \).

Similar to presheaves and sheaves of Abelian groups one defines presheaves and sheaves with other algebraic structures, e.g. rings or modules, see Definition 4.25.

**Example 2.14 (Sheaves and presheaves).**

1. Let \( X \) be a topological space.

   - **Sheaf \( C \) of continuous functions:** For any open set \( U \subset X \) define
     \[
     C(U) := \{ f : U \to \mathbb{C} \mid f \text{ continuous} \}
     \]
     as the complex vector space of complex-valued, continuous functions on \( U \).
     The presheaf
     \[
     C(U), \ U \subset X \text{ open},
     \]
     with the restriction of functions
     \[
     \rho^U_V : C(U) \to C(V), \ f \mapsto f|V, \ V \subset U,
     \]
     is a sheaf. It is named the sheaf \( C \) of continuous functions on \( X \).

   - **Sheaf \( \mathbb{Z} \) of locally constant functions:** Consider a topological space \( X \), and provide \( \mathbb{Z} \) with its discrete topology. For each open \( U \subset X \) define
     \[
     \mathcal{F}(U) := \{ f : U \to \mathbb{Z} \mid f \text{ constant} \}
     \]
     and take the canonical restriction morphisms. The family
     \[
     \mathcal{F} := \mathcal{F}(U), \ U \subset X \text{ open},
     \]
     is a presheaf.
     In general, the presheaf \( \mathcal{F} \) is not a sheaf: Assume
     \[
     X = X_1 \cup X_2
     \]
     with two connected components. Then the family
     \[
     (f_1, f_2) \in \mathcal{F}(X_1) \times \mathcal{F}(X_2)
     \]
     with
     \[
     f_1 := 1 \in \mathbb{Z} \text{ and } f_2 := 2 \in \mathbb{Z}
     \]
     does not arise from a global section.
\[ f \in \mathcal{F}(X) = \mathbb{Z} \]

as
\[ f_1 = f|X_1 \text{ and } f_2 = f|X_2 \]

with a constant section \( f \in \mathcal{F}(X) \).

A slight change in the definition of \( \mathcal{F} \) provides a sheaf on \( X \): A function on an open set \( U \subset X \) is \textit{locally constant} if each point \( x \in U \) has a neighbourhood \( V \), such that the restriction \( f|V \) is constant. One defines

\[ \mathbb{Z}(U) := \{ f : U \to \mathbb{Z} \mid f \text{ locally constant} \}. \]

Then

\[ \mathbb{Z}(U), U \subset X \text{ open}, \]

with the canonical restrictions is a sheaf. The sheaf is often denoted \( \mathbb{Z} \) like the ring of integers. The context has to clarify whether the symbol denotes the ring of integers or the sheaf of locally constant integer-valued functions.

Similarly one defines the sheaf \( \mathbb{C} \) of locally constant complex-valued functions. Note that both sheaves \( \mathbb{Z} \) and \( \mathbb{C} \) are named \textit{constant} sheaves - although they are defined by locally constant sections.

2. Let \( X \) be a complex manifold.

- \textit{Sheaf} \( \mathcal{O} \) of holomorphic functions: Consider for each open \( U \subset X \) the ring

\[ \mathcal{O}(U) := \{ f : U \to \mathbb{C} \mid f \text{ holomorphic} \} \]

the ring of holomorphic functions on \( U \). The presheaf

\[ \mathcal{O}(U), U \subset X \text{ open}, \]

with the canonical restriction of functions is a sheaf of rings. It is named the sheaf \( \mathcal{O} \) of holomorphic functions on \( X \) or the \textit{structure sheaf} of \( X \).

- \textit{Sheaf} \( \mathcal{O}^* \) of holomorphic functions without zeros: Consider for each open \( U \subset X \) the multiplicative Abelian group

\[ \mathcal{O}^*(U) := \{ f \in \mathcal{O}(U) : f(x) \neq 0 \text{ for all } x \in U \}. \]

The presheaf

\[ \mathcal{O}^*(U), U \subset X \text{ open}, \]

with the canonical restriction of functions is a sheaf. It is named the sheaf \( \mathcal{O}^* \) of holomorphic functions without zeros on \( X \). Apparently \( \mathcal{O}^* \) is the sheaf of units of \( \mathcal{O} \).
2.2 Sheaves and their stalks

- **Sheaf \( \mathcal{E} \) of smooth functions**: Consider for each open \( U \subset X \) the ring

\[
\mathcal{E}(U) := \{ f : U \to \mathbb{C} | f \text{ smooth} \}
\]

The presheaf

\[
\mathcal{E}(U), \ U \subset X \text{ open},
\]

with the canonical restriction of functions is a sheaf of rings. It is named the sheaf \( \mathcal{E} \) of smooth functions on \( X \) or the *smooth structure sheaf*.

**Definition 2.15 (Stalk of a presheaf)**. Consider a presheaf \( \mathcal{F} \) of Abelian groups on a topological space \( X \), and a point \( x \in X \). The *stalk* \( \mathcal{F}_x \) of \( \mathcal{F} \) at \( x \) is the set of equivalence classes with respect to the following equivalence relation on the union of all \( \mathcal{F}(U) \), \( U \) open neighbourhood of \( x \):

\[
f_1 \in \mathcal{F}(U_1) \sim f_2 \in \mathcal{F}(U_2)
\]

if for a suitable open neighbourhood \( V \) of \( x \) with \( V \subset U_1 \cap U_2 \)

\[
f_1|_V = f_2|_V.
\]

Apparently, the stalk \( \mathcal{F}_x \) is an Abelian group in a canonical way, and each canonical map

\[
\pi^U_x : \mathcal{F}(U) \to \mathcal{F}_x
\]

is a group homomorphism. The elements of the stalk \( \mathcal{F}_x \) are named the *germs* of \( \mathcal{F} \) at \( x \), see Figure 2.3.

The stalk \( \mathcal{F}_x \) of a presheaf \( \mathcal{F} \) at a point \( x \in X \) is the *inductive limit* of the sections from \( \mathcal{F}(U) \) for all neighbourhoods \( U \) of \( x \):

\[
\mathcal{F}_x = \lim_{U} \mathcal{F}(U)
\]

A sheaf is when you do vertically algebra and horizontally topology, see Figure 2.3.
Each subpresheaf of a sheaf is again a sheaf. The quotient of a presheaf $\mathcal{F}$ with respect to a subpresheaf is again a presheaf. But even if $\mathcal{F}$ is a sheaf the quotient subsheaf is not necessarily a sheaf: A family of quotients is not necessarily the quotient of a global sections, see the counter example of the logarithm in Proposition 2.24. Sheafification from Theorem 2.16 attaches to every presheaf a sheaf. The method is used to define the quotient of a sheaf as the sheafification of the quotient presheaf, see Definition 2.17.

**Theorem 2.16 (Sheafification of a presheaf).** For each presheaf $\mathcal{F}$ on a topological space exists a sheaf $\hat{\mathcal{F}}$ on $X$ and a presheaf morphism

$$\mathcal{F} \to \hat{\mathcal{F}}$$

which induces for each $x \in X$ an isomorphism of stalks

$$\mathcal{F}_x \to \hat{\mathcal{F}}_x$$

The sheaf $\hat{\mathcal{F}}$ is named the sheafification of the presheaf $\mathcal{F}$.

**Proof.** i) For an open set $U \subset X$ define the Abelian group

$$w(\mathcal{F})(U) := \prod_{x \in U} \mathcal{F}_x$$

as the Cartesian product of the corresponding stalks. Together with the canonical restrictions

$$w(\mathcal{F})(U) \to w(\mathcal{F})(V), \; V \subset U \text{ open},$$
we obtain a sheaf \( w(\mathcal{F}) \) on \( X \). The sheaf \( w(\mathcal{F}) \) is an example of a \textit{flabby} sheaf (deutsch: welke Garbe).

ii) For an open set \( U \subset X \) a family of germs
\[
\phi = (\phi_x)_{x \in U} \in w(\mathcal{F})(U)
\]
is \textit{compatible} if the family locally arises from sections, i.e. if each \( x \in U \) has an open neighbourhood \( V \subset U \) and a section
\[
f \in \mathcal{F}(V)
\]
satisfying \( f_y = \phi_y \) for all \( y \in V \).

For each open \( U \subset X \) define the presheaf
\[
\hat{\mathcal{F}}(U) := \{ \phi \in w(\mathcal{F})(U) : \phi \text{ compatible} \}
\]
with the canonical restrictions
\[
\hat{\mathcal{F}}(U) \to \hat{\mathcal{F}}(V), \ V \subset U \text{ open}.
\]

iii) The presheaf \( \hat{\mathcal{F}} \) is a sheaf with the same stalks as \( \mathcal{F} \): One checks that the construction via stalks and compatible sections of \( w(\mathcal{F}) \) satisfies the two sheaf axioms. One also checks that the induced maps on stalks \( \mathcal{F}_x \to \hat{\mathcal{F}}_x \) are isomorphisms. \( \square \)

**Definition 2.17 (Quotient sheaf).** Consider a topological space \( X \) and two sheaves
\[
\mathcal{F} \subset \mathcal{G}
\]
of Abelian groups. The \textit{quotient sheaf}
\[
\mathcal{F} / \mathcal{G}
\]
is defined as the sheafification \( \mathcal{F} \) of the quotient presheaf
\[
U \mapsto \mathcal{F}(U) / \mathcal{G}(U), \ U \subset X \text{ open},
\]
with the canonical restrictions.

A section of the quotient sheaf \( \mathcal{F} / \mathcal{G} \) is represented by a family of local sections from \( \mathcal{F} \).

A meromorphic function on a complex manifold is a compatible family of quotients of holomorphic functions which are locally defined. Because the denominator
functions may have zeros a meromorphic function does not have necessarily a value at each point of definition.

From a general perspective the concept of a meromorphic function uses the algebraic construction of localisation, i.e. taking for rings of holomorphic functions the modules of fractions. Here one has to exclude as denominators those holomorphic functions which vanish identically on a component of their domain of definition. Hence one localises each ring of holomorphic functions with respect to its subring of non-zero-divisors. To obtain a sheaf in the end, one has to sheafify in a final step the algebraic construction. As so often, the whole procedure is to carry over a construction from commutative algebra to a corresponding construction with sheaves, see [1, Chap. 3].

**Definition 2.18 (Sheaf of fractions).** Consider a topological space $X$ and a sheaf of rings $\mathcal{R}$ on $X$.

1. Consider the multiplicatively closed family of sections with non-zero divisor germs

$$S(U) := \{g \in \mathcal{R}(U) : g_x \in \mathcal{R}_x \text{ non-zero divisor for all } x \in U\}, \quad U \subset X \text{ open.}$$

On each $\mathcal{R}(U) \times S(U)$ consider the equivalence relation

$$(f_1, g_1) \equiv (f_2, g_2) \iff \exists n \in S(U) : n \cdot (g_2 f_1 - g_1 f_2) = 0 \in \mathcal{R}(U).$$

Denote by $\mathcal{R}(U)_{S(U)}$ the ring of equivalence classes, and by $\frac{f}{g} \in \mathcal{R}(U)_{S(U)}$ the equivalence class of a given pair $(f, g) \in \mathcal{R}(U) \times S(U)$.

2. The sheafification $Q(\mathcal{R})$ of the presheaf

$$U \mapsto \mathcal{R}(U)_{S(U)}, \quad U \subset X \text{ open,}$$

is named the sheaf of fractions of $\mathcal{R}$ with respect to the family $S(U)_U$.

**Definition 2.19 (Sheaf of meromorphic functions).** For a complex $n$-dimensional manifold $X$ each ring $\mathcal{O}_x$, $x \in X$, is isomorphic to $\mathbb{C}\{z_1, \ldots, z_n\}$, hence an integral domain. The sheaf of fractions

$$\mathcal{M} := Q(\mathcal{O})$$

is named the sheaf of meromorphic functions on $X$. A section

$$f \in \mathcal{M}(U), \quad U \subset X,$$

is named a meromorphic function on $U$. 
Hence a meromorphic function on $X$ is a family of fractions $(f_i/g_i)_{i \in I}$ of locally defined holomorphic functions with $g_i \neq 0$ under the identification

$$f_1/g_1 = f_2/g_2 : \iff g_2 f_1 = g_1 f_2$$

**Proposition 2.20 (Field of meromorphic functions).** Consider a complex manifold $X$. For a domain $G \subset X$ the ring $\mathcal{M}(G)$ of global meromorphic functions is a field.

**Proof.** The claim follows because $\mathcal{O}(G)$ is an integral domain, cf. Corollary 1.11.

The sheaves $\mathcal{O}$ and $\mathcal{M}$ are sheaves of rings. The sheaves $\mathcal{O}^*$ and $\mathcal{M}^*$ are sheaves of multiplicative groups. They are the sheaves of units of respectively $\mathcal{O}$ and $\mathcal{M}$.

Sheaves of locally constant functions like $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ are important for homology and cohomology in the context of algebraic topology. While sheaves of holomorphic and meromorphic functions are the basic objects on complex manifolds.

**Remark 2.21 (Stalks of the structure sheaf and quotient fields).**

1. Let $X$ be a complex manifold. For a point $x \in X$ consider the stalks

$$R := \mathcal{O}_x \text{ and } K := \mathcal{M}_x.$$ 

Using a chart around $x$ shows

$$R = \mathbb{C}\{z\}, \text{ the ring of convergent power series with center } = 0,$$

and

$$K = Q(R), \text{ the quotient field of } R,$$

a statement about germs.

The ring $R$ is a local ring, i.e. it has exactly one maximal ideal, namely

$$m := \langle z_1, \ldots, z_n \rangle \subset R,$$

the ideal of non-units of $R$.

2. In general, the local statement about meromorphic germs does not necessarily generalize to a global statement: On one hand, for $X = \mathbb{C}$ one has

$$\mathcal{M}(X) := (Q(\mathcal{O}))(X) = Q(\mathcal{O}(X))$$
due to the Weierstrass product theorem, a statement about global sections. The same statement holds even for any domain \( X \subset \mathbb{C} \). On the other hand, on a compact complex manifold \( X \) one has

\[ \mathcal{O}(X) = \mathbb{C} \]

but possibly

\[ \mathcal{M}(X) \neq \mathcal{Q} (\mathcal{O}(X)) = \mathbb{C} \text{, e.g., } \mathcal{M}(\mathbb{P}^1) = \mathbb{C}(z). \]

For more advanced results see [24, Kapitel 4* §1.5 Satz, §2.4].

3. Any morphism of presheaves on \( X \)

\[ f : \mathcal{F} \to \mathcal{G} \]

induces for any \( x \in X \) a morphism

\[ f_x : \mathcal{F}_x \to \mathcal{G}_x \]

of the corresponding stalks such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\
\downarrow \pi^U_x & & \downarrow \tau^U_x \\
\mathcal{F}_x & \xrightarrow{f_x} & \mathcal{G}_x
\end{array}
\]

Here the vertical maps are the canonical group homomorphisms from Definition 2.15. In general, these maps are not surjective. But each germ \( f_x \in \mathcal{F}_x \) has an open neighbourhood \( V \subset X \) and a representative \( f \in \mathcal{F}(V) \). The neighbourhood \( V \) may depend on the germ \( f_x \).

4. On a complex manifold \( X \) sections of a sheaf like the holomorphic structure sheaf \( \mathcal{O} \) can be considered at least from the following different topological viewpoints:

- At a point \( x \in X \) one considers the value \( f(x) \in \mathbb{C} \) of a function \( f \) which is holomorphic in an open neighbourhood of \( x \).

- At a point \( x \in X \) one considers the germ \( f_x \in \mathbb{C}\{z\} \) of a function \( f \) which is holomorphic in an open neighbourhood of \( x \).

- In a given open neighbourhood \( U \subset X \) of a point \( x \in X \) one considers a holomorphic function \( f \in \mathcal{O}(U) \).
2.2 Sheaves and their stalks

- One considers a globally defined holomorphic function \( f \in \mathcal{O}(X) \).

**Definition 2.22 (Exact sheaf sequence resp. presheaf sequence).** Consider a topological space \( X \).

1. A **sequence of sheaves** on \( X \) is a family
   \[
   (f_i : \mathcal{F}_i \to \mathcal{F}_{i+1})_{i \in \mathbb{Z}}
   \]
   of morphisms of sheaves. The family is a **complex** if for all \( x \in X \) on the level of stalks the induced family of morphisms of Abelian groups
   \[
   (f_{i,x} : \mathcal{F}_{i,x} \to \mathcal{F}_{i+1,x})_{i \in \mathbb{Z}}
   \]
   satisfies for all \( i \in \mathbb{Z} \)
   \[
   f_{i,x} \circ f_{i-1,x} = 0.
   \]
   The family is **exact** if for all \( x \in X \) on the level of stalks the induced family of morphisms of Abelian groups
   \[
   (f_{i,x} : \mathcal{F}_{i,x} \to \mathcal{F}_{i+1,x})_{i \in \mathbb{Z}}
   \]
   is exact, i.e. if for all \( i \in \mathbb{Z} \)
   \[
   \ker [f_{i,x} : \mathcal{F}_{i,x} \to \mathcal{F}_{i+1,x}] = \text{im} [f_{i-1,x} : \mathcal{F}_{i-1,x} \to \mathcal{F}_{i,x}].
   \]

2. A **short exact sequence of sheaves** is an exact sheaf sequence of the form
   \[
   0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0.
   \]

3. A morphism of sheaves
   \[
   f : \mathcal{F} \to \mathcal{G}
   \]
   is respectively, **injective** or **surjective** or **bijective** if the corresponding property holds on the level of stalks
   \[
   f_x : \mathcal{F}_x \to \mathcal{G}_x
   \]
   for all \( x \in X \).

4. A **sequence of presheaves** on \( X \) is a family
   \[
   (f_i : \mathcal{F}_i \to \mathcal{F}_{i+1})_{i \in \mathbb{Z}}
   \]
   of morphisms between presheaves. The sequence of presheaves is **exact** if for each open \( U \subset X \) and each \( i \in \mathbb{Z} \) on the level of sections
   \[
   \ker f_{i,U} = \text{im} f_{i-1,U}
   \]
for
\[ F_{i-1}(U) \xrightarrow{f_{i-1}} F_i(U) \xrightarrow{f_i} F_{i+1}(U) \]

**Remark 2.23 (Exactness of a sheaf sequence).**

1. A sequence of sheaf morphisms is studied on stalks, while a sequence of presheaf morphisms is studied on the level of sections.

2. An exact sequence of sheaves satisfies for all \( i \in \mathbb{Z} \)
   \[ f_{i+1} \circ f_i = 0. \]
   A sequence of morphisms between sheaves may be exact when considered as sequence of sheaves, but not when considered as sequence of presheaves. Exactness of a sheaf sequence is a statement about the induced morphisms of the stalks. It is not required that the corresponding sequence of morphisms of the groups of sections
   \[ f_i : \mathcal{F}_i(U) \rightarrow \mathcal{F}_{i+1}(U), \ U \text{ open neighbourhood of } x \in X, \ i \in \mathbb{Z}, \]
   is exact.

3. One has to distinguish between a statement on the level of germs and a local statement on the level of sections. It is exactly the task of cohomology theory, see Chapter 3, to measure the difference between exactness on the level of germs and exactness on the level of sections, in particular on the level of global sections.

**Proposition 2.24 (Exponential sequence).** The exponential sequence on a complex manifold \( X \) is the following exact sequence of sheaves of Abelian groups

\[ 0 \rightarrow \mathbb{Z} \xrightarrow{j} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0 \]

Here the morphism \( j \) is the canonical inclusion. And the exponential

\[ \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \]

is defined for open sets \( U \subset X \) as

\[ \exp_U : \mathcal{O}(U) \rightarrow \mathcal{O}^*(U), \ f \mapsto \exp(2\pi i \cdot f). \]

**Proof.** To prove that the sheaf sequence is exact we consider for arbitrary but fixed \( x \in X \) the sequence of stalks
\[ 0 \to \mathbb{Z}_x = \mathbb{Z} \xrightarrow{h} \mathcal{O}_x^* \xrightarrow{e_x} \mathcal{O}_x^* \to 0 \]

i) **Exactness at \( \mathcal{O}_x \):** Each holomorphic function \( f \) defined on a domain and satisfying

\[ e^{2\pi i f} = 1 \]

is an integer constant and vice versa.

ii) **Exactness at \( \mathcal{O}_x^* \):** The surjectivity of the morphism \( e_x \) follows from the fact, that any holomorphic function without zeros, which is defined on a simply connected domain, has a holomorphic logarithm. \( \Box \)

Note that in general the exponential sequence from Proposition 2.24 is not exact on the level of global sections: For \( X = \mathbb{C}^* \) the morphism

\[ \mathcal{O}(X) \xrightarrow{e} \mathcal{O}^*(X), \ f \mapsto exp(2\pi i \cdot f) \]

is not surjective, because the holomorphic function

\[ 1/z \in \mathcal{O}^*(X) \]

has no logarithm.

**Definition 2.25 (Direct image).** Consider a continuous map

\[ f : X \to Y \]

between topological spaces and a sheaf \( \mathcal{F} \) on \( X \). The presheaf on \( Y \)

\[(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}V), \ V \subset Y \text{ open}, \]

is a sheaf, named the **direct image** \( f_*\mathcal{F} \) of \( \mathcal{F} \) with respect to \( f \).

Note. Taking the direct image is a covariant functor:

\[ id_*\mathcal{F} = \mathcal{F} \text{ and } (g \circ f)_*\mathcal{F} = g_*(f_*\mathcal{F}) \]

**Definition 2.26 (Ringed space).**

1. A **ringed space** is a pair \((X, \mathcal{R})\) with a topological space \( X \) and a subsheaf \( \mathcal{R} \subset \mathcal{C} \) of rings of complex-valued continuous functions on \( X \).
2. A morphism of ringed spaces

\((f, \tilde{f}) : (X, \mathcal{A}) \to (Y, \mathcal{B})\)

is a pair \((f, \tilde{f})\) with a continuous map \(f : X \to Y\) and a morphism of sheaves of rings

\(\tilde{f} : \mathcal{B} \to f_*\mathcal{A}\)

3. The morphism \((f, \tilde{f})\) is an isomorphism of ringed spaces if there exists a morphism of ringed

\((g, \check{g}) : (Y, \mathcal{B}) \to (X, \mathcal{A})\)

- satisfying for the continuous maps
  
  \[g \circ f = \text{id}_X, \quad f \circ g = \text{id}_Y\]

- and satisfying for the composition of the sheaf morphisms

\[\mathcal{A} \xrightarrow{\check{g}} g_*\mathcal{B} \xrightarrow{g_*f} g_*(f_*\mathcal{A}) = (g \circ f)_*\mathcal{A} = \mathcal{A} = \text{id}_\mathcal{A}\]

and analogously for the composition in the opposite order.

**Proposition 2.27 (Local models as ringed spaces).**

1. Local model of complex manifolds: For each open \(U \subset \mathbb{C}^n\) the pair \((U, \mathcal{O}_U)\) is a ringed space. If

   \[U \subset \mathbb{C}^n \text{ and } V \subset \mathbb{C}^m\]

   are open, then a continuous map

   \(\phi : U \to V\)

   is holomorphic iff \(\phi\) induces a sheaf morphism

   \(\tilde{\phi} : \mathcal{O}_V \to \phi_*\mathcal{O}_U\)

   by pullback of holomorphic functions

   \[\mathcal{O}_V(W) = \mathcal{O}_{\mathbb{C}^n}(W) \to (\phi_*\mathcal{O}_U)(W) = \mathcal{O}_U(\phi^{-1}W) = \mathcal{O}_{\mathbb{C}^n}(\phi^{-1}W)\]

   \[f \mapsto f \circ \phi, \quad W \subset V \text{ open.}\]

2. Local model of smooth manifolds: For each open subset \(U \subset \mathbb{R}^n\) the pair \((U, \mathcal{E}_U)\) is a ringed space. If

   \[U \subset \mathbb{R}^n \text{ and } V \subset \mathbb{R}^m\]

   are open, then a continuous map

   \(\phi : U \to V\)
is smooth iff $\phi$ induces a sheaf morphism

$$\tilde{\phi} : \mathcal{E}_V \to \phi_\ast \mathcal{E}_U$$

by pullback of smooth functions.

3. Local model of reduced complex spaces: Consider an open subset $U \subset \mathbb{C}^n$ and an analytic subset $A \subset U$. Its ideal sheaf $\mathcal{I}_A \subset \mathcal{O}_U$ defines the quotient sheaf

$$\mathcal{O}_A := \mathcal{O}_U / \mathcal{I}_A,$$

induced from the exact sequence

$$0 \to \mathcal{I}_A \to \mathcal{O}_U \to \mathcal{O}_A \to 0$$

The pair $(A, \mathcal{O}_A)$ is a ringed space.

**Proof.** 1. If $\phi$ induces by pullback the sheaf morphisms

$$\mathcal{O}_V \to \phi_\ast \mathcal{O}_U$$

then in particular for the holomorphic coordinate functions

$$z_j \in \mathcal{O}_V(V), \ j = 1, \ldots, m :$$

the functions

$$\tilde{\phi}(z_j) = z_j \circ \phi$$

are holomorphic, i.e. all component functions of $\phi$ are holomorphic. Hence

$$\phi : U \to V$$

is a holomorphic map in the sense of Definition 2.2.

2. The proof is analogous using the real coordinate functions. □

**Corollary 2.28 (Complex manifolds as ringed space).** Consider a connected topological manifold $X$ with second countable topology, and on $X$ a subsheaf of rings

$$\mathcal{R} \subset \mathcal{C}$$

of complex-valued, continuous functions. Then for each open covering $\mathcal{U} = (U_i)_{i \in I}$ of $X$ with homeomorphisms

$$\phi : U_i \to \tilde{\mathcal{V}}_i, \ V_i \subset \mathbb{C}^n \text{ open}, \ i \in I,$$

are equivalent:
1. The family \((\phi_i)_{i \in I}\) is an atlas of a complex structure on \(X\) with holomorphic structure sheaf \(\mathcal{O}_X = \mathcal{R}\).

2. Each homeomorphism
\[
\phi_i : U_i \rightarrow V_i, \ i \in I,
\]
induces by its pullback \(\tilde{\phi}_i\) an isomorphism of ringed spaces
\[
(\phi_i, \tilde{\phi}_i) : (U_i, \mathcal{R}|U_i) \sim (V_i, \mathcal{O}_V)
\]

Corollary 2.28 literally carries over to smooth structures on the topological manifold \(X\) and its structure sheaf \(\mathcal{E}_X\).

Moreover, a reduced complex space is defined by using the local models of analytic sets from Proposition 2.27, see [5, Section 0.14].

### 2.3 The Cousin problems

**Definition 2.29 (Cousin problems).** Consider a complex manifold \(X\).

1. An **additive Cousin distribution** on \(X\) is an open covering \(\mathcal{U} = (U_i)_{i \in I}\) of \(X\) and a corresponding family of locally defined meromorphic functions
\[
h_i \in \mathcal{M}(U_i), i \in I,
\]
such that for all pairs \(i, j \in I\) the differences are holomorphic, i.e. they satisfy
\[
h_i|_{U_{ij}} - h_j|_{U_{ij}} \in \mathcal{O}(U_{ij}), \quad U_{ij} := U_i \cap U_j.
\]

A solution of the additive Cousin distribution is a global meromorphic function \(h \in \mathcal{M}(X)\) with holomorphic differences
\[
h|_{U_i} - h_i \in \mathcal{O}(U_i), \ i \in I.
\]

2. A **multiplicative Cousin distribution** on \(X\) is an open covering \(\mathcal{U} = (U_i)_{i \in I}\) of \(X\) and a corresponding family of meromorphic functions
\[
h_i \in \mathcal{M}^*(U_i), i \in I,
\]
such that for all pairs \(i, j \in I\) the quotients are holomorphic without zeros, i.e. they satisfy
\[
h_i|_{U_{ij}} h_j|_{U_{ij}} \in \mathcal{O}^*(U_{ij}), \quad U_{ij} := U_i \cap U_j,
\]
A solution of the multiplicative Cousin distribution is a global meromorphic function \( h \in \mathcal{M}^*(X) \) with holomorphic quotients without zeros

\[
\frac{h|U_i}{h_i} \in O^*(U_i), \quad i \in I.
\]

Remark 2.30 shows: The additive resp. multiplicative Cousin problems are the analogue to the problems of Mittag-Leffler resp. Weierstrass from complex analysis in one variable. The Cousin problems ask for a global meromorphic function, which satisfies certain conditions referring to the sheaf \( O \) resp. referring to the sheaf \( O^* \).

**Remark 2.30 (Solution of the Cousin problems).**

1. Each solution of an additive Cousin distribution defines a global meromorphic function

   \[
   h \in \mathcal{M}(X)
   \]

   with the given “principal parts” \( (h_i)_{i \in I} \). Each two solutions differ by a holomorphic function from \( O(X) \).

2. Each solution of a multiplicative Cousin distribution defines a global meromorphic function

   \[
   h \in \mathcal{M}^*(X)
   \]

   with the same pole and zero orders as \( (h_i)_{i \in I} \). Each two solutions differ by a holomorphic function without zeros from \( O^*(X) \).

3. The main difference to the 1-dimensional case is the fact that the zeros and poles of holomorphic resp. meromorphic functions of several variables are no longer isolated points.

Our first goal is to solve the Cousin problems for polydics \( X \subset \mathbb{C}^n \).

We start the solution with a step *bottom-up*.

**Definition 2.31 (Adjacent product domains).**

1. A *product domain* in an open set \( X \subset \mathbb{R}^n \) is a set of the form

   \[
   Q = I_1 \times \ldots \times I_n \subset X \text{ with intervals } I_j \subset \mathbb{R} \text{ for } j = 1, \ldots, n.
   \]

   Depending on the type of all intervals one distinguishes *open* product domains resp. *compact* product domains.
2. Two compact product domains $Q_1, Q_2 \subset X$ are \textit{adjacent} if after renumbering

$$Q_1 = I_{1,1} \times I_2 \times \ldots \times I_n$$

and

$$Q_2 = I_{2,1} \times I_2 \times \ldots \times I_n$$

with adjacent first intervals

$$I_{1,1} = [a, b], \quad I_{2,1} = [b, c] \subset \mathbb{R},$$

see Figure 2.4.

\textbf{Notation 2.32 (Set of compact product domains).} For an open set $X \subset \mathbb{C}^n$ denote by $\mathcal{Q}(X)$ the set of all compact product domains contained in $X$.

\textbf{Proposition 2.33 (Cousin’s principle of induction).} Consider an open set $X \subset \mathbb{R}^N$ and a map

$$A : \mathcal{Q}(X) \rightarrow \{ \text{TRUE}, \text{FALSE} \}.$$ 

Assume that the truth-function $A$ satisfies the following two properties:

- \textbf{Small product domains:} Each point $x \in X$ has an open neighbourhood $U \subset X$ with $A(Q) = \text{TRUE}$ for all $Q \in \mathcal{Q}(U)$

- \textbf{Fusing adjacent product domains:} For each pair of adjacent product domains $Q_1, Q_2 \in \mathcal{Q}(X)$ holds

$$A(Q_1) = A(Q_2) = \text{TRUE} \implies A(Q_1 \cup Q_2) = \text{TRUE}.$$ 

Then for all $Q \in \mathcal{Q}(X)$

$$A(Q) = \text{TRUE}.$$
2.3 The Cousin problems

Proof. For a given product domain

\[ Q = I_1 \times \ldots I_N \in \mathcal{P}(X) \]

we have to show \( A(Q) = TRUE \).

i) Cutting \( Q \) into small adjacent product domains \( Q' \) with \( A(Q') = TRUE \): For each \( k \in \mathbb{N} \) and for all \( j = 1, \ldots, N \) we split each interval \( I_j = [a_j, b_j] \) into \( k \) adjacent intervals

\[ I_j = I_{j1} \cup \ldots \cup I_{jk} \text{ with } I_{jl} = [a_j + (l-1) \cdot h_j, a_j + l \cdot h_j] \]

\[ h_j := \frac{b_j - a_j}{k}, \quad l = 1, \ldots, k, \]

and define for each \( N \)-tuple \( (i_1, \ldots, i_N) \) of indices \( i_j \leq k \) the small product domains

\[ Q_{i_1 \ldots i_N} := I_{i_1} \times \ldots \times I_{i_N} \]

from the induced splitting of \( Q \). Due to the first property of the function \( A \) there exists a finite open covering \( \mathcal{U} \) of the compact set \( Q \) such that each element \( U \) of the covering \( \mathcal{U} \) satisfies

\[ A(Q') = TRUE \text{ for all } Q' \in \mathcal{P}(U). \]

The covering \( \mathcal{U} \) of the compact metric space \( Q \) has a Lebesgue number \( \varepsilon > 0 \). For large \( k \) each of the small product domain \( Q_{i_1 \ldots i_N} \) has diameter \( \leq \varepsilon \), and is therefore contained in an element \( U \) of the covering \( \mathcal{U} \). Choose a fixed \( k \in \mathbb{N} \) with this property. Then each of the small product domains

\[ Q_{i_1 \ldots i_N} \]

is contained in an element \( U \) of the covering \( \mathcal{U} \). As a consequence

\[ A(Q_{i_1 \ldots i_N}) = TRUE \]

for all \( N \)-tuples \( (i_1, \ldots, i_N) \).

ii) Edge fusing bottom-up: We now enlarge step by step the size of the product domains \( Q' \subset Q \) with \( A(Q') = TRUE \). For \( m = 0, \ldots, N \) set

\[ Q_{i_1\ldots i_m} := I_{i_1} \times \ldots \times I_{i_m} \times I_{m+1} \times \ldots \times I_N \]

as a product domain, built within \( Q \) from \( m \) small product domains as first factors and \( N - m \) large product domains as last factors. We prove by descending induction on \( m \)

\[ A(Q_{i_1\ldots i_m}) = TRUE \text{ for all tuples } (i_1, \ldots, i_{m-1}) \]

Induction start \( m = N \): The result has been proved in part i).

Induction step \( m + 1 \mapsto m \): We illustrate the induction step for the induction steps
\[ 2 \mapsto m = 1 \text{ and } 1 \mapsto m = 0 \]
in the 2-dimensional case
\[ Q = I_1 \times I_2 \subset X \subset \mathbb{R}^2 \]
and \( k = 3 \) partitions, see Figure 2.5: The bold lines indicate the common borders across which the adjacent product domains are fused during the induction step \( m \mapsto m - 1 \). The fusing is possible due to the second property of the function \( A \).

For a formalization of the general step see [6, § 4, Satz 3]. □

Fig. 2.5 Cousin principle of induction

A first application of Cousin’s principle of induction to complex analysis is Proposition 2.34.

**Proposition 2.34 (Cartan’s lemma for holomorphic functions).** Consider two adjacent product domains
\[ Q', Q'' \subset \mathbb{C}^n, \]
and in an open neighbourhood \( U \) of \( Q' \cap Q'' \) a holomorphic function
\[ f \in \mathcal{O}(U) \text{ (additive case)} \]
resp. a holomorphic function without zeros
\[ g \in \mathcal{O}^*(U) \text{ (multiplicative case)}. \]
Then there exist open neighbourhoods
\[ \tilde{U}' \supset Q' \text{ and } \tilde{U}'' \supset Q'' \]
and an open set \( \tilde{U} \) satisfying...
(\bar{U}' \cap \bar{U}'') \supset \bar{U} \supset (\bar{Q}' \cap \bar{Q}'')

with

• holomorphic functions

\[ f' \in O(\bar{U}') \text{ and } f'' \in O(\bar{U}'') \]

satisfying over \( \bar{U} \)

\[ f = f' - f'' \]

• resp. holomorphic functions without zeros

\[ g' \in O^*(\bar{U}') \text{ and } g'' \in O^*(\bar{U}'') \]

satisfying over \( \bar{U} \)

\[ g = \frac{g'}{g''} \]

Fig. 2.6 Cartan’s lemma for holomorphic functions, view from the first factor \( \mathbb{C} \) of \( \mathbb{C}^n \)

Proof. We may assume the following geometric situation, see Figure 2.6:

\[ Q' = R' \times Q \text{ and } Q'' = R'' \times Q \]
with a product domain $Q \subset \mathbb{C}^{n-1}$ and two adjacent product domains $R', R'' \subset \mathbb{C}$. Moreover
$$U = U_0 \times V$$
is an open neighbourhood of $Q_1 \cap Q_2$ with $V \subset \mathbb{C}^{n-1}$ open and $U_0 \subset \mathbb{C}$ an open neighbourhood of $Q' \cap Q''$.

i) *Additive case:* Consider the holomorphic function
$$f \in U_0 \times V$$
Compactness of $R' \cap R''$ and openness of $U_0$ provide a product domain $R \subset \mathbb{C}$ satisfying
$$R' \cap R'' \subset \mathcal{R} \subset R \subset U_0 \subset \mathbb{C}$$
Split the boundary as
$$\partial R = C' \cup C''$$
with the orientation from Figure 2.6, and choose open neighbourhoods
$$U' \supset R', U'' \supset R''$$
with
$$C' \cap U' = C'' \cap U'' = \emptyset.$$ 
For each point $(z, s) \in \mathcal{R} \times V$ the Cauchy integral formula with respect to the 1-dimensional integration $d\zeta$ shows
$$f(z,s) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(\zeta, s)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{C'} \frac{f(\zeta, s)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C''} \frac{f(\zeta, s)}{\zeta - z} d\zeta$$
Define the functions
$$f' : U' \times V \to \mathbb{C}, \quad f'(z,s) := \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{f(\zeta, s)}{\zeta - z} d\zeta, \quad \mathcal{U}' := U' \times V,$$
and
$$f'' : U'' \times V \to \mathbb{C}, \quad f''(z,s) := \frac{1}{2\pi i} \int_{\mathcal{C}''} \frac{f(\zeta, s)}{\zeta - z} d\zeta, \quad \mathcal{U}'' := U'' \times V,$$
Both functions are holomorphic due to Theorem 1.6. Due to construction they have on
$$\mathcal{U} := (U' \cap U'') \times V$$
the difference
$$f = f' - f''.$$

ii) *Multiplicative case:* The case of the given holomorphic function $g \in \mathcal{O}^*(U_0 \times V)$ reduces to the additive case from part i). Choose
The Cousin problems

\[ U = U_0 \times V \]

simply connected and consider the holomorphic function

\[ \log g \in \mathcal{O}(U_0 \times V) \]
e.g., choosing the principal branch of the holomorphic logarithm in \( \mathbb{C}^* \). By part i) there exist holomorphic functions \( \tilde{f}' \) in an open neighborhood of \( Q' \) and \( \tilde{f}'' \) in an open neighborhood of \( Q'' \) satisfying in an open neighborhood of \( Q' \cap Q'' \)

\[ \log g = \tilde{f}' - \tilde{f}'' \]

Then the holomorphic functions without zeros

\[ g' := \exp \tilde{f}' \quad \text{and} \quad g'' := \exp \tilde{f}'' \]
satisfy in an open neighborhood of \( Q' \cap Q'' \)

\[ g = \frac{g'}{g''}. \]

\[ \square \]

As a consequence of Cartan’s lemma from Proposition 2.34 one obtains with Proposition 2.35 a solution of the Cousin problems after shrinking.

**Proposition 2.35 (Solving the Cousin problems in a an open neighbourhood of a product domain).** Consider an open set \( X \subset \mathbb{C}^n \). Then each additive or multiplicative Cousin distribution

\[ \mathcal{C} := (U_i, f_i)_{i \in I} \]
on \( X \) has for each product domain \( Q \in \mathcal{Q}(X) \) a solution in a neighborhood of \( Q \).

**Proof.** The proof applies Cousin’s principle of induction, Proposition 2.33, together with the Cartan Lemma, Proposition 2.34. We consider the map

\[ A : \mathcal{Q}(X) \to \{ TRUE, FALSE \}, \]
defined as

\[ A(Q) := \begin{cases} TRUE & \mathcal{C} \text{ is solvable in a neighborhood of } Q \\ FALSE & \text{otherwise} \end{cases} \]

The map \( A \) satisfies the assumptions of Proposition 2.33:

- For a given \( x \in X \) choose an index \( i \in I \) with \( x \in U_i \). The function \( f_i \in \mathcal{M}(U_i) \) solves the Cousin problem in \( U_i \), in particular for all \( Q \in \mathcal{Q}(U_i) \).
• Consider two adjacent product domains $Q', Q'' \in \mathcal{Q}(X)$ and assume open neighbourhoods

$$U' \supset Q' \text{ and } U'' \supset Q''$$

with solutions

$$f' \in \mathcal{H}(U'), \ f'' \in \mathcal{H}(U'')$$

of the induced Cousin distributions on $U'$ respectively $U''$.

i) *Additive case*: On $U' \cap U''$ the function

$$f' - f''$$

is holomorphic because for each $i \in I$

$$(f' - f'')(U' \cap U'' \cap U_i = (f' - f_i) - (f'' - f_i) \in \mathcal{O}(U' \cap U'' \cap U_i)$$

Proposition 2.34 provides two open neighbourhoods

$$V' \supset Q' \text{ and } V'' \supset Q'' \text{ with } V' \cap V'' \subset U' \cap U''$$

with corresponding holomorphic functions

$$g' \in \mathcal{O}(V') \text{ and } g'' \in \mathcal{O}(V'')$$

satisfying on $V' \cap V''$

$$f' - f'' = g' - g'' \text{ i.e. } f' - g' = f'' - g''$$

They define a meromorphic function

$$f = (f' - g', f'' - g'') \in \mathcal{H}(V' \cup V'')$$

which solves $\mathcal{C}$ in the neighbourhood $V' \cup V''$ of $Q' \cup Q''$. Hence Proposition 2.33 implies

$$A(Q) = \text{TRUE} \text{ for all } Q \in \mathcal{Q}(X).$$

ii) *Multiplicative case*: On $U' \cap U''$ the function

$$\frac{f'}{f''}$$

is holomorphic without zeros because for each $i \in I$

$$\frac{f'}{f''} = \frac{f'/f_i}{f''/f_i} \in \mathcal{O}^*(U' \cap U'' \cap U_i)$$

The multiplicative case of Proposition 2.34 provides two open neighbourhoods
2.3 The Cousin problems

\[ V' \supset Q' \text{ and } V'' \supset Q'' \text{ with } V' \cap V'' \subset U' \cap U'' \]

with corresponding holomorphic functions

\[ g' \in \mathcal{O}^+(V') \text{ and } g'' \in \mathcal{O}^+(V'') \]

satisfying on \( V' \cap V'' \)

\[ \frac{f'}{f''} = \frac{g'}{g''} \text{ i.e. } \frac{f'}{g'} = \frac{f''}{g''} \]

They define a meromorphic function

\[ f = \left( \frac{f'}{g'}, \frac{f''}{g''} \right) \in \mathcal{M}^+(V' \cup V'') \]

Again Proposition 2.33 implies

\[ A(Q) = \text{TRUE} \text{ for all } Q \in \mathcal{Q}(X). \]

\[ \square \]

**Theorem 2.36 (The Cousin problems in polydiscs).** In an \( n \)-dimensional polydisc

\[ \Delta(r) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_\nu| < r_\nu, \ \nu = 1, \ldots, n\} \]

with polyradius

\[ r = (r_1, \ldots, r_n), \ 0 < r_\nu \leq \infty, \ \nu = 1, \ldots, n, \]

each additive or multiplicative Cousin problem has a solution.

The proof rests on the existence of solutions after shrinking from Proposition 2.35 and the subsequent extension step thanks to the Mittag-Leffler exhaustion from Proposition 1.26.

**Proof.** Consider a given Cousin distribution on \( \Delta(r) \)

\[ \mathcal{C} = (U_i, f_i)_{i \in I}. \]

Choose an exhaustion

\[ (X_j)_{j \in \mathbb{N}} \]

of \( \Delta(r) \) by relatively compact, open polydiscs satisfying \( X_j \subset \subset X_{j+1}, \ j \in \mathbb{N}. \)

i) **Solution on \( X_j \)**: For two polydiscs

\[ \Delta \subset \subset \tilde{\Delta} \subset \mathbb{C}^n \]
the Riemann mapping theorem applies to each factor of $\tilde{\Delta}$. It provides a biholomorphic map
\[ \phi : \tilde{\Delta} \cong \tilde{Q} \]
with a product domain
\[ \tilde{Q} \subseteq C^n \cong \mathbb{R}^{2n}. \]

The compact subset
\[ \phi(\Delta) = \overline{\phi(\Delta)} \subseteq \tilde{Q} \]
has an open neighbourhood
\[ Q \subset \subset \tilde{Q}, \]
which is a product domain $Q$. Proposition 2.35 provides a solution of the induced Cousin problem on $\tilde{Q}$ in a neighbourhood of $Q$. It pulls back via $\phi$ to a solution of the original Cousin distribution over $\Delta$.

As a consequence, there exists for each $j \in \mathbb{N}$ a solution of the Cousin distribution $\mathcal{C}$ over the polydisc $X_j$. For each $j \in \mathbb{N}^*$ denote by $M_j$ the set of solutions of $\mathcal{C}$ on $X_j$ and by
\[ \rho_j : M_j \to M_{j-1} \]
the canonical restriction.

i) *Additive Cousin problem*: For each $j \in \mathbb{N}$ the set $M_j$ is the affine space
\[ M_j = f_j + \mathcal{O}(X_j) \]
with an arbitrary, fixed solution $f_j \in M_j$. In particular the map
\[ \tau_j : \mathcal{O}(X_j) \to M_j, \phi \mapsto f_j + \phi, \]
is bijective. The translation-invariant Fréchet space structure of $\mathcal{O}(X_j)$ carries over to $M_j$ and provides $M_j$ with the structure of a complete metric space. The metric structure is independent from the choice of $f_j$: Consider a second solution $f'_j$ and the corresponding map
\[ \tau'_j : \mathcal{O}(X_j) \to M_j, \phi \mapsto f'_j + \phi, \]
Choose a fixed translation-invariant metric $\delta$ on $\mathcal{O}(X_j)$. Then the induced metrics $d$ resp. $d'$ satisfy:
\[
d(f, g) := \delta(f - f_j, g - f_j) = \delta(f - f_j + (f_j - f'_j), g - f_j + (f_j - f'_j)) = \\
= \delta(f - f'_j, g - f'_j) =: d'(f, g)
\]
In order to show that each restriction $\rho_j : M_j \to M_{j-1}$ has dense image one considers the commutative diagram
2.3 The Cousin problems

\[ \mathcal{O}(X_j) \longrightarrow \mathcal{O}(X_{j-1}) \]
\[ \tau_j \downarrow \quad \tau_{j-1} \downarrow \]
\[ M_j \quad \rho_j \rightarrow M_{j-1} \]

with the horizontal map

[\mathcal{O}(X_j) \rightarrow \mathcal{O}(X_{j-1})]

as the canonical restriction. The latter map is continuous with dense image, because holomorphic functions on \(X_{j-1}\) can be approximated on each compact set by their Taylor polynomials, which are defined on all of \(\mathbb{C}^n\). By construction of the topologies the restrictions

[\rho_j : M_j \rightarrow M_{j-1}]  

are continuous with dense image. The Mittag-Leffler principle of exhaustion, Proposition 1.26, applies to the sequence \((M_j, \rho_j)_{j \in \mathbb{N}}\) and provides a sequence \((f_j)_{j \in \mathbb{N}}\) of local solutions \(f_j \in M_j\) such that

[\rho_j(f_j) = f_{j-1}]  

for all \(j \in \mathbb{N}\). The sequence defines a global meromorphic function

[\mathcal{H}(\Delta(r)) \ni f \ni f_j|X_j = f_j]  

satisfying for all \(j \in \mathbb{N}\)

\[ f|X_j = f_j. \]

iii) \textit{Multiplicative Cousin problem}: Because each polydisc \(X_j, j \in \mathbb{N}\), is a simply connected domain, the exponential and its inverse, the logarithm, control the space of local solutions, i.e. the local meromorphic functions which define the Cousin distribution \(\mathcal{C}\). For each \(j \in \mathbb{N}\) the map

[\psi_j : \mathcal{O}(X_j) \rightarrow M_j, \phi \mapsto f_j \cdot e^{\phi}]  

is surjective. The twisted restriction

[\sigma_j : \mathcal{O}(X_j) \rightarrow \mathcal{O}(X_{j-1}), \phi \mapsto \phi|X_{j-1} + h_{j-1}]  

with

[h_{j-1} := \log \left( \frac{f_j|X_{j-1}}{f_{j-1}} \right) \in \mathcal{O}(X_{j-1})]  

is continuous. Here \(h_{j-1}\) expresses as a quotient the difference between local solutions. The map \(\sigma_j\) has dense image, because holomorphic functions over a given compact \(K \subset X_{j-1}\) can be uniformly approximated by their Taylor polynomials. One checks the commutativity of the following diagram
The Mittag-Leffler principle of exhaustion, Proposition 1.26, applies to the sequence of morphism between complete metric spaces

\[(\mathcal{O}(X_j), \sigma_j)_{j \in \mathbb{N}}.\]

It provides a sequence \((g_j)_{j \in \mathbb{N}}\) of compatible holomorphic functions

\[g_j \in \mathcal{O}(X_j), \quad \text{with} \quad \sigma_j(g_j) = \sigma_{j-1}, \quad j \in \mathbb{N}.\]

The commutativity of the diagram above concludes

\[(f_j \cdot e^{g_j})|_{X_{j-1}} = f_{j-1} \cdot e^{g_{j-1}}.\]

Hence the sequence \((f_j \cdot e^{g_j})_{j \in \mathbb{N}}\) defines a global meromorphic function

\[f \in \mathcal{M}^*(\Delta(r)) \text{ with } f|_{X_j} = f_j \cdot e^{g_j}, \quad j \in \mathbb{N},\]

which solves the multiplicative Cousin problem. \(\square\)

**Remark 2.37 (Solving the Cousin problems).**

1. Theorem 2.36 solves the Cousin problems on \(n\)-dimensional polydiscs. Cousin’s proof constructs the solution bottom-up over neighbourhoods of adjacent product domains (Cousin’s principle of induction). For each step Cousin uses a mild version of a result, which was later generalized as Cartan’s lemma.

A second proof for the solvability of Cousin’s problems in polydiscs will be given in Corollary 3.24. The second proof relies on Dolbeault’s lemma and uses basic results from cohomology theory.

2. Complex analysis in the 20th century was driven by the search for solutions of the Cousin problems for more general complex manifolds \(X\). For an intermediate result see Corollary 3.24. The Corollary gives cohomological conditions for the solvability of the Cousin problems. The final result is Corollary 6.25. The main tool is the cohomological formulation of the Cousin problems as a statement about the cohomology of the structure sheaf \(\mathcal{O}_X\) and about the topological cohomology group \(H^2(X, \mathbb{Z})\).
Chapter 3
Sheaf cohomology

For a complex manifold $X$ the functor of sections over a fixed open set $U \subset X$

$$\text{Sh}_X \rightarrow \text{Ab}$$

from the category of sheaves of Abelian groups on $X$ to the category of Abelian
groups is left exact, i.e. for any exact sequence of sheaf morphisms

$$0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

and for any open set $U \subset X$ the sequence of morphisms of Abelian groups of sections

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U) \xrightarrow{g_U} \mathcal{H}(U)$$

is exact. But in general, for a surjective sheaf morphism $g$ the morphism on sections

$$\mathcal{G}(U) \xrightarrow{g_U} \mathcal{H}(U)$$

is not surjective, see Remark after Proposition 2.24.

Cohomology, or right-derivation of the functor of sections, is the means to extend
for an exact sequence of sheaf morphisms

$$0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \rightarrow 0$$

the exact sequence of Abelian groups

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U) \xrightarrow{g_U} \mathcal{H}(U)$$

to the right-hand. One defines groups

$$H^q(U, \mathcal{F}), H^q(U, \mathcal{G}), H^q(U, \mathcal{H}), q \geq 0,$$
to obtain a long exact sequence in the category of Abelian groups. The length of the extended sequence measures the failing of right-exactness of the functor of sections.

3.1 Čech cohomology groups

Definition 3.1 (Cochains, cocycles, coboundaries and Čech cohomology classes). Consider a topological space $X$, a presheaf $\mathcal{F}$ of Abelian groups on $X$ and an open covering $\mathcal{U} = (U_i)_{i \in I}$ of $X$.

- For each $q \in \mathbb{N}$ the $q$-th cochain group of $\mathcal{F}$ with respect to $\mathcal{U}$ is the Abelian group
  \[ C^q(\mathcal{U}, \mathcal{F}) := \prod_{(i_0, \ldots, i_q) \in \mathbb{N}^{q+1}} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_q}), \quad C^{-1}(\mathcal{U}, \mathcal{F}) := 0. \]
  Hence a $q$-cochain is a family
  \[ f = (f_{i_0, \ldots, i_q})_{i_0, \ldots, i_q \in \mathbb{N}^{q+1}} \]
  of sections $f_{i_0, \ldots, i_q} \in \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_q})$ over the $q + 1$-fold intersections
  \[ U_{i_0, \ldots, i_q} := U_{i_0} \cap \ldots \cap U_{i_q} \]
  of the open sets of the covering. The group structure on $C^q(\mathcal{U}, \mathcal{F})$ derives from the group structure of the factors.

- For each $q \in \mathbb{N}$ the coboundary operator
  \[ \delta := \delta^q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F}) \]
  is defined for $f \in C^q(\mathcal{U}, \mathcal{F})$ as
  \[ \delta f := g := (g_{i_0, \ldots, i_{q+1}})_{(i_0, \ldots, i_{q+1}) \in \mathbb{N}^{q+2}} \]
  with the cross sum of restrictions
  \[ g_{i_0, \ldots, i_{q+1}} := \sum_{k=0}^{q+1} (-1)^k \cdot f_{i_0, i_k, i_{k+1} \ldots i_{q+1}}|U_{i_0, \ldots, \hat{i}_k, \ldots, i_{q+1}} \]
  Here $\hat{i}_k$ means to omit the index $i_k$.

- For each $q \in \mathbb{N}$ one defines the group of $q$-cocycles
  \[ Z^q(\mathcal{U}, \mathcal{F}) := \ker[C^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^q} C^{q+1}(\mathcal{U}, \mathcal{F})], \]
the group of \( q \)-coboundaries
\[
B^q(U, F) := \text{im} \left[ C^{q-1}(U, F) \xrightarrow{\partial^{q-1}} C^q(U, F) \right],
\]
and after checking the chain condition, see Remark 3.2,
\[
\delta^q \circ \delta^{q-1} = 0
\]
the \( q \)-th cohomology group
\[
H^q(U, F) := \frac{Z^q(U, F)}{B^q(U, F)}
\]
Elements from \( H^q(U, F) \) are named \( q \)-th Čech cohomology classes of \( F \) with respect to the covering \( U \). Two cocycles from \( Z^q(U, F) \) with determine the same cohomology class in \( H^q(U, F) \) are named cohomologous.

Remark 3.2 (Cohomology).
1. **Cocycle relation**: The meaning of cocycles of dimension \( q = 0, 1 \) can be stated explicitly as follows:
   - \( q = 0 \): A family \( (f_i)_i \in C^0(U, F) \) is a 0-cocycle iff for all \( i, j \in I \)
     \[
f_j - f_i = 0 \text{ on } U_i \cap U_j,
\]
i.e. if the cochain satisfies on the intersections
   \[
   U_{ij} := U_i \cap U_j
\]
   the cocycle condition
   \[
f_i = f_j.
\]
   If \( F \) is a sheaf, then 0-cocycles correspond bijectively to global sections \( f \in \mathcal{F}(X) \), because
   \[
   B^0(U, F) = 0
   \]
   and because the two sheaf axioms imply that the canonical map
   \[
   \mathcal{F}(X) \to Z^0(U, F) = H^0(U, F), \quad f \mapsto (f_i := f|_{U_i})_{i \in I},
   \]
is an isomorphism.
   - \( q = 1 \): A family \( (f_{ij})_{ij} \in C^1(U, F) \) is a 1-cocycle iff for all \( i, j, k \in I \)
     \[
     0 = f_{jk} - f_{ik} + f_{ij}
     \]
i.e. the cochain satisfies on the 3-fold intersections
Sheaf cohomology

\[ U_{ijk} := U_i \cap U_j \cap U_k \]

the cocycle condition

\[ f_{ik} = f_{ij} + f_{jk}. \]

2. Iteration of the coboundary operator: One checks that the composition of the coboundary operator from Definition 3.1 satisfies for each \( q \in \mathbb{N} \) the chain condition

\[ \delta^q \circ \delta^{q-1} = 0, \]

i.e. cochains and coboundary form a complex of Abelian groups. For the proof one uses that the sum in the definition of the coboundary operator is an alternating sum. For example for \( q = 1 \):

\[ C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^1} C^2(\mathcal{U}, \mathcal{F}) \]

satisfies

\[ \delta^0((f_i)_{i \in I}) = (g_{ij} := f_j - f_i)_{i,j \in I} \in C^1(\mathcal{U}, \mathcal{F}) \]

and

\[ \delta^1((g_{ij})_{i,j \in I}) := ((h_{ijk} = g_{jk} - g_{ik} + g_{ij})_{i,j,k \in I}) \in C^2(\mathcal{U}, \mathcal{F}). \]

As a consequence

\[ h_{ijk} = (f_k - f_j) - (f_k - f_i) + (f_j - f_i) = 0. \]

Hence

\[ B^1(\mathcal{U}, \mathcal{F}) \subset Z^1(\mathcal{U}, \mathcal{F}) \]

and the first cohomology group is well-defined as

\[ H^1(\mathcal{U}, \mathcal{F}) := \frac{Z^1(\mathcal{U}, \mathcal{F})}{B^1(\mathcal{U}, \mathcal{F})} \]

Cocycles and coboundaries are a suitable language to express the Cousin problems from Definition 2.29.

**Proposition 3.3 (The Cousin problems as problems for Čech cohomology).**

Consider a complex manifold \( X \) and an open covering \( \mathcal{U} = (U_i)_{i \in I} \) of \( X \).

1. An additive Cousin distribution with respect to \( \mathcal{U} \) is a 0-cochain with values in the sheaf \( \mathcal{M} \)

\[ c := (h_i)_{i \in I} \in C^0(\mathcal{U}, \mathcal{M}) \]

providing a 1-cocycle with values in the sheaf \( \mathcal{O} \)

\[ \Delta c := (h_{ij} = h_j - h_i)_{i,j} \in Z^1(\mathcal{U}, \mathcal{O}) \]

The Cousin distribution \( c \) has a solution iff \( \Delta c \) is even a coboundary, i.e. iff
3.1 Čech cohomology groups

\[ \Delta c \in B^1(\mathcal{U}, \mathcal{O}) \text{ or } [\Delta c] = 0 \in H^1(\mathcal{U}, \mathcal{O}). \]

2. A multiplicative Cousin distribution with respect to \( \mathcal{U} \) is a 0-cochain with values in the sheaf \( \mathcal{M}^* \)

\[ c := (h_i)_{i \in I} \in C^0(\mathcal{U}, \mathcal{M}^*) \]

providing a 1-cocycle with values in the multiplicative sheaf \( \mathcal{O}^* \)

\[ \Delta c := \left( \frac{h_j}{h_i} \right)_{i j} \in Z^1(\mathcal{U}, \mathcal{O}^*) \]

The Cousin distribution \( c \) has a solution iff \( \Delta c \) is even a coboundary, i.e. iff

\[ \Delta c \in B^1(\mathcal{U}, \mathcal{O}^*) \text{ or } [\Delta c] = 0 \in H^1(\mathcal{U}, \mathcal{O}^*). \]

**Proof.** 1. The cochain

\[ \Delta c = (h_{ij})_{ij} \in C^1(\mathcal{U}, \mathcal{O}) \]

is a 1-cocycle:

\[ h_{ij} + h_{jk} = (h_j - h_i) + (h_k - h_j) = h_k - h_i = h_{ik} \]

Assume that

\[ \Delta c \in Z^1(\mathcal{U}, \mathcal{O}) \]

is even a coboundary, i.e.

\[ \Delta c = \delta^0 g \in B^1(\mathcal{U}, \mathcal{O}) \]

with a suitable

\[ g = (g_i)_{i} \in C^0(\mathcal{U}, \mathcal{O}), \]

then on each \( U_{ij} \)

\[ h_{ij} = h_j - h_i = (\delta g)_{ij} = g_j - g_i \]

i.e.

\[ h_j - g_j = h_i - g_i. \]

Hence

\[ h = (h_i - g_i)_{i \in I} \in Z^0(\mathcal{U}, \mathcal{M}) = \mathcal{M}(X) \]

is a solution of the additive Cousin distribution. Reverting the argument proves the opposite direction of the claim.

2. Analogous to the first part. \( \square \)

The question whether a given Cousin distribution has a solution translates into the question: Is the 1-cocycle induced by the Cousin distribution a coboundary? Hence
the solvability of a Cousin distribution is equivalent to the vanishing of the induced cohomology class 

\[ [\Delta c] \in H^1(\mathcal{U}, \mathcal{O}) \] 

resp. 

\[ [\Delta c] \in H^1(\mathcal{U}, \mathcal{O}^*) \].

The cohomological formulation of the Cousin problems provides an example that the first cohomology group 

\[ H^1(\mathcal{U}, \mathcal{F}) \]

collects the obstructions against glueing local solutions of a problem to a global solution of the problem. Proposition 3.21 will show that on a smooth manifold all obstructions with values in the smooth structure sheaf or in the sheaf of smooth differential forms vanish. An analogous result does not hold in the category of holomorphic functions on an arbitrary complex manifold. Hence it is an important result to prove the vanishing theorems on Stein manifolds in Chapter 6.

Our next aim is to remove the dependency of the cohomology on a given open covering of the complex manifold \( X \). We show how to abstract from the covering to obtain a cohomology theory which only depends on \( X \) and on the sheaf \( \mathcal{F} \).

**Definition 3.4 (Refinement).** Consider a topological space \( X \) and two open coverings of \( X \) 

\[ \mathcal{U} = (U_i)_{i \in I} \] 

and \( \mathcal{V} = (V_\alpha)_{\alpha \in A} \)

The covering \( \mathcal{V} \) is a refinement of the covering \( \mathcal{U} \), notation 

\[ \mathcal{V} < \mathcal{U} \],

if there exists a map, named refinement map, 

\[ \tau : A \to I \]

such that for all \( \alpha \in A : V_\alpha \subset U_{\tau(\alpha)}. \)

The refinement map attaches to each open subset \( V \) of the finer covering an open subset \( U \) of the coarser covering with \( V \subset U \). In general, the refinement map of a refinement \( \mathcal{V} < \mathcal{U} \) is not uniquely determined. But any two refinement maps induce homotopic maps between the cochain groups with respect to the coverings \( \mathcal{V} \) and \( \mathcal{U} \). And due to the homotopy, both maps induce the same map in cohomology.

**Proposition 3.5 (Chain homotopy).** Consider a topological space \( X \) and a presheaf \( \mathcal{F} \) on \( X \). For each pair of refinements 

\[ \mathcal{U} = (U_i)_{i \in I} < \mathcal{V} = (V_\alpha)_{\alpha \in A} \]
the refinement map

\[ \tau : A \to I \]
induces a family of morphisms

\[ t^q_V : H^q(U, F) \to H^q(V, F), \quad q \in \mathbb{N}. \]

These morphisms are independent from the choice of \( \tau \).

Proof. i) Construction of the morphisms: For each \( q \in \mathbb{N} \) the refinement map \( \tau \) defines a restriction on the level of cochains, namely the group morphism

\[ t^q : C^q(U, F) \to C^q(V, F), \quad \xi = (\xi_{\alpha_0} \ldots, \xi_{\alpha_q}) \mapsto t(\xi) = (\eta_{\alpha_0} \ldots, \eta_{\alpha_q}), \]

with

\[ \eta_{\alpha_0} \ldots, \eta_{\alpha_q} := \xi_{\tau(\alpha_0) \ldots, \tau(\alpha_q)}|_{V_{\alpha_0} \cap \ldots \cap V_{\alpha_q}} \]

The group morphism is compatible with the coboundary operator, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
C^q(U, F) & \xrightarrow{t} & C^q(V, F) \\
\delta \downarrow & & \delta \downarrow \\
C^{q+1}(U, F) & \xrightarrow{t} & C^{q+1}(V, F)
\end{array}
\]

Hence the morphism induces a group morphism on the level of cocycles

\[ t : Z^q(U, F) \to Z^q(V, F), \]

on the level of coboundaries

\[ t : B^q(U, F) \to B^q(V, F), \]

and on the level of cohomology

\[ t : H^q(U, F) \to H^q(V, F). \]

ii) Chain homotopy: Assume two refinement maps

\[ \tau_j : A \to I, \quad j = 1, 2, \]

and denote by \( t_j, \ j = 1, 2 \), the induced families of restriction maps. We construct a family of group morphisms

\[ h^q : H^{q+1}(U, F) \to H^q(V, F), \quad q \in \mathbb{N}, \]
as homotopies between the family $t_1$ and the family $t_2$, i.e. such that with respect to the following diagram

$$
\begin{array}{ccc}
C^{q-1}(\mathcal{U}, F) & \delta & C^q(\mathcal{U}, F) \\
\downarrow h^q \quad t_1 & \quad & \downarrow h^q + 1 \\
C^{q-1}(\mathcal{V}, F) & \delta & C^q(\mathcal{V}, F)
\end{array}
$$

holds

$$t_1 - t_2 = \delta \circ h^q + h^{q+1} \circ \delta.$$

Define

$$h^q : C^q(\mathcal{U}, F) \to C^{q-1}(\mathcal{V}, F), \quad f \mapsto h^q(f),$$

with

$$(h^q f)_{a_0 \ldots a_{q-1}} := \sum_{k=0}^{q-1} (-1)^k \cdot f_{\tau_1(a_0) \ldots \tau_2(a_k) \ldots \tau_2(a_{q-1})} | V$$

for

$$V := V_{a_0} \cap \ldots \cap V_{a_{q-1}} \subset (U_{\tau_1(a_0)} \cap \ldots \cap U_{\tau_1(a_k)} \cap U_{\tau_2(a_k)} \cap \ldots \cap U_{\tau_2(a_{q-1})}),$$

and check

$$t_1 - t_2 = \delta \circ h^q + h^{q+1} \circ \delta : C^q(\mathcal{U}, F) \to C^q(\mathcal{V}, F).$$

As a consequence, for a cocycle $f \in Z^q(\mathcal{U}, F)$

$$t_1(f) - t_2(f) = \delta(h^q(f)) + h^{q+1}(\delta(f)) = \delta(h^q(f)) \in B^q(\mathcal{V}, F)$$

or

$$[t_1(f)] - [t_2(f)] = 0 \in H^q(\mathcal{V}, F).$$

□

**Proposition 3.6 (Injectivity of the refinement).** Consider a topological space $X$ and a sheaf $\mathcal{F}$ on $X$. Each refinement of two open coverings $\mathcal{V} < \mathcal{U}$ induces an injection on the level of the first cohomology

$$t_{\mathcal{V}} : H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{V}, \mathcal{F}).$$

The refinement in higher dimensions is not necessarily injective. Moreover, in general the claim does not hold for a presheaf.
Proof. i) Refinement: For $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_a)_{a \in A}$ the refinement map

$$\tau : A \to I$$

with $V_a \subset U_{\tau(a)}$

defines on the level of cochains the restriction

$$t : C^1(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{V}, \mathcal{F}), (\xi_{ij}) \mapsto (\eta_{a\beta}),$$

with

$$\eta_{a\beta} := \xi_{\tau(a),\tau(b)}|_{V_{a\beta}}, V_{a\beta} := V_a \cap V_\beta.$$ 

The map is also a restriction on the level of cocycles

$$t : Z^1(\mathcal{U}, \mathcal{F}) \to Z^1(\mathcal{V}, \mathcal{F}).$$

ii) Injectivity: Consider a cocycle

$$\xi = (\xi_{ij}) \in Z^1(\mathcal{U}, \mathcal{F})$$

and assume that

$$t(\xi) = (\eta_{a\beta}) \in Z^1(\mathcal{V}, \mathcal{F})$$

splits as a coboundary

$$(\eta_{a\beta}) = \delta((g_a)) \text{ with } (g_a) \in C^0(\mathcal{V}, \mathcal{F}).$$

We have to construct a cochain

$$f = (f_k) \in C^0(\mathcal{U}, \mathcal{F}) \text{ with } \xi = \delta f :$$

Consider an arbitrary but fixed index $k \in I$. Using the sheaf axioms for $\mathcal{F}$ we construct a section $f_k \in \mathcal{F}(U_k)$ by gluing local sections on open subsets of $U_k$; Consider the open covering of $U_k$ induced by $\mathcal{V}$

$$U_k \cap \mathcal{V}$$

and the corresponding cochain

$$(g_a|_{U_k \cap V_a}) \in C^0(U_k \cap \mathcal{V}, \mathcal{F}),$$

see Figure 3.1.
Fig. 3.1 Covering a given $U_k$ by $\mathcal{V}$, dashed $U_k \cap V_\alpha$

On $U_k \cap (V_\alpha \cap V_\beta)$ holds

$$g_\alpha - g_\beta = \eta_{\beta\alpha} = \xi_{\tau(\beta) \tau(\alpha)} = \xi_{\tau(\beta)k} + \xi_{\tau(\alpha)k} = \xi_{\tau(\beta)k} - \xi_{\tau(\alpha)k}$$

Hence for all $\alpha, \beta \in A$

$$g_\alpha + \xi_{\tau(\alpha)k} = g_\beta + \xi_{\tau(\beta)k}$$

The second sheaf axiom for $\mathcal{F}$ provides the section

$$f_k := (g_\alpha + \xi_{\tau(\alpha)k}) \in \mathcal{F}(U_k)$$

Then for each $\alpha \in A$ holds on $(U_i \cap U_j) \cap V_\alpha$

$$\xi_{ij} = \xi_{\tau(\alpha)} + \xi_{\tau(\alpha)j} = (g_\alpha - f_i) + (f_j - g_\alpha) = f_j - f_i$$

Hence the first sheaf axiom implies on $U_i \cap U_j$

$$\xi_{ij} = f_j - f_i$$

As a consequence
\[ \xi = \delta f. \]

\[ \square \]

For a topological space \( X \) and a presheaf \( \mathcal{F} \) on \( X \) the family of all open coverings of \( X \) and restriction maps

\[ H^q(\mathcal{U}, \mathcal{F}) \xrightarrow{t_{\mathcal{U}}^\mathcal{V}} H^q(\mathcal{V}, \mathcal{F}) \text{ for } \mathcal{V} < \mathcal{U} \]

is an inductive family: The refinement map

\[ id : \mathcal{U} \to \mathcal{U} \]

induces the identity \( t_{\mathcal{U}}^{\mathcal{U}} = id \), and for refinements

\[ \mathcal{W} < \mathcal{V} < \mathcal{U} \]

holds

\[ t_{\mathcal{W}}^{\mathcal{V}} \circ t_{\mathcal{V}}^{\mathcal{U}} = t_{\mathcal{W}}^{\mathcal{U}}. \]

Moreover, for each pair of open coverings \( \mathcal{U} = (U_i) \) and \( \mathcal{V} = (V_a) \) the covering

\[ \mathcal{U} \cap \mathcal{V} := (U_i \cap V_a)_{i,a} \]

satisfies

\[ (\mathcal{U} \cap \mathcal{V}) < \mathcal{U} \text{ and } (\mathcal{V} \cap \mathcal{V}) < \mathcal{V}. \]

**Definition 3.7 (Čech cohomology as inductive limit).** Consider a topological space \( X \) and a presheaf \( \mathcal{F} \) on \( X \). With respect to the inductive family of all open coverings of \( X \) and their refinement maps one defines for each \( q \in \mathbb{N} \) the \( q \)-th cohomology group of \( X \) with values in \( \mathcal{F} \) as the inductive limit

\[ H^q(X, \mathcal{F}) := \lim_{\rightarrow} H^q(\mathcal{U}, \mathcal{F}) \]

The cohomology from Definition 3.7, defined as inductive limit with respect to all open coverings \( \mathcal{U} \) of \( X \), is named Čech cohomology of \( X \). The corresponding cohomology groups are often written with the Čech accent like in \( \check{H}^q(X, \mathcal{F}) \), but we will not use this notation. For cohomology based on resolutions by flabby sheaves see [11, Chap. II, §4.3].

**Corollary 3.8 (Injection of the first cohomology).** Consider a topological space \( X \) and a sheaf \( \mathcal{F} \) on \( X \). For each open covering \( \mathcal{U} \) of \( X \) the canonical map

\[ H^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \]
is injective.

Proof. The proof follows from Proposition 3.6. □

3.2 The long exact cohomology sequence

Proposition 3.9 (Long exact cohomology sequence for presheaf morphisms and a fixed covering). Consider a topological space $X$ and an open covering $\mathcal{U}$ of $X$. Each short exact sequence of presheaf morphisms on $X$

$$0 \rightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \rightarrow 0$$

induces a sequence of morphisms, named connecting morphisms,

$$\delta^*: (\delta^*_q: H^q(\mathcal{U}, \mathcal{H}) \rightarrow H^{q+1}(\mathcal{U}, \mathcal{F}))_{q \in \mathbb{N}},$$

such that the following sequence, named long cohomology sequence, is exact

$$0 \rightarrow H^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\alpha_U} H^0(\mathcal{U}, \mathcal{G}) \xrightarrow{\beta_U} H^0(\mathcal{U}, \mathcal{H}) \xrightarrow{\delta^*_0}$$

$$\xrightarrow{\delta^*_1} H^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\alpha_1} H^1(\mathcal{U}, \mathcal{G}) \xrightarrow{\beta_1} H^1(\mathcal{U}, \mathcal{H}) \xrightarrow{\delta^*_1} ...$$

$$\xrightarrow{\delta^*_q} H^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\alpha_q} H^q(\mathcal{U}, \mathcal{G}) \xrightarrow{\beta_q} H^q(\mathcal{U}, \mathcal{H}) \xrightarrow{\delta^*_q} ...$$

Proof. For each open set $U \subset X$ the assumed exactness of the presheaf morphisms implies the exact sequence

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U) \rightarrow 0$$

Generalizing the result to cochain groups provides the following commutative diagram with exact rows for each $q \in \mathbb{N}$

$$\begin{array}{ccccccccc}
0 & \rightarrow & C^q(\mathcal{U}, \mathcal{F}) & \xrightarrow{\alpha_q} & C^q(\mathcal{U}, \mathcal{G}) & \xrightarrow{\beta_q} & C^q(\mathcal{U}, \mathcal{H}) & \rightarrow & 0 \\
\downarrow{\delta} & & \downarrow{\delta} & & \downarrow{\delta} & & \downarrow{\delta} & & \\
0 & \rightarrow & C^{q+1}(\mathcal{U}, \mathcal{F}) & \xrightarrow{\alpha_{q+1}} & C^{q+1}(\mathcal{U}, \mathcal{G}) & \xrightarrow{\beta_{q+1}} & C^{q+1}(\mathcal{U}, \mathcal{H}) & \rightarrow & 0 \\
\downarrow{\delta} & & \downarrow{\delta} & & \downarrow{\delta} & & \downarrow{\delta} & & \\
0 & \rightarrow & C^{q+2}(\mathcal{U}, \mathcal{F}) & \xrightarrow{\alpha_{q+2}} & C^{q+2}(\mathcal{U}, \mathcal{G}) & \xrightarrow{\beta_{q+2}} & C^{q+2}(\mathcal{U}, \mathcal{H}) & \rightarrow & 0
\end{array}$$

i) Definition of $\delta^*$: Define the map
3.2 The long exact cohomology sequence

\[ \delta^*_q : H^q(U, \mathcal{H}) \to H^{q+1}(U, \mathcal{F}) \]

by descending stairs backwards along the arrows highlighted in the above diagram to increase level \( q \) to level \( q + 1 \). Consider a given class \( \xi \in H^q(U, \mathcal{H}) \) represented by the cocycle \( c \in Z^q(U, \mathcal{H}) \). The surjectivity of

\[ \beta_q : C^q(U, \mathcal{G}) \to C^q(U, \mathcal{H}) \]

provides a cochain \( b \in C^q(U, \mathcal{G}) \) satisfying

\[ \beta_q(b) = c \]

Because

\[ 0 = \delta(c) = (\delta \circ \beta_q)(b) = \beta_{q+1}(\delta(b)) \]

the exactness of the middle row provides a cochain \( a \in C^{q+1}(U, \mathcal{F}) \) satisfying

\[ \alpha_{q+1}(a) = \delta(b) \]

Because

\[ \alpha_{q+2}(\delta(a)) = (\alpha_{q+2} \circ \delta)(a) = (\delta \circ \alpha_{q+1})(a) = (\delta \circ \delta)(b) = 0 \]

the injectivity of \( \alpha_{q+2} \) implies \( \delta(a) = 0 \), i.e.

\[ a \in Z^{q+1}(U, \mathcal{F}) \]

is a cocycle. Define now

\[ \delta^*_q(\xi) := [a] \in H^{q+1}(U, \mathcal{F}) \]

One checks that the class \( [a] \) does not depend on the choices made during its definition.

ii) Exactness at \( H^q(U, \mathcal{H}) \): We have to show the exactness of

\[ H^q(U, \mathcal{G}) \xrightarrow{\beta_q} H^q(U, \mathcal{H}) \xrightarrow{\delta^*_q} H^{q+1}(U, \mathcal{F}) \]

- **Claim ker \( \delta^*_q \subset \text{im } \beta_q \):** Consider a given class

\[ \xi = \in H^q(U, \mathcal{H}) \text{ with } \delta^*_q(\xi) = 0 \in H^{q+1}(U, \mathcal{F}) \]

We have to find a cocycle

\[ \eta \in Z^q(U, \mathcal{G}) \text{ with } \beta_q([\eta]) = \xi. \]

Choose a representing cocycle...
$c \in Z^q(\mathcal{U}, \mathcal{H})$ with $[c] = \xi$.

By construction from part i) there exists a cochain

$$b \in C^q(\mathcal{U}, \mathcal{G}) \text{ with } \beta_q(b) = c$$

and a cocycle

$$a \in Z^{q+1}(\mathcal{U}, \mathcal{F}) \text{ with } \alpha_{q+1}(a) = \delta(b) \in B^{q+1}(\mathcal{U}, \mathcal{G})$$

Then

$$\delta^*(\xi) = [a] = 0 \implies \exists a' \in C^q(\mathcal{U}, \mathcal{F}) \text{ with } \delta(a') = a \in B^{q+1}(\mathcal{U}, \mathcal{F})$$

The vanishing

$$\delta(b - \alpha_q(a')) = \delta(b) - (\delta \circ \alpha_q)(a') = \delta(b) - \alpha_{q+1}(a) = 0$$

implies

$$b - \alpha_q(a') \in Z^q(\mathcal{U}, \mathcal{G}).$$

Set

$$\eta := [b - \alpha_q(a')] \in H^q(\mathcal{U}, \mathcal{G})$$

Then

$$\beta_q(b - \alpha_q(a')) = \beta_q(b) - (\beta_q \circ \alpha_q)(a') = \beta_q(b) = c.$$ 

Hence

$$\beta_q([\eta]) = \xi.$$ 

• Claim im $\beta_q \subset \ker \delta^*_q$: Consider a class $\xi \in H^q(\mathcal{U}, \mathcal{H})$ represented as

$$\xi = [c] \text{ with } c \in Z^q(\mathcal{U}, \mathcal{H})$$

and assume the existence of a cocycle

$$b \in Z^q(\mathcal{U}, \mathcal{G}) \text{ with } \beta_q(b) = c.$$

By construction exists

$$a \in Z^{q+1}(\mathcal{U}, \mathcal{F}) \text{ with } \alpha_q(a) = \delta(b)$$

From

$$\delta(b) = 0 \in C^{q+1}(\mathcal{U}, \mathcal{G})$$

follows due to the injectivity of $\alpha_{q+1}$ the vanishing

$$a = 0 \in Z^{q+1}(\mathcal{U}, \mathcal{F}).$$

Hence
3.2 The long exact cohomology sequence

\[ \delta_q^*(\xi) = [a] = 0 \in H^{q+1}(\mathcal{U}, \mathcal{F}). \]

iii) Exactness at \( H^q(\mathcal{U}, \mathcal{G}) \): We have to show the exactness of

\[ H^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\alpha_q} H^q(\mathcal{U}, \mathcal{G}) \xrightarrow{\beta_q} H^q(\mathcal{U}, \mathcal{H}) \]

- **Claim ker \( \beta_q \subset im \alpha_q \):** Consider a class

\[ [b] \in H^q(\mathcal{U}, \mathcal{G}) \] with \( \beta_q([b]) = 0 \in H^q(\mathcal{U}, \mathcal{H}) \)

Then \( \beta_q(b) \in B^q(\mathcal{U}, \mathcal{H}) \), i.e. there exists

\[ c' \in C^{q-1}(\mathcal{U}, \mathcal{H}) \] with \( \delta(c') = \beta_q(b) \)

The surjectivity of \( \beta_q \) provides a cochain

\[ b' \in C^{q-1}(\mathcal{U}, \mathcal{G}) \] with \( \beta_{q-1}(b') = c' \)

The commutativity

\[ \delta \circ \beta_{q-1} = \beta_q \circ \delta \]

implies

\[ \beta_q(b - \delta(b')) = 0. \]

The exactness at \( C^q(\mathcal{U}, \mathcal{G}) \) provides an element

\[ a \in C^q(\mathcal{U}, \mathcal{F}) \] with \( \alpha_q(a) = b - \delta(b') \)

Using the commutativity

\[ \alpha_{q+1} \circ \delta = \delta \circ \alpha_q \]

and the injectivity of \( \alpha_{q+1} \) shows \( \delta(a) = 0 \), hence \( a \in Z^q(\mathcal{U}, \mathcal{F}) \) and

\[ \alpha_q([a]) = [a - \delta(b')] = [b] \]

- **Claim im \( \alpha_q \subset ker \beta_q \):** The claim follows from the cochain property \( \beta_q \circ \alpha_q = 0 \).

iv) Exactness at \( H^q(\mathcal{U}, \mathcal{F}) \): We have to show the exactness of

\[ H^{q-1}(\mathcal{U}, \mathcal{H}) \xrightarrow{\delta^{-1}_q} H^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\alpha_q} H^q(\mathcal{U}, \mathcal{G}) \]

- **Claim ker \( \alpha_q \subset im \delta^{-1}_q \):** Consider a class \([a] \in H^q(\mathcal{U}, \mathcal{F}) \) with

\[ \alpha_q([a]) = [\alpha_q(a)] = 0 \in H^q(\mathcal{U}, \mathcal{G}) \]

Then the cocycle \( \alpha_q(a) \in Z^q(\mathcal{U}, \mathcal{G}) \) is a coboundary, i.e.

\[ \alpha_q(a) = \delta(b) \in B^q(\mathcal{U}, \mathcal{G}) \]
for a suitable $b \in C^{q-1}(\mathcal{U}, G)$. As a consequence

$$\delta (\beta_{q-1}(b)) = (\delta \circ \beta_{q-1})(b) = (\beta_{q} \circ \delta)(b) = \beta_{q} (\alpha_{q} \circ \beta_{q})(a) = 0,$$

which implies $\beta_{q-1}(b) \in Z^{q-1}(\mathcal{U}, \mathcal{H})$. By construction

$$[a] = \delta_{q-1}^*(\beta_{q-1}(b))$$

• **Claim im $\delta_{q-1}^* \subset \ker \alpha_{q}$:** Consider a class

$$\delta_{q-1}^*(\xi) = [a] \in H^q(\mathcal{U}, \mathcal{H})$$

satisfying

$$\alpha_{q}(a) = \delta(b)$$

for a suitable $b \in C^{q-1}(\mathcal{U}, G)$

Then

$$\alpha_{q}([a]) = [\alpha_{q}(a)] = [\delta(b)] = 0 \in H^q(\mathcal{U}, \mathcal{H})$$

\[ \square \]

**Corollary 3.10 (Long exact cohomology sequence for presheaf morphisms).**

Each short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of presheaf morphisms on a topological space $X$ induces a long exact cohomology sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \xrightarrow{\delta_0^*} H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \xrightarrow{\delta_1^*} \cdots$$

$$\cdots \to H^{q+1}(X, \mathcal{F}) \to H^{q+1}(X, \mathcal{G}) \to H^{q+1}(X, \mathcal{H}) \xrightarrow{\delta_{q+1}^*} \cdots$$

**Proof.** One applies the functor inductive limit to the long exact sequence from Proposition 3.9. The functor is exact, see [1, Chap. 2, Ex. 19]. \[ \square \]

Lemma 3.11 prepares the proof that on a paracompact space $X$ the long exact cohomology sequence from Corollary 3.10 also originates from an exact sequence of sheaf morphisms - not only presheaf morphisms.

**Lemma 3.11 (Sheafification and cohomology).** On a paracompact topological space $X$ a presheaf $\mathcal{F}$ and its sheafification $\hat{\mathcal{F}}$ have the same cohomology, i.e. the canonical map

$$\mathcal{F} \to \hat{\mathcal{F}}$$

induces for all $q \in \mathbb{N}$ isomorphisms

$$H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X, \hat{\mathcal{F}}).$$
3.2 The long exact cohomology sequence

Proof. See [23, Satz 33.7], the book uses the term “Garbendatum” which means “presheaf”. The proof relies on the following properties:

- For a presheaf $\mathcal{G}$ on a paracompact topological space $X$ with sheafification $\mathcal{G} = 0$ all cohomology groups vanish, i.e.
  
  $$H^q(X, \mathcal{G}) = 0 \text{ for all } q \in \mathbb{N},$$

  see [23, Satz 33.6].

- Sheafification is an exact functor on short exact sequences of presheaf morphisms. □

**Theorem 3.12 (Long exact cohomology sequence for sheaf morphisms).** Consider a paracompact topological space $X$. Each short exact sequence of sheaf morphisms on $X$

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$$

induces a long exact cohomology sequence

$$0 \to H^0(X, \mathcal{F}) \xrightarrow{\delta_0^*} H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \xrightarrow{\delta_0^*}$$

$$\xrightarrow{\delta^*_1} H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \xrightarrow{\delta^*_1} \cdots$$

$$\cdots \xrightarrow{\delta^*_q} H^{q+1}(X, \mathcal{F}) \to H^{q+1}(X, \mathcal{G}) \to H^{q+1}(X, \mathcal{H}) \xrightarrow{\delta^*_q} \cdots$$

Note that the given short exact sequence of sheaf morphisms is not necessarily exact when considered sequence of presheaf morphisms. Hence Theorem 3.12 is not a particular case of Corollary 3.10.

Proof. We consider the quotient presheaf $\mathcal{Q}$ on $X$

$$\mathcal{Q}(U) := \text{coker} [\mathcal{F}(U) \xrightarrow{\alpha^*} \mathcal{G}(U)] = \mathcal{G}(U) / \alpha^*(\mathcal{F}(U)), \ U \subset X \text{ open},$$

with the induced restrictions, and obtain the short exact sequence of presheaves on $X$

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$$

Corollary 3.10 provides the corresponding long exact cohomology sequence which contains the cohomology groups $H^q(X, \mathcal{Q})$. Because

$$\mathcal{H} = \mathcal{Q}$$

Lemma 3.11 concludes
\[H^q(X, \mathcal{H}) = H^q(X, \hat{\mathcal{L}}) \simeq H^q(X, \mathcal{L}),\]

which finishes the proof. \(\square\)

### 3.3 Acyclic sheaves and applications

**Definition 3.13 (Acyclic sheaf, flabby sheaf).** Consider a topological space \(X\).

1. A sheaf \(\mathcal{F}\) on \(X\) is **acyclic** if for all \(q \geq 1\)
   \[H^q(X, \mathcal{F}) = 0.\]

2. A sheaf \(\mathcal{F}\) on \(X\) is **flabby** or **flasque** (deutsch: welk) if for each open set \(U \subset X\) the canonical restriction
   \[\mathcal{F}(X) \rightarrow \mathcal{F}(U), \ f \mapsto f|_U,\]
   is surjective, i.e. if each section over an open subset extends to all of \(X\).

**Proposition 3.14 (Flabby sheaves are acyclic).** Each flabby sheaf \(\mathcal{F}\) on a topological space \(X\) satisfies for each open covering \(\mathcal{U} = (U_i)_{i \in I}\) of \(X\) and for all \(q \geq 1\)
\[H^q(\mathcal{U}, \mathcal{F}) = 0.\]

In particular \(\mathcal{F}\) is acyclic.

**Proof.** Assume \(q \geq 1\) and consider a cocycle
\[\xi = (\xi_{i_0, \ldots, i_q})_{(i_0, \ldots, i_q) \in I_{q+1}} \in Z^q(\mathcal{U}, \mathcal{F}).\]

i) **Splitting over small open sets:** If \(i^* \in I\) and \(Y \subset U_{i^*}\) open, then \(\xi\) splits over \(Y\):
Define
\[\eta := (\eta_{i_0, \ldots, i_{q-1}})_{(i_0, \ldots, i_{q-1})} \in C^{q-1}(\mathcal{U} \cap Y, \mathcal{F})\]
as the family of sections
\[\eta_{i_0, \ldots, i_{q-1}} := \xi_{i^*, i_0, \ldots, i_{q-1}}|_Y,\]
see Figure 3.2.
Then
\[(\delta \eta)_{i_0, \ldots, i_q} = \sum_{\alpha=0}^{q} (-1)^{\alpha} \cdot \xi_{i_0^{\alpha}, \ldots, i_q^{\alpha}}\]
The cocycle condition with respect to $U$
\[0 = (\delta \xi)_{i_0, \ldots, i_q} = \xi_{i_0, \ldots, i_q} - \sum_{\alpha=0}^{q} (-1)^{\alpha} \cdot \xi_{i_0^{\alpha}, \ldots, i_q^{\alpha}}\]
implies
\[\sum_{\alpha=0}^{q} (-1)^{\alpha} \cdot \xi_{i_0^{\alpha}, \ldots, i_q^{\alpha}} = \xi_{i_0, \ldots, i_q}\]
hence
\[\delta \eta = \xi|_Y\]

ii) Maximal splittings due to Zorn’s lemma: Consider the set of local splittings
\[\text{Split}(\xi) := \{(Y, \eta) : Y \subset X \text{ open}, \eta \in C^{q-1}(\mathcal{U} \cap Y, \mathcal{F}) \text{ with } \delta \eta = \xi|_Y\}\]

Due to part i) the set $\text{Split}(\xi)$ is not empty. The set is partially ordered by
\[(Y, \eta) \leq (Y', \eta') : \iff Y \subset Y' \text{ and } \eta'|_Y = \eta.\]

For each totally ordered subset $\tilde{S} \subset \text{Split}(\xi)$ consider the open set
\[\tilde{Y} := \bigcup\{Y \subset X : \exists \eta_Y \text{ with } (Y, \eta_Y) \in \tilde{S}\}\]

The definition of the partial order and the second sheaf axiom provide a cochain on $\tilde{Y}$
\[\tilde{\eta} \in C^{q-1}(\mathcal{U} \cap \tilde{Y}, \mathcal{F})\]
satisfying for all $Y$ with $(Y, \eta_Y) \in \tilde{S}$

$$\tilde{\eta}|Y = \eta_Y \text{ and } \delta \tilde{\eta}|Y = \delta (\tilde{\eta})|Y = \delta \eta_Y = \xi|Y$$

The first sheaf axiom ensures

$$\delta \tilde{\eta} = \xi|\Y,$$

which shows that $\tilde{S}$ has the upper bound $(\Y, \tilde{\eta})$. Zorn’s lemma provides a maximal element

$$\tag{\text{Zorn’s lemma}} (Z, \eta) \in \text{Split} (\xi).$$

iii) Global splitting: We now apply the assumption that the sheaf $\mathcal{F}$ is flabby. We give an indirect proof that a maximal element $(Z, \eta) \in \text{Split} (\xi)$ from part ii) provides a splitting of $\xi$ on $X$.

If $Z \neq X$ then exists an index $i_0 \in I$ with $U_{i_0} \subsetneq Z$. Set

$$Y' := Z \cup U_{i_0}$$

Part i) provides a cochain $a \in C^{q-1}(\mathcal{U} \cap U_{i_0}, \mathcal{F})$ satisfying

$$\delta a = \xi|U_{i_0}$$

As a consequence on $Z \cup U_{i_0} \subset U_{i_0}$ holds $\delta (\eta - a) = 0$. We now consider the cocycle

$$\eta - a \in Z^{q-1}(\mathcal{U} \cap (Z \cap U_{i_0}), \mathcal{F}).$$

**Case $q \geq 2$:** The cocycle

$$\eta - a \in Z^{q-1}(\mathcal{U} \cap (Z \cap U_{i_0}), \mathcal{F})$$

does not extend to the cover of $Y'$

$$\mathcal{U} \cap Y'$$

necessarily as a cocycle. Because extending sections does not preserve the cocycle condition. Therefore we extend a suitable cochain from $C^{q-2}(\mathcal{U} \cap (Z \cap U_{i_0}), \mathcal{F})$.

Part i) provides an element $\tilde{b} \in C^{q-2}(\mathcal{U} \cap (Z \cap U_{i_0}), \mathcal{F})$ satisfying

$$\delta \tilde{b} = \eta - a$$

Because $(U_{i_0} \cap Z) \subset U_{i_0}$ the flabyness of $\mathcal{F}$ allows to extend $\tilde{b}$ to an element

$$b \in C^{q-2}(\mathcal{U} \cap U_{i_0}, \mathcal{F}) \text{ with } b|U_{i_0} \cap Z = \tilde{b}$$

We define an element $\eta' \in C^{q-1}(\mathcal{U} \cap Y', \mathcal{F})$ as
3.3 Acyclic sheaves and applications

\[ \eta' := \begin{cases} 
\eta & \text{on } Z \\
\eta + \delta b & \text{on } U_0 
\end{cases} \]

It satisfies

\[ \eta'|_Z = \eta \text{ and } \delta \eta' = \xi|_Y \text{ because } \delta(\eta'|_{U_0}) = \delta a|_{U_0} = \xi|_{U_0} \]

- Case \( q = 1 \): By assumption

\[ \eta - a \in Z^0(\mathcal{U} \cap (Z \cap U_0), \mathcal{F}) = \mathcal{F}(Z \cap U_0) \]

The flabbyness of \( \mathcal{F} \) extends the section \( \eta - a \) to a section

\[ b \in \mathcal{F}(U_0) \text{ with } b|_{Z \cap U_0} = \eta - a \]

Using

\[ b \in Z^0(\mathcal{U} \cap U_0, \mathcal{F}) \subset C^0(\mathcal{U} \cap U_0, \mathcal{F}) \]

we define the cochain \( \eta' \in C^1(\mathcal{U} \cap Y', \mathcal{F}) \) as

\[ \eta' := \begin{cases} 
\eta & \text{on } Z \\
\eta + b & \text{on } U_0 
\end{cases} \]

obtaining

\[ \eta'|_Z = \eta \text{ and } \delta \eta' = \xi|_Y \text{ because } \delta(a + b)|_{Z \cap U_0} = \delta \eta|_{Z \cap U_0} = \xi|_{Z \cap U_0} \]

In both cases we obtain a proper extension \((Y', \eta')\) satisfying

\[ (Y', \eta') \geq (Z, \eta) \text{ but } Y' \supset Z, \ Y' \neq Z, \]

a contradiction to the maximality of \((Z, \eta)\). \( \square \)

**Theorem 3.15 (Canonical flabby resolution).** Each sheaf \( \mathcal{F} \) on a topological space \( X \) has a canonical resolution by flabby sheaves

\[ 0 \to \mathcal{F} \to \mathcal{W}^0 \to \mathcal{W}^1 \to ... \]

**Proof.** We construct the resolution from left step by step by induction.

i) Induction start \( \mathcal{W}^0 \): Set

\[ \mathcal{W}^0 := w(\mathcal{F}), \]

the sheaf from the proof of Theorem 2.16. The sheaf \( w(\mathcal{F}) \) is flabby because each section \( s_U \in w(\mathcal{F})(U) \) extends by zero to a global section \( s \in w(\mathcal{F})(X) \). The maps

\[ \mathcal{F} \to w(\mathcal{F}), \phi \mapsto (\phi_s)_{s \in U}, \ U \subset X \text{ open}, \]
define a morphism of sheaves $\mathcal{F} \to \mathcal{W}(\mathcal{F})$ which is injective. Hence the sequence of sheaf morphisms

$$0 \to \mathcal{F} \to \mathcal{W}$$

is exact.

ii) Induction step $k \mapsto k + 1$: Assume an exact sequence of length $k$ of sheaf morphisms with flabby sheaves $\mathcal{W}^i$, $i = 0, \ldots, k$

$$0 \to \mathcal{F} \to \mathcal{W}^0 \to \ldots \to \mathcal{W}^{k-1} \xrightarrow{\alpha_{k-1}} \mathcal{W}^k$$

The sheaf $coker \alpha_{k-1}$, the sheafification of the corresponding presheaf according to Theorem 2.16, fits into the canonical exact sequence of sheaf morphisms

$$\mathcal{W}^{k-1} \xrightarrow{\alpha_{k-1}} \mathcal{W}^k \xrightarrow{\pi_k} coker \alpha_{k-1} \to 0$$

Set

$$\mathcal{W}^{k+1} := w(coker \alpha_{k-1})$$

and consider the canonical injection, a sheaf morphism, $j_k : coker \alpha_{k-1} \hookrightarrow \mathcal{W}^{k+1}$

Then the sequence

$$\mathcal{W}^{k-1} \xrightarrow{\alpha_{k-1}} \mathcal{W}^k \quad \mathcal{W}^{k+1} := j_k \circ \pi_k \xrightarrow{\alpha_k} \mathcal{W}^{k+1}$$

is exact and extends the assumed exact sequence of flabby sheaves. □

**Proposition 3.16 (Abstract de Rahm theorem).** Consider a paracompact space $X$ and assume that the sheaf $\mathcal{F}$ on $X$ has a resolution by acyclic sheaves $\mathcal{G}^j$, $j \in \mathbb{N}$,

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \ldots$$

Then the cohomology of $\mathcal{F}$ can be computed as

$$H^q(X, \mathcal{F}) \simeq \ker [\mathcal{G}^q(X) \to \mathcal{G}^{q+1}(X)] \simeq \text{im} [\mathcal{G}^{q-1}(X) \to \mathcal{G}^q(X)], \quad q \in \mathbb{N}.$$  

Here $\mathcal{G}^{-1} := 0$.

**Proof.** i) Case $q = 0$: The resolution implies

$$\mathcal{F} \simeq \ker [\mathcal{G}^0 \to \mathcal{G}^1].$$

The left exactness of the functor $\Gamma(X, -)$ with
for a sheaf $\mathcal{G}$ on $X$ ensures the exactness of

$$0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{G}^0(X) \rightarrow \mathcal{G}^1(X)$$

which implies

$$H^0(X, \mathcal{F}) = \mathcal{F}(X) \simeq \ker [\mathcal{G}^0(X) \rightarrow \mathcal{G}^1(X)]$$

and proves the case $q = 0$.

ii) Case $q \geq 1$: The idea of the proof is to split the long exact sequence of sheaf morphisms from the resolution into a family of short exact sequences. For each short exact sequence one considers the long exact cohomology sequence, and takes into account that the sheaves $\mathcal{G}^n$ are acyclic. For $n \in \mathbb{N}$ set

$$\mathcal{K}^n := \ker [\mathcal{G}^n \rightarrow \mathcal{G}^{n+1}]$$

and consider the canonical short exact sequence of sheaf morphisms

$$0 \rightarrow \mathcal{K}^n \rightarrow \mathcal{G}^n \rightarrow \mathcal{K}^{n+1} \rightarrow 0.$$ 

Theorem 3.12 provides for each $q \geq 1$ the following section of its long exact sequence

$$0 = H^q(X, \mathcal{G}^n) \rightarrow H^q(X, \mathcal{K}^{n+1}) \xrightarrow{\delta_q^*} H^{q+1}(X, \mathcal{K}^n) \rightarrow H^{q+1}(X, \mathcal{G}^n) = 0.$$ 

Hence the connection morphisms are isomorphisms

$$\delta_q^*: H^q(X, \mathcal{K}^{n+1}) \simeq H^{q+1}(X, \mathcal{K}^n),$$

which implies successively for $q \geq 1$

$$H^q(X, \mathcal{F}) = H^q(X, \mathcal{K}^0) \simeq H^{q-1}(X, \mathcal{K}^1) \simeq H^1(X, \mathcal{K}^{q-1}),$$

and for

$$0 \rightarrow \mathcal{K}^{q-1} \rightarrow \mathcal{G}^{q-1} \rightarrow \mathcal{K}^q \rightarrow 0$$

the exact sequence

$$0 = H^q(X, \mathcal{G}^{q-1}) \rightarrow H^q(X, \mathcal{K}^q) \xrightarrow{\delta_q^*} H^{q+1}(X, \mathcal{K}^{q-1}) \rightarrow H^{q+1}(X, \mathcal{G}^{q-1}) = 0$$

The last exact sequence implies

$$H^1(X, \mathcal{K}^{q-1}) \simeq \frac{H^0(X, \mathcal{K}^q)}{\text{im} [H^0(X, \mathcal{G}^{q-1}) \rightarrow H^0(X, \mathcal{K}^q)]}$$

Inserting the result into the second last isomorphy
Here we use the apparent equalities

\[ H^0(X, \mathcal{K}_q) = \ker \left[ H^0(X, \mathcal{G}_q) \to H^0(X, \mathcal{G}_{q+1}) \right] \]

and

\[ \text{im} \left[ \mathcal{G}_q^{-1}(X) \to \mathcal{K}_q(X) \right] = \text{im} \left[ \mathcal{G}_q^{-1}(X) \to \mathcal{G}_q(X) \right] \]

\[ \vdash \]

**Corollary 3.17 (Cohomology via flabby resolutions).** Consider a paracompact topological space \( X \) and a sheaf \( \mathcal{F} \) on \( X \). Each resolution of \( \mathcal{F} \) by flabby sheaves

\[ 0 \to \mathcal{F} \to \mathcal{K}^0 \to \mathcal{K}^1 \to \ldots \]

induces for each \( q \in \mathbb{N} \) an isomorphism

\[ \alpha_q : \frac{\ker \left[ \mathcal{K}_q(X) \to \mathcal{K}_{q+1}(X) \right]}{\text{im} \left[ \mathcal{K}_q^{-1}(X) \to \mathcal{K}_q(X) \right]} \cong H^q(X, \mathcal{F}), \mathcal{K}_q^{-1} = 0. \]

**Proof.** The proof follows from Proposition 3.16, because flabby sheaves are acyclic due to Proposition 3.14. \( \square \)

**Lemma 3.18 (Navigation in a double complex).** Consider a double complex of morphisms of Abelian groups
Assume: With the possible exception of the first column and the first row all other columns and rows are exact. Then the cohomology of the first column is isomorphic to the cohomology of the first row, i.e. for all \( q \in \mathbb{N} \)

\[
H^q_I := \ker \left[ C^q_I \to C^{q+1}_I \right] / \text{im} \left[ C^{q-1}_I \to C^q_I \right] \cong H^q_{II} := \ker \left[ C^q_{II} \to C^{q+1}_{II} \right] / \text{im} \left[ C^{q-1}_{II} \to C^q_{II} \right]
\]

**Proof.** All squares of a double complex are commutative. The proof of the lemma is by stair climbing from the first column to the first row. The above diagram highlights the path on the level of cochains to define the morphism

\[
H^q_I \to H^q_{II}
\]

for \( q = 2 \), and also to define its inverse. \( \square \)

Next we deal with the question: Under which assumptions can the cohomology of a topological space be already computed by considering only a single distinguished covering?

**Theorem 3.19 (Leray’s theorem).** Consider a metrizable topological space \( X \) with an open covering \( \mathcal{U} = (U_i)_{i \in I} \) and a sheaf \( \mathcal{F} \) on \( X \). Assume: On each intersection

\[
U_{i_0, \ldots, i_q} := U_{i_0} \cap \ldots \cap U_{i_q}, \ q \in \mathbb{N}, \ i_0, \ldots, i_q \in I^{q+1},
\]
the restricted sheaf

\[ \mathcal{F}|_{U_{i_1 \cdots i_k}} \]

is acyclic. Then for all \( q \in \mathbb{N} \) the canonical maps

\[ \rho_q : H^q(\mathcal{U}, \mathcal{F}) \to H^q(X, \mathcal{F}) \]

are isomorphisms.

Theorem 3.19 makes the assumption that \( X \) is metrizable. The assumption ensures that the open sets of the covering \( \mathcal{U} \) and their finite intersections are also metrizable and therefore paracompact. Note: In general paracompactness is not inherited by open subspaces. The assumption of metrizability is satisfied in the context of complex analysis, because each complex manifold is a regular space and has to satisfy the axiom of second countability, see also Remark 2.3.

**Proof.** i) *Construction of a double complex:* Theorem 3.15 provides a flabby resolution of \( \mathcal{F} \)

\[ 0 \to \mathcal{F} \to \mathcal{U}^0 \to \mathcal{U}^1 \to \ldots \]

The corresponding morphisms extend to a double complex of global sections and cochain groups: The horizontal maps derive from the morphisms of the flabby resolution, the vertical maps are the coboundary maps.

![Double Complex Diagram](image)

Because all sheaves \( \mathcal{U}^j, j \in \mathbb{N} \), are flabby, Proposition 3.14 ensures that with the possible exception of the first column all other columns are exact, while the first column with the cochain groups of \( \mathcal{F} \) is a complex.
ii) Exactness of the rows with index $\geq 2$: For given $q \in \mathbb{N}$ and tuple $(i_0, ..., i_q) \in I^{q+1}$ consider the intersection of the covering sets

$$U_i := U_{i_0} \cap ... \cap U_{i_q+1}$$

The open set $U_i$ is metrizable because $X$ is assumed to be metrizable. In particular $U_i$ is paracompact. The restriction of the original resolution to $U_i$

$$0 \to \mathcal{F}|U_i \to \mathcal{W}^0|U_i \to \mathcal{W}^1|U_i \to ...$$

is a flabby resolution of the restricted sheaf $\mathcal{F}|U_i$. Proposition 3.16 implies for $j \in \mathbb{N}$

$$H^j(U_i, \mathcal{F}) = \ker \left[ \mathcal{W}^j(U_i) \to \mathcal{W}^{j+1}(U_i) \right] / \im \left[ \mathcal{W}^{j-1}(U_i) \to \mathcal{W}^j(U_i) \right], \mathcal{W}^{-1} := 0$$

Due to the acyclicity of $\mathcal{F}|U_i$ holds for each $j \geq 1$

$$H^j(U_i, \mathcal{F}) = 0$$

Hence the sequence

$$0 \to \mathcal{F}(U_i) \to \mathcal{W}^0(U_i) \to \mathcal{W}^1(U_i) \to ...$$

is exact. Varying $q \in \mathbb{N}$ and the tuple $(i_0, ..., i_q) \in I^{q+1}$ implies the exactness of the complex of cochain groups of the corresponding row

$$0 \to C^q(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{W}^0) \to C^q(\mathcal{U}, \mathcal{W}^1) \to ...$$

iii) Navigation in the double complex: Corollary 3.17 provides for each $q \in \mathbb{N}$ an isomorphism

$$\alpha_q : \ker \left[ \mathcal{W}^q(X) \to \mathcal{W}^{q+1}(X) \right] / \im \left[ \mathcal{W}^{q-1}(X) \to \mathcal{W}^q(X) \right] \cong H^q(X, \mathcal{F})$$

Lemma 3.18 applies to the double complex from part i) and provides isomorphisms

$$\phi_q : H^q(\mathcal{U}, \mathcal{F}) \cong \ker \left[ \mathcal{W}^q(X) \to \mathcal{W}^{q+1}(X) \right] / \im \left[ \mathcal{W}^{q-1}(X) \to \mathcal{W}^q(X) \right], q \in \mathbb{N},$$

such that the following diagram commutes

$$\begin{array}{ccc}
H^q(\mathcal{U}, \mathcal{F}) & \xrightarrow{\rho_q} & H^q(X, \mathcal{F}) \\
\downarrow{\phi_q} & & \downarrow{\alpha_q} \\
\ker \left[ \mathcal{W}^q(X) \to \mathcal{W}^{q+1}(X) \right] / \im \left[ \mathcal{W}^{q-1}(X) \to \mathcal{W}^q(X) \right] & & \\
\end{array}$$
Because $\alpha_q$ and $\phi_q$ are isomorphisms, also $\rho_q$ is an isomorphism. □

**Theorem 3.20 (Acyclicity of the smooth structure sheaf).** The structure sheaf $\mathcal{E}$ of a smooth manifold $(X, \mathcal{E})$ is acyclic with respect to each open covering

$$\mathcal{U} = (U_i)_{i \in I}$$

of $X$, i.e.

$$H^q(\mathcal{U}, \mathcal{E}) = 0, \; q \geq 1.$$ In particular

$$H^q(X, \mathcal{E}) = 0, \; q \geq 1.$$\n
**Proof.** i) **Partition of unity:** Because $X$ is paracompact there exists a partition of unity $(\eta_i)_{i \in I}$ subordinate to $\mathcal{U}$, i.e. the family of smooth functions

$$\eta_i \in \mathcal{E}(X), \; i \in I,$$

satisfies

- **Range:** For each $i \in I$ \(\eta_i(X) \subset [0, 1] \subset \mathbb{R}\)
- **Adapted to $\mathcal{U}$:** $\text{supp } \eta_i \subset U_i$, $i \in I$.
- **Locally finite:** The family $(\text{supp } \eta_i)_{i \in I}$ is locally finite.
- **Decomposing the identity:**

$$\sum_{i \in I} \eta_i = 1$$

The sum is well-defined because the family $(\text{supp } \eta_i)_{i \in I}$ of supports is locally finite.

ii) **Cocycles are coboundaries:** Consider a given cocycle

$$\xi = (\xi_{i_0, \ldots, i_q}) \in Z^q(\mathcal{U}, \mathcal{E})$$

Set

$$U_{i_0, \ldots, i_q} := U_{i_0} \cap \ldots \cap U_{i_q}$$

with last index $i_{q-1}$. For each $i \in I$ the function

$$\eta_i \cdot \xi_{i_0, \ldots, i_{q-1}} \in \mathcal{E}(U_i \cap U_{i_0, \ldots, i_{q-1}})$$

extends by zero to a smooth function on $U_{i_0, \ldots, i_{q-1}}$, see Figure 3.3. Here one uses $\text{supp } \eta_i \subset \subset U_i$. 


3.3 Acyclic sheaves and applications

Varying $i \in I$ provides a covering of $U_{i_0 \ldots i_{q-1}}$. The extended functions define the $q-1$-cochain

$$
\zeta = (\xi_{i_0 \ldots i_{q-1}})_{(i_0 \ldots i_{q-1}) \in I^q} \in C^{q-1}(\mathcal{U}, \mathcal{E})
$$

with

$$
\xi_{i_0 \ldots i_{q-1}} := \sum_{i \in I} \eta_i \cdot \xi_{i_0 \ldots i_{q-1}}
$$

Note that the last sum is well-defined.

Claim: $\delta \zeta = \xi$. Compute

$$
(\delta \zeta)_{i_0 \ldots i_q} := \sum_{k=0}^q (-1)^k \cdot \xi_{i_0 \ldots i_k \ldots i_q} = \sum_{k=0}^q (-1)^k \cdot \left( \sum_{i \in I} \eta_i \cdot \xi_{i_0 \ldots i_k \ldots i_q} \right) = \sum_{i \in I} \eta_i \cdot \left( \sum_{k=0}^q (-1)^k \cdot \xi_{i_0 \ldots i_k \ldots i_q} \right)
$$

The cocycle condition $\delta \xi = 0$ implies
\[0 = \xi_{i_0 \ldots i_q} - \sum_{k=0}^{q} (-1)^k \xi_{i_{0 \ldots \hat{i}_k \ldots i_q}}\]

Hence
\[(\delta \zeta)_{i_0 \ldots i_q} = \sum_{i \in I} (\eta_i \cdot \xi_{i_0 \ldots i_q}) = \xi_{i_0 \ldots i_q} \sum_{i \in I} \eta_i = \xi_{i_0 \ldots i_q}\]

\[\square\]

**Proposition 3.21 (Sheaf of smooth differential forms).** On a smooth manifold \(X\) the sheaves \(E^{p,q}\) of smooth \((p,q)\)-forms are acyclic with respect to each open covering \(U\) of \(X\).

The proof is analogous to the proof of Theorem 3.20.

The most prominent link between a complex manifold and its underlying smooth structure is Dolbeault’s theorem, Theorem 3.22.

**Theorem 3.22 (Dolbeault’s theorem).** On a complex manifold \(X\) for all indices \(p,q \in \mathbb{N}\)

\[H^q(X, \Omega^p) \cong \ker \left[ \mathcal{E}^{p,q}(X) \xrightarrow{d''} \mathcal{E}^{p,q+1}(X) \right] / \operatorname{im} \left[ \mathcal{E}^{p,q-1}(X) \xrightarrow{d''} \mathcal{E}^{p,q}(X) \right]\]

**Proof.** The sheaves \(\mathcal{E}^{p,q}\) are acyclic due to Proposition 3.21. Dolbeault’s lemma, Theorem 1.27, implies: For each \(p \in \mathbb{N}\) the sequence of sheaf morphisms

\[0 \to \Omega^{p,0} \to \mathcal{E}^{p,0} \xrightarrow{d''} \mathcal{E}^{p,1} \xrightarrow{d''} \ldots \xrightarrow{d''} \mathcal{E}^{p,n} \to 0, \quad n := \dim X,\]

is exact. Hence the claim follows from the abstract de Rahm theorem, Proposition 3.16.

\[\square\]

**Corollary 3.23 (Acyclicity of the sheaf of holomorphic differential forms on a polydisc).** The structure sheaf \(\mathcal{O}\) and all sheaves \(\Omega^p\) on an open polydisc \(\Delta \subset \mathbb{C}^n\) are acyclic.

**Proof.** The Corollary follows from Theorem 3.22 and Dolbeault’s lemma, Theorem 1.27.

\[\square\]

**Corollary 3.24 (Solution of the Cousin problems).** Consider a complex manifold \(X\).

1. If
\[H^1(X, \mathcal{O}) = 0\]
then each additive Cousin problem on \(X\) is solvable.
2. If
\[ H^1(X, \mathcal{O}) = 0 \text{ and } H^2(X, \mathbb{Z}) = 0 \]
then each multiplicative Cousin problem on \( X \) is solvable.

3. On the polydisc \( \Delta \subset \mathbb{C}^n \) each additive and each multiplicative Cousin problem is solvable.

**Proof.**

1. Due to Definition 2.29: An additive Cousin distribution
\[ c := (h_i)_{i \in I} \]
with respect to an open covering \( \mathcal{U} = (U_i)_{i \in I} \) is a cochain \( h \in C^0(\mathcal{U}, \mathcal{M}) \). By assumption its coboundary is holomorphic, i.e.
\[ \delta h \in Z^1(\mathcal{U}, \mathcal{O}) \].

The canonical map
\[ H^1(\mathcal{U}, \mathcal{O}) \to H^1(X, \mathcal{O}) \]
is injective due to Proposition 3.6. Hence
\[ H^1(X, \mathcal{O}) = 0 \implies H^1(\mathcal{U}, \mathcal{O}) = 0 \].

Proposition 3.3 implies that the Cousin distribution \( c \) is solvable.

2. The exponential sequence on \( X \), see Proposition 2.24,
\[ 0 \to \mathbb{Z} \to \mathcal{O} \overset{\epsilon}{\to} \mathcal{O}^* \to 0 \]
provides the following segment of the long exact cohomology sequence
\[ H^1(X, \mathcal{O}) \overset{\epsilon}{\to} H^1(X, \mathcal{O}^*) \overset{\delta^1}{\to} H^2(X, \mathbb{Z}) \]

If the two cohomology groups at the left and right end vanish then
\[ H^1(X, \mathcal{O}^*) = 0. \]

Analogous to part 1) Proposition 3.6 implies for each open covering \( \mathcal{U} \) of \( X \)
\[ H^1(\mathcal{U}, \mathcal{O}^*) = 0. \]

Proposition 3.3 implies that each multiplicative Cousin distribution on \( X \) is solvable.

3. Corollary 3.23 states
\[ H^1(\Delta, \mathcal{O}) = 0 \]
Because \( \Delta \) is contractible also \( H^2(\Delta, \mathbb{Z}) = 0 \). Hence the claim follows from part 1) and 2). \( \square \)
Corollary 3.24 gives a second proof for the solvability of the Cousin problems in polydiscs. Theorem 2.36 gave a first proof.

Applying cohomology theory to the exponential sequence provides a logarithm on simply connected domains \( X \subset \mathbb{C}^n \). Hence Corollary 3.25 provides a second proof for this result, independent from covering theory.

**Corollary 3.25 (Existence of a logarithm).** Consider a complex manifold \( X \) with \( H^1(X, \mathbb{Z}) = 0 \).

Then each holomorphic function \( f \in \mathcal{O}^*(X) \) has a holomorphic logarithm, i.e. there exists \( g \in \mathcal{O}(X) \) satisfying

\[
e^{2\pi i g} = f.
\]

**Proof.** The exponential sequence on \( X \)

\[
0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\text{ex}} \mathcal{O}^* \rightarrow 0
\]

provides the following segment of the long exact cohomology sequence

\[
H^0(X, \mathcal{O}) \xrightarrow{\delta^*_0} H^0(X, \mathcal{O}^*) \rightarrow H^1(X, \mathbb{Z})
\]

The vanishing \( H^1(X, \mathbb{Z}) = 0 \) implies the surjectivity of

\[
H^0(X, \mathcal{O}) \xrightarrow{\delta^*_0} H^0(X, \mathcal{O}^*)
\]

\( \Box \)

**Remark 3.26 (Cohomology theory).** Consider a topological space \( X \).

1. A **cohomology theory** for sheaves of Abelian groups on \( X \) is a family of covariant functors

\[
(H^q(X, -))_{q \in \mathbb{N}} : \text{Sh}_X \rightarrow \text{Ab}
\]

which satisfies the following properties:

- **Connecting morphisms:** For each short exact sequence of sheaf morphisms on \( X \)

\[
0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0
\]

exists a family \( \delta = (\delta^*_q)_{q \in \mathbb{N}} \) of morphisms

\[
\delta^*_q : H^q(X, \mathcal{H}) \rightarrow H^{q+1}(X, \mathcal{F})
\]

such that the induced long cohomology sequence - analogous to Theorem 3.12 - is exact and functorial with respect to morphisms between short exact sequences of sheaf morphisms.
• **Normalization:** There is a canonical isomorphism of global sections

\[ H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F}) := \mathcal{F}(X) \]

• **Acyclicity:** For a flabby sheaf \( \mathcal{F} \) on \( X \)

\[ H^q(X, \mathcal{F}) = 0, \quad q \geq 1. \]

2. A cohomology theory for sheaves of Abelian groups on \( X \) is uniquely determined up to canonical isomorphisms, see [23, Satz 19.3].

3. For each sheaf \( \mathcal{F} \) of Abelian groups on \( X \) the canonical flabby resolution from Theorem 3.15

\[ 0 \to \mathcal{F} \to \mathcal{W}^\bullet \]

provides a cohomology theory for sheaves of Abelian groups on \( X \) by defining for \( q \in \mathbb{N} \)

\[ H^q(X, \mathcal{F}) := \ker \left[ \mathcal{W}^q(X) \to \mathcal{W}^{q+1}(X) \right] / \text{im} \left[ \mathcal{W}^{q-1}(X) \to \mathcal{W}^q(X) \right], \quad \mathcal{W}^{-1} := 0, \]

see [11, Chap. II, §4].

4. For paracompact \( X \) also Čech cohomology from Definition 3.7 is a cohomology theory.
Chapter 4
Local theory and coherence of sheaves

The chapter introduces the concept of coherent $\mathcal{O}$-modules over the structure sheaf of a complex manifold. The concept allows to extend results from the local analytic geometry of a given stalk of a coherent sheaf to all stalks in a neighbourhood. Insofar coherence allows to sheafify commutative algebra.

4.1 The ring of convergent power series and finitely generated modules

The present section deals with some results from Local Analytic Geometry which are needed for the global theory of coherent sheaves.

We start with the investigation of the local ring

$$R_n := \mathbb{C}\{z_1, \ldots, z_n\}$$

of convergent power series in $n$ complex variables and its maximal ideal

$$m := <z_1, \ldots, z_n> \subset R.$$  

As usual, a unit $u \in R_n$ is an element with an inverse $u^{-1} \in R_n$. The maximal ideal $m$ is the ideal of all non-units. The ring $R_n$ is an integral domain as a consequence of the identity theorem, analogous to Corollary 1.11.

In the 1-dimensional case the ring of convergent power series

$$R_1 = \mathbb{C}\{z\}$$

is a principal ideal domain: Each non-zero ideal $a \subset R_1$ is generated by a well defined monomial $z^k$, $k \in \mathbb{N}$. Because each $f \in R_1$, $f \neq 0$, has a unique representation
\[ f(z) = \sum_{j=k}^{\infty} a_j \cdot z^j = t^k \cdot \sum_{j \in \mathbb{N}} a_{j+k} \cdot z^j \]

with \( a_k \neq 0 \). The number \( k \in \mathbb{N} \) is the order of the zero of \( f \) in \( 0 \in \mathbb{C} \).

The situation is different for \( n \geq 2 \): A non-zero ideal \( a \subset R_n \) is not necessary a principal ideal. And a non-zero principal ideal \( a \subset R_n \) is not necessarily generated by monomials.

First we investigate the question about the generators of non-zero principal ideals in \( R_n \). It’s good practice in complex analysis to visualize an ideal by its zero set. Hence we take a geometric viewpoint and ask for the zero set of a holomorphic function, or more precisely: For the zero set of the germ of a holomorphic function. It turns out that the zero set displays an algebraic character. This property is brought out by the concept of a Weierstrass polynomial and its role in the Weierstrass theorems, Theorem 4.7 and 4.9.

**Definition 4.1** (Power series distinguished in \( z_n \) and Weierstrass polynomial).

1. A power series \( f \in R_n \) is distinguished in \( z_n \) of order \( k \in \mathbb{N} \) if its restriction satisfies

\[ f(0, \ldots, 0, z_n) = \sum_{y=k}^{\infty} c_{ny} z_n^y, \]

i.e if the restriction \( f(0, \ldots, 0, -) \in R_1 \) has a zero of order \( k \) in \( 0 \in \mathbb{C} \).

2. A polynomial \( f \in R_{n-1}[z_n] \) is a Weierstrass polynomial in \( z_n \) of degree \( k \in \mathbb{N} \) if

\[ f(z_1, \ldots, z_n) = \sum_{j=1}^{k} a_j(z_1, \ldots, z_{n-1}) \cdot z_n^j \in R_{n-1}[z_n] \]

with coefficients \( a_j \in R_{n-1} \) satisfying

\[ a_j(0, \ldots, 0) = 0, \ j = 0, \ldots, k-1, \ \text{while} \ a_k = 1 \in \mathbb{C}. \]

Hence a power series \( f \in R_n \) is distinguished in \( z_n \) of order \( k \) if the restriction

\[ f(0, \ldots, 0, z_n) \]

has in \( 0 \in \mathbb{C} \) a zero of exact order \( k \). The identity theorem from complex analysis of one variable implies that \( f \in R_n \) is distinguished in \( z_n \) for a suitably order \( k \in \mathbb{N} \) if and only if the restriction \( f(0, \ldots, 0, -) \in R_1 \) is not zero.

Each Weierstrass polynomial \( f \in R_n \) of degree \( k \) is a monic polynomial distinguished in \( z_n \) of degree \( k \). All coefficients
4.1 The ring of convergent power series and finitely generated modules

\[ a_j \in R_{n-1}, \ j = 0, \ldots, k - 1, \]

vanish at the origin of \( \mathbb{C}^{n-1} \). The polynomial character in \( z_n \) with coefficients from \( R_{n-1} \) makes a Weierstrass polynomial a suitable tool to prove results about \( R_n \) by induction on \( n \): As a first result Theorem 4.7 shows that a power series from \( R_n \) distinguished in \( z_n \) of degree \( k \) agrees up to a unit from \( R_n \) with a Weierstrass polynomial from \( R_{n-1}[z_n] \) of degree \( k \).

Figure 4.1 shows the zeros of a Weierstrass polynomial of degree \( k \) as branches over the variables \((z_1, \ldots, z_{n-1})\), which vary in a neighbourhood of \( 0 \in \mathbb{C}^{n-1} \). The zeros form a branched covering over an open set in \( \mathbb{C}^{n-1} \).

Fig. 4.1 Zero set of a Weierstrass polynomial

A counter example against being distinguished in one of the variables is the element \( f(z_1, z_2) \in R_2 \). It satisfies \( f \neq 0 \), but it is neither distinguished in \( z_1 \) nor in \( z_2 \). Lemma 4.2 shows how this shortage can easily be cured by a linear coordinate transformation.

**Lemma 4.2 (Linear coordinate transformation).** For each finite set of non-zero power series

\[ f_1, \ldots, f_r \in R_n \]

exists a linear isomorphism
\( \phi : \mathbb{C}^n \to \mathbb{C}^n \)

such that for each \( j = 1, \ldots, r \) the transformed power series

\( f_j \circ \phi \in \mathbb{R}^n \)

is distinguished in \( z_n \) with a degree \( k_j \geq 1 \).

**Proof.** First, choose an open zero-neighbourhood \( U \subset \mathbb{C}^n \) and a point \( a \in U \setminus \{0\} \) satisfying the following property: For each \( j = 1, \ldots, s \) the power series \( f_j \) defines a holomorphic function in \( U \) and \( f_j(a) \neq 0 \). Secondly, choose a linear isomorphism

\( \phi : \mathbb{C}^n \to \mathbb{C}^n \)

which maps the \( z_n \)-axis to the line passing through \( a \) and the origin. \( \square \)

The main input from complex analysis for the local theory is the Cauchy formula in the form of the Laurent splitting in annuli, depending on holomorphic parameters, see Theorem 4.3.

**Theorem 4.3 (Laurent splitting in annuli).** Consider a domain \( U \subset \mathbb{C}^{n-1} \) with \( 0 \in U \), two radii \( 0 < r < R < \infty \), and a holomorphic function

\( f : U \times (\Delta(R) \setminus \overline{\Delta}(r)) \to \mathbb{C}, \ (t,z) \mapsto f(t,z). \)

1. There exist two uniquely determined holomorphic functions

\( f_1 : U \times \Delta(R) \to \mathbb{C} \) and \( f_2 : U \times (\mathbb{C} \setminus \overline{\Delta}(r)) \to \mathbb{C} \)

with the following properties:

\begin{itemize}
  \item On \( U \times (\Delta(R) \setminus \overline{\Delta}(r)) \) the function \( f \) splits additively as
    \[ f = f_1 + f_2. \]
  \item For each \( t \in U \) holds
    \[ \lim_{z \to \infty} f_2(t,z) = 0. \]
\end{itemize}

2. For each \( t \in U \) and for each \( \rho \) with \( r < \rho < R \): For each \( z \in \mathbb{C} \) with \( |z| < \rho \) holds the integral representation

\( f_1(t,z) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(t,\zeta)}{\zeta - z} \, d\zeta, \)

and for each \( z \in \mathbb{C} \) with \( |z| > \rho \) holds the integral representation

\( f_2(t,z) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(t,\zeta)}{\zeta - z} \, d\zeta. \)
3. The functions $f_1$ and $f_2$ have the convergent power series

$$f_1(t, z) = \sum_{\nu=0}^{\infty} c_{\nu}(t) \cdot z^\nu, \ |z| < R, \text{ and } f_2(t, z) = \sum_{\nu=1}^{\infty} c_{-\nu}(t) \cdot z^{-\nu}, \ |z| > r,$$

with coefficients $c_{\nu}(t), \nu \in \mathbb{Z}, \ t \in U$, obtained for arbitrary $\rho > 0$ with $r < \rho < R$ as

$$c_{\nu}(t) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(t, \zeta)}{\zeta^{\nu+1}} d\zeta.$$

**Proof.** For fixed $t \in U$ the proof uses from complex analysis of a single variable the unique Laurent representation by the Cauchy formula for holomorphic functions defined in an annulus, see [30, Chap. 3]. To prove the uniqueness: Assume an analogous second pair $f_1'$ and $f_2'$ with

$$f = f_1' + f_2'.$$

Then on $U \times (\Delta(R) \setminus \overline{\Delta(r)})$ holds

$$f_1 - f_1' = f_2 - f_2'.$$

Hence the family $(f_1 - f_1', f_2 - f_2')$ defines a function $h$ on $U \times \mathbb{C}$ satisfying for each $t \in U$

$$\lim_{z \to \infty} h(t, z) = 0.$$

The maximum principle for one complex variable implies that for each $t \in U$ the holomorphic function $h(t, -)$ vanishes. Hence $h = 0$ and $f_j = f_j', j = 1, 2$.

Because the integrand of the Cauchy integral from Theorem 4.3 depends holomorphically on the parameter $t \in U$, both functions $f_1$ and $f_2$ are holomorphic. \[\square\]

**Remark 4.4 (Counting zeros).** Consider two radii $0 < r < R$ and a holomorphic function of one variable

$$f : \Delta(R) \to \mathbb{C}$$

which has no zeros in the annulus $\Delta(R) \setminus \overline{\Delta(r)}$. Then the number $k$ of zeros of $f$ can be computed for any $\rho > 0$ with $r < \rho < R$ by the integral formula

$$k = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f'(\zeta)}{f(\zeta)} d\zeta.$$

For the proof see [30, Chap. 6].

We use this equality from Remark 4.4 to define the global order for the holomorphic functions considered in the following.
Definition 4.5 (Order in $z_n$). Consider a domain $U \subset \mathbb{C}^{n-1}$, two radii $0 < r < R$ and a holomorphic function

$$f : U \times \Delta(G) \to \mathbb{C}$$

with no zeros in $U \times (\Delta(R) \setminus \overline{A}(r))$. The number

$$k := \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f_{[z_n]}(t, \zeta)}{f(t, \zeta)} d\zeta, \quad r < \rho < R,$$

is named the order of $f$ in $U \times \Delta(R)$ with respect to $z_n$. Here $f_{[z_n]}$ in the numerator of the integrand denotes the partial derivative of $f$ with respect to $z_n$.

For each $t \in U$ the value of the integral is an integer which depends continuously on $t$. Because $U$ is connected, the value is constant.

Theorem 4.6 (Weierstrass preparation theorem: Analytic version). Consider a domain $U \subset \mathbb{C}^{n-1}$, two radii $0 < r < R$, and a holomorphic function

$$f : U \times \Delta(R) \to \mathbb{C}$$

with no zeros in $U \times (\Delta(R) \setminus \overline{A}(r))$. Denote by $k$ the order of $f$ with respect to $z_n$. Then exist

- a monic polynomial in $z_n$ of degree $k$

  $$p : U \times \mathbb{C}, \quad p(t, z_n) = z_n^k + \sum_{j=0}^{k-1} a_j(t) \cdot z_n^j$$

  with holomorphic coefficients

  $$a_j : U \to \mathbb{C}, \quad j = 1, \ldots, k-1,$$

- and a holomorphic function

  $$g : U \times \Delta(R) \to \mathbb{C}^*$$

such that in $U \times \Delta(R)$ holds

$$f = u \cdot p.$$

Both functions $p$ and $g$ are uniquely determined by this equation.

The following proof is due to Stickelberger (1887). It uses the Laurent splitting.

Proof. i) Existence: Because the function $f$ has order $= k$ with respect to $z_n$ the induced function
4.1 The ring of convergent power series and finitely generated modules

\[ U \times (\Delta(R) \setminus \tilde{A}(r)) \rightarrow \mathbb{C}, \ (t, z_n) \mapsto \frac{1}{z_n} \cdot f(t, z_n), \]

has a well-defined logarithm: During one loop of

\[ z_n \in \Delta(R) \setminus \tilde{A}(r) \]

the logarithm of the first factor of \( F \) changes by the additive summand \(-2k\pi i\) and the second factor by the additive summand \( 2k\pi i \). Theorem 4.3 provides the Laurent splitting of the function

\[ F : U \times (\Delta(R) \setminus \tilde{A}(r)) \rightarrow \mathbb{C}, \ F(t, z_n) := \log \frac{1}{z_n} \cdot f(t, z_n), \]

in the form

\[ F = F_1 + F_2 \]

with two holomorphic functions

\[ F_1 : U \times \Delta(R) \rightarrow \mathbb{C} \text{ and } F_2 : U \times (\mathbb{C} \setminus \tilde{A}(r)) \rightarrow \mathbb{C} \text{ satisfying } \lim_{z_n \rightarrow \infty} F_2(t, z_n) = 0, \ t \in U. \]

Taking the exponential of \( F \) provides the multiplicative splitting

\[ f(t, z_n) = \exp F_1(t, z_n) \cdot z_n^k \cdot \exp F_2(t, z_n) \]

On one hand, define

\[ u := \exp F_1(t, z_n) \]

which is holomorphic on \( U \times \Delta(R) \) and has no zeros. On the other hand, the result

\[ \lim_{z_n \rightarrow \infty} \exp F_2(t, z_n) = 1 \text{ due to } \lim_{z_n \rightarrow \infty} F_2(t, z_n) = 0, \ t \in U, \]

implies

\[ \lim_{z_n \rightarrow \infty} \exp F_2(t, z_n) = 1 \]

and for \( (t, z_n) \in U \times (\Delta(R) \setminus \tilde{A}(r)) : \)

\[ \exp F_2(t, z_n) = 1 + \sum_{v=1}^{\infty} a_v(t) \cdot z_n^{-v} \text{ resp. } \exp F_2(t, z_n) = z_n^k \cdot (1 + \sum_{v=1}^{\infty} a_v(t) \cdot z_n^{-v}) \]

Define the polynomial in \( z_n \) of degree = \( k \)

\[ p(t, z_n) := \frac{z_n^k}{z_n^k + \sum_{v=1}^{k} a_v(t) \cdot z_n^{-v}} \]

and the rest

\[ h(t, z_n) := \sum_{v=1}^{\infty} a_{k+v}(t) \cdot z_n^{-v}. \]
Then
\[ f = u \cdot (p + h) \]
The representation
\[ \left( \frac{f}{u} - p \right) - h = 0 \]
is a Laurent splitting of zero. The uniqueness of the Laurent splitting, see Theorem 4.3, implies the vanishing of both summands, hence \( h = 0 \) and \( f = u \cdot p \).

ii) **Uniqueness:** For each \( t \in U \) the two holomorphic functions of one variable
\[ f(t, -) : \Delta(R) \to \mathbb{C} \text{ and } p(t, -) : \Delta(R) \to \mathbb{C} \]
have the same \( k \) zeros. The monic polynomial \( p(t, -) \in \mathbb{C}[z_n] \) of degree \( k \) is determined by its zeros. In the open complement of the zeros holds
\[ \frac{f(t, -)}{p(t, -)} = u(t, -) \]
Hence the identity theorem determines \( u(t, -) \) and a posteriori \( u \).

Theorem 4.7 states: Each power series from \( R_n \) distinguished in \( z_n \) is associated in \( R_n \) to a Weierstrass polynomial from \( R_{n-1}[z_n] \).

**Theorem 4.7 (Weierstrass preparation theorem: Algebraic version).** For each power series \( f \in R_n \) distinguished in \( z_n \) of order \( k \) exist
• a Weierstrass polynomial \( p \in R_{n-1}[z_n] \) of degree \( k \)
• and a unit \( u \in R_n \)
such that
\[ f = u \cdot p. \]

**Proof.** The given element \( f \in R_n \) can be represented by a holomorphic function
\[ f : U \times \Delta(R) \to \mathbb{C} \]
with \( U \subset \mathbb{C}^n \) a domain containing \( 0 \in \mathbb{C}^{n-1} \) and a radius \( R > 0 \). Because the restriction \( f(0, ..., 0, z_n) \) has a zero at \( 0 \in \mathbb{C} \) of finite order \( k \), we may assume - possibly after shrinking \( U \) and \( R \) - a second radius \( r > 0 \) with \( 0 < r < R \) such that the restriction
\[ f|U \times (\Delta(R) \setminus \Delta(r)) \]
satisfies the assumptions of Theorem 4.6. The theorem provides a holomorphic function
4.1 The ring of convergent power series and finitely generated modules

\[ u : U \times \Delta(R) \to \mathbb{C}^* \]

and a polynomial

\[ p : U \times \mathbb{C} \to \mathbb{C}, \quad p(t, z_n) = z_n^k + \sum_{j=1}^{k-1} a_j \cdot z_n^j, \]

with

\[ f = u \cdot p \]

And both functions are uniquely determined by these properties. □

Lemma 4.8 (Multiples of a Weierstrass polynomial). Consider a Weierstrass polynomial \( p \in R_{n-1}[z_n] \) of degree \( k \) and a polynomial \( f \in R_{n-1}[z_n] \) satisfying \( f = q \cdot p \) with a power series \( q \in R_n \). Then also \( q \) is a polynomial, i.e.

\[ q \in R_{n-1}[z_n] \]

Proof. Set

\[ R := R_{n-1}. \]

We apply the Euclidean algorithm of division in the polynomial ring \( R[z_n] \) with the monic polynomial \( p \in R[z_n] \). We obtain

\[ g = q \cdot p + r \]

with suitable polynomials \( q, r \in R[z_n] \) and \( \deg r < k \). There exist \( \epsilon > 0 \) and a radius \( \rho > 0 \) such that

- all coefficients of the polynomials \( p, q, r \) are holomorphic in \( \Delta(r) \subset \mathbb{C}^{n-1} \),

- and for each \( t \in \Delta(\rho) \) all zeros \( z_n \) of the Weierstrass polynomial \( p(t, -) \) satisfy \( |z_n| < \epsilon \).

As a consequence, for each \( t \in \Delta(\rho) \) the function \( g(t, -) \) has at least \( k \) zeros, and a posteriori also \( r(t, -) \). But

\[ \deg r(t, -) \leq k - 1, \]

which implies \( r(t, -) = 0 \). As a consequence

\[ f = q \cdot p \]

which proves the claim. □

Theorem 4.9 proves the analogue of Euclidean division for power series distinguished in \( z_n \). The theorem is due to H. Späth (1929). In general it is named after Weierstrass.
Theorem 4.9 (Weierstrass division theorem). Consider a power series \( f \in \mathbb{R}^n \) distinguished in \( z_n \) of order \( k \).

Then for each power series \( \phi \in \mathbb{R}^n \) exist

- a power series \( q \in \mathbb{R}^n \)
- and a polynomial \( r \in \mathbb{R}^{n-1}[z_n] \) of degree \(< k \)

such that

\[
\phi = q \cdot f + r.
\]

Both elements \( q \) and \( r \) are uniquely determined.

Proof. i) Uniqueness: Two different representations of \( \phi \) provide a representation of zero

\[
0 = q \cdot f + r
\]

with \( q \in \mathbb{R}^n \) and \( r \in \mathbb{R}^{n-1}[z_n] \) a polynomial of degree \(< k \). We consider the elements \( f, q, r \) as holomorphic functions

\[
U \times \Delta(R) \to \mathbb{C}, \quad U \subset \mathbb{C}^{n-1} \text{ a domain with } 0 \in U, \quad R > 0.
\]

If \( r \neq 0 \) then exists \( t \in U \) such that the polynomial \( r(t, z_n) \in \mathbb{C}[z_n] \) does not vanish identically and therefore has at most \( k-1 \) zeros. But the function

\[
q(t, -) \cdot f(t, -) \in \mathbb{C}\{z_n\}
\]

has at least \( k \) zeros, a contradiction. As a consequence \( r = 0 \), which implies \( q = 0 \) because \( \mathbb{R}^n \) is a domain of integrity.

ii) Reduction to the case of \( f \) a Weierstrass polynomial: Theorem 4.7 applies to \( f \) and provides a representation

\[
f = u \cdot p
\]

with a unit \( u \in \mathbb{R}^n \) and a Weierstrass polynomial \( p \in \mathbb{R}^{n-1}[z_n] \) of degree \( k \). It suffices to prove the claim for the particular case of a Weierstrass polynomial

\[
f = p \in \mathbb{R}^{n-1}[z_n]:
\]

Because a representation

\[
\phi = q \cdot p + r
\]

implies

\[
\phi = u^{-1} \cdot (u \cdot p) + r = u^{-1} \cdot f + r
\]

with the unit \( u^{-1} \in \mathbb{R}^n \).

iii) Proof for \( f \) a Weierstrass polynomial: We represent \( f \) and \( \phi \) by holomorphic functions

\[
U \times \Delta(R) \to \mathbb{C}, \quad U \subset \mathbb{C}^{n-1} \text{ a domain with } 0 \in U,
\]
4.1 The ring of convergent power series and finitely generated modules

with suitable radii $0 < r < R$ such that $f$ has no zeros in

$$V := U \times (\Delta(R) \setminus \overline{\Delta}(r))$$

Theorem 4.3 applies to the holomorphic function

$$\frac{\phi}{f} : V \to C$$

and provides the Laurent splitting

$$\frac{\phi}{f} = q + f_2$$

with the two holomorphic summands

$q : U \times \Delta(R) \to C$ and $f_2 : U \times (C \setminus \overline{\Delta}(r)) \to C$ satisfying $\lim_{z_n \to \infty} f_2(t, z_n) = 0, t \in U$.

Expanding

$$f(t, z_n) = z_n^k + \sum_{j=1}^{k} a_j(t) \cdot z_n^{-j}$$

and

$$f_2(t, z_n) = \sum_{v=1}^{\infty} c_v(t) \cdot z_n^{-v}$$

implies

$$f(t, z_n) \cdot f_2(t, z_n) = \left( z_n^k + \sum_{j=1}^{k} a_j(t) \cdot z_n^{-j} \right) \cdot \left( \sum_{v=1}^{\infty} c_v(t) \cdot z_n^{-v} \right) =$$

$$= \left( b_1(t) \cdot z_n^{k-1} + \ldots + b_k(t) \right) + \sum_{v=-1}^{\infty} b_v(t) \cdot z_n^v$$

with holomorphic coefficients $b_v : U \to C, \ v \in \mathbb{Z}$. Splitting the last equation into the two summands

$$r(t, z_n) := \sum_{v=1}^{k} b_v(t) \cdot z_n^{k-v}, \ h(t, z_n) := \sum_{v=-1}^{\infty} b_v(t) \cdot z_n^v$$

provides the representation

$$\phi = q \cdot f + r + h$$

with the polynomial $r \in R_{n-1}[z_n]$ of degree $< k$. The vanishing $h = 0$ follows from the uniqueness of the Laurent splitting

$$0 = (-\phi + q \cdot f + r) + h$$
Corollary 4.10 (Weierstrass theorem on module finiteness). Each element \( q \in R_{n-1}[z_n] \) distinguished in \( z_n \) of degree \( k \) induces a surjective morphism of \( R_{n-1} \)-modules

\[
\gamma : R_n \to R_{n-1}^k, \quad f \mapsto (a_0, \ldots, a_{k-1}).
\]

Here the tuple \((a_0, \ldots, a_{k-1})\) results from the division with rest

\[
f = g \cdot q + r
\]

with

\[
g \in R_n, \quad r = \sum_{j=0}^{k-1} a_j \cdot z^j_n.
\]

The map \( \gamma \) induces an isomorphism of \( R_{n-1} \)-modules

\[
R_n/(q \cdot R_n) \cong R_{n-1}^k
\]

Proof. The claim follows from Theorem 4.9. \( \square \)

While Corollary 4.10 considered two \( R_{n-1} \)-module structures, the following Corollary 4.11 compares two \( \mathbb{C} \)-algebra structures.

Corollary 4.11 (Weierstrass isomorphism of \( \mathbb{C} \)-algebras). Consider an element \( q \in R_n \) distinguished in \( z_n \) and denote by \( \omega \in R_{n-1}[z_n] \) the well-defined Weierstrass polynomial with

\[
q = u \cdot \omega
\]

and \( u \in R_n \) a unit. The injection

\[
R_{n-1}[z_n] \hookrightarrow R_n
\]

induces an isomorphism of \( \mathbb{C} \)-algebras

\[
R_{n-1}/\omega \cdot R_{n-1} \cong R_n/q \cdot R_n
\]

Proof. Because \( u \in R_n \) is a unit there holds the equality of ideals

\[
q \cdot R_n \simeq \omega \cdot R_n \subset R_n
\]

The injection

\[
R_{n-1}[z_n] \hookrightarrow R_n
\]

induces a surjective morphism of \( \mathbb{C} \)-algebras

\[
\alpha : R_{n-1}[z_n] \to R_n/\omega \cdot R_n \simeq R_n/q \cdot R_n
\]
4.1 The ring of convergent power series and finitely generated modules

because due to Theorem 4.9 each \( f \in R_n \) is congruent modulo \( q \) to a polynomial \( r \in R_{n-1}[z_n] \).

One has

\[
\ker \alpha = (\omega \cdot R_n) \cap R_{n-1}[z_n] = \omega \cdot R_{n-1}.
\]

Hence

\[
R_{n-1}/\omega \cdot R_{n-1} \cong R_n/q \cdot R_n.
\]

\( \square \)

**Corollary 4.12 (Weierstrass polynomials prime in \( R_n \) respectively in \( R_{n-1}[z_n] \)).**

A Weierstrass polynomial \( f \in R_{n-1}[z_n] \) is a prime in \( R_{n-1}[z_n] \) if and only if \( f \in R_n \) is a prime in \( R_n \).

**Proof.** We use the well-known characterization of primes in a ring \( A \): An element

\[
a \in A, \ a \neq 0,
\]

is prime in \( A \) if and only if the quotient \( A/\langle a \rangle \) is an integral domain. Hence Corollary 4.11 proves the claim. \( \square \)

**Theorem 4.13 (Hilbert-Rückert ideal basis theorem).** The ring \( R_n, \ n \in \mathbb{N}, \) is Noetherian.

**Proof.** The proof is by induction on \( n \in \mathbb{N} \).

Induction start \( n = 0 \): The ring \( R_0 = \mathbb{C} \) is a field and has no proper ideal different from \( \{0\} \).

Induction step \( n - 1 \mapsto n \): Consider a proper ideal \( \mathfrak{a} \neq \{0\} \). Due to Lemma 4.2 we may assume w.l.o.g the existence of an element \( 0 \neq p \in \mathfrak{a} \) which is distinguished in \( z_n \). Denote its order by \( k \). For each \( f \in R_n \) Theorem 4.9 provides a representation

\[
f = q \cdot p + r
\]

with \( r \in R_{n-1}[z_n] \) of order \( k \). Hence the ring

\[
R := R_n/\langle p \rangle \cong R_{n-1}^k
\]

is a free \( R_{n-1} \)-module of finite rank. Hence \( R \) is a Noetherian \( R_{n-1} \)-module because the ring \( R_{n-1} \) is Noetherian by induction assumption. Therefore the quotient ring \( R/\mathfrak{a} \) is a Noetherian \( R_{n-1} \)-module, generated by finitely many residue classes

\[
\overline{f}_1, \ldots, \overline{f}_r, \ f_j \in \mathfrak{a}, \ j = 1, \ldots, r.
\]

As a consequence, the ideal \( \mathfrak{a} \subset R_n \) is generated by the elements \( (f, f_1, \ldots, f_r) \). \( \square \)
Recall that an integral domain $A$ is \textit{factorial} if each non-zero $a \in A$ is product of finitely many prime elements from $A$.

**Proposition 4.14 (Factoriality).** The ring $R_n$, $n \in \mathbb{N}$, is factorial.

\textit{Proof.} The proof is by induction on $n$, see [13, Chap. I, §5, Satz 5].

Induction start $n = 0$: Because $R_0 = \mathbb{C}$ is a field, each non-zero element of $R_0$ is a unit.

Induction step $n - 1 \rightarrow n$: Assume that $R_{n-1}$ is factorial. By Gauss’ theorem also the ring $R_{n-1}[z_n]$ is factorial.

To factor a given element $f \in R_n$, $f \neq 0$, as a product of primes, we may assume

$$f \in R_{n-1}[z_n]$$

a Weierstrass polynomial according to Theorem 4.7. In particular $f \in R_{n-1}[z_n]$ is not a unit in $R_{n-1}[z_n]$: By induction assumption

$$f = \prod_{j=1}^{r} f_j \in R_{n-1}[z_n]$$

with primes $f_j \in R_{n-1}[z_n]$, $j = 1, \ldots, r$. Because $f$ is a monic polynomial we may arrange that $f_j$ monic for each $j = 1, \ldots, r$. Then the monic polynomial

$$f_j \in R_{n-1}[z_n], \quad j = 1, \ldots, r,$$

is a Weierstrass polynomial, prime in $R_{n-1}[z_n]$. Corollary 4.12 implies that $f_j$, $j = 1, \ldots, r$, is also prime in $R_n$. Hence the given element $f \in R_n$ factorizes into a product of prime elements. \hfill \qed

**Corollary 4.15 (Normality).** The ring $R_n$, $n \in \mathbb{N}$, is normal, i.e. it is closed in its quotient field with respect to integral extensions of $R_n$.

\textit{Proof.} The claim holds for each factorial ring: Denote by $Q(R_n)$ the quotient field of $R_n$. Consider an element $f \in Q$ which is integral over $R_n$, i.e. $f$ satisfies an integral equation

$$f^k = \sum_{j=0}^{k-1} a_j \cdot f^j$$

with $a_j \in R_n$ for all $j = 1, \ldots, k - 1$. The factoriality of $R_n$ provides a representation

$$f = \frac{g}{h}$$

with $g, h \in R_n$ without common prime factor. Then
4.1 The ring of convergent power series and finitely generated modules

\[ g^k = \sum_{j=0}^{k-1} a_j \cdot g^j \cdot h^{k-j} = h \cdot \left( \sum_{j=0}^{k-1} a_j \cdot g^j \cdot h^{k-1-j} \right) \in R_n \]

Hence \( h \) divides \( g^k \), contradicting the fact that \( h \) and \( g \) have no common prime factor.

\[ \square \]

**Proposition 4.16 (Hensel’s Lemma).** Consider a monic polynomial \( f \in R_{n-1}[z_n] \), and assume the restriction factorizing as

\[ f(0, \ldots, 0, z_n) = \prod_{j=1}^{m} (z_n - \alpha_j)^{b_j}, \quad b_j \in \mathbb{N} \text{ for } j = 1, \ldots, m, \]

with zeros \( \alpha_1, \ldots, \alpha_j \in \mathbb{C} \). Then exist uniquely determined, pairwise coprime, monic polynomials

\[ f_1, \ldots, f_m \in R_{n-1}[z_n] \]

satisfying

\[ f_j(0, \ldots, 0, z_n) = (z_n - \alpha_j)^{b_j}, \quad j = 1, \ldots, m, \]

such that \( f \) factorizes as

\[ f = \prod_{j=1}^{m} f_j \]

**Proof.**

1) **Existence:** The proof is by induction on \( m \). The induction start \( m = 1 \) is obvious.

Induction step \( m - 1 \mapsto m \): The monic polynomial

\[ g(t, z_n) := f(t, z_n + \alpha_1) \in R_{n-1}[z_n] \]

is distinguished in \( z_n \) of order \( b_1 \). Theorem 4.7 provides a representation

\[ g = u \cdot p \]

with a unit \( u \in R_n \) and a Weierstrass polynomial \( p \in R_{n-1}[z_n] \) of order \( b_1 \).

Lemma 4.8 ensures \( u \in R_{n-1}[z_n] \). Define two polynomials from \( R_{n-1}[z_n] \)

\[ f_1(t, z_n) := p(t, z_n - \alpha_1) \text{ and } \tilde{f} := e(t, z_n - \alpha_1) \]

The polynomials \( f_1, \tilde{f} \in R_{n-1}[z_n] \) are monic and satisfy

\[ f_1(0, \ldots, 0, z_n) = (z_n - \alpha_1)^{b_1} \text{ and } \tilde{f}(0, \ldots, 0, z_n) = \prod_{j=2}^{m} (z_n - \alpha_j)^{b_j} \]

The induction assumptions applies to \( \tilde{f} \) and provides monic polynomials
\[ f_2, \ldots, f_m \in R_{n-1}[z_n] \]
satisfying
\[ f_j(0, \ldots, 0, z_n) = (z_n - \alpha_j)^{b_j}, \quad j = 2, \ldots, m, \]
and \( \tilde{f} = \prod_{j=2}^{m} f_j \)
Hence
\[ f = f_1 \cdot \tilde{f} \]
which terminates the induction step.

ii) Uniqueness and coprimality: The ring \( R_{n-1}[z_n] \) is factorial according to Proposition 4.14 and Gauss’ theorem. Consider the prime decomposition
\[ f = \pi_1 \cdot \ldots \cdot \pi_s \]
Because \( f \) is monic we may arrange that the prime elements \( \pi_i, \quad i = 1, \ldots, s \) are also monic polynomials. Then the factors are uniquely determined up to numbering. The restriction provides in the factorial ring \( \mathbb{C}[z_n] \) the representation
\[ \prod_{i=1}^{s} \pi_j(0, \ldots, 0, z_n) = \prod_{j=1}^{t} (z_n - \alpha_j)^{b_j} \in \mathbb{C}[z_n] \]
Because each monomial \( (z_n - \alpha_j) \in \mathbb{C}[z_n] \) on the right-hand side is a prime element for each \( i = 1, \ldots, s \) holds
\[ \pi_i(0, \ldots, 0, z_n) = \prod_{j=1}^{t} (z_n - \alpha_j)^{b_j} \text{ with } 0 \leq b_{ij} \leq b_j \]
If \( b_{ij} \neq 0 \) for at least two indices \( j_1 \neq j_2 \) then part i) of the proof would produce a reducible representation of \( \pi_i \in R_{n-1}[z_n] \), contradicting the primality of \( \pi_i \). Hence \( \pi_i(0, \ldots, 0, z_n) \) vanishes for exactly one \( j \in \{1, \ldots, t\} \). Accordingly, for \( j = 1, \ldots, t \) we define
\[ F_j \in R_{n-1}[z_n] := \prod_{i: \pi_i(0, \ldots, 0, \alpha_i) = 0} \pi_i \]
The monic polynomials \( F_1, \ldots, F_t \) are pairwise coprime. Their product satisfies
\[ \prod_{j=1}^{t} F_j = F = \prod_{j=1}^{t} f_j \]
Hence the monic prime elements on both sides coincide. As a consequence, for each \( j = 1, \ldots, t \) the prime factors of \( F_j \) and \( f_j \) on both sides coincide, which implies \( F_j = f_j \). \( \square \)
Corollary 4.17 (Irreducibility and Weierstrass polynomial). Each normed, irreducible polynomial \( f \in R_{n-1}[z_n] \) with \( f(0, \ldots, 0) = 0 \) is a Weierstrass polynomial.

Proof. The restriction of the given polynomial \( f \in R_{n-1}[z_n] \) factorizes as
\[
f(0, \ldots, 0, z_n) = \prod_{j=1}^{m} (z_n - \alpha_j)^{b_j}, \quad b_j \in \mathbb{N} \text{ for } j = 1, \ldots, m,
\]
with pairwise distinct roots
\[
\alpha_j \in \mathbb{C}, \quad j = 1, \ldots, m, \quad \alpha_1 = 0 \text{ and } b_1 \neq 0.
\]
Proposition 4.16 implies the existence of normed polynomials
\[
f_1, \ldots, f_m \in R_{n-1}[z_n]
\]
satisfying for \( j = 1, \ldots, m \)
\[
f_j(0, \ldots, 0, z_n) = (z_n - \alpha_j)^{b_j}
\]
and
\[
f = \prod_{j=1}^{m} f_j \in R_{n-1}[z_n]
\]
The irreducibility of \( f \in R_{n-1}[z_n] \) implies that for each \( j = 2, \ldots, m \) the factor
\[
f_j, \quad j = 2, \ldots, m,
\]
is a unit, hence \( b_j = 0 \). As a consequence \( f = f_1 \) is a Weierstrass polynomial. \( \square \)

Corollary 4.18 (Parameter representation of the germ of hypersurface). Consider an open set \( U \subset \mathbb{C}^{n-1} \) and a holomorphic function \( f \in \mathcal{O}(U)[z_n] \) of the form
\[
f : U \times \mathbb{C} \to \mathbb{C}, \quad f(z', z_n) = \frac{1}{z_n} + \sum_{j=1}^{k} a_j(z') \cdot \frac{1}{z_n^{j-1}}, \quad a_j \in \mathcal{O}(U), j = 1, \ldots, k,
\]
Assume the existence of a point \( c \in U \) such that the restricted polynomial
\[
f(c, -) \in R_1
\]
has pairwise distinct zeros \( \alpha_j \in \mathbb{C}, \quad j = 1, \ldots, k. \) Then exists an open neighbourhood \( V \subset U \) of \( c \) and holomorphic functions
\[
\phi_j : V \to \mathbb{C}, \quad j = 1, \ldots, k, \text{ satisfying } \phi_j(c_j) = \alpha_j, \quad j = 1, \ldots, k,
\]
such that for all \( (z', z_n) \in U \times \mathbb{C} \) holds
\begin{equation*}
  f(z', z_n) = \prod_{j=1}^{k} (z_n - \phi_j(z')).
\end{equation*}

Figure 4.2 visualizes geometrically Corollary 4.18 as the representation of a hypersurface in non-singular points.

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{fig42.png}
  \caption{The germ of a non-singular hypersurface as a non-branched covering}
  \label{fig:4.2}
\end{figure}

**Proof.** We assume \( c = 0 \in \mathbb{C}^{n-1} \). Then \( f \) restricts to

\begin{equation*}
  f(0, ..., 0, z_n) = \prod_{j=1}^{k} (z_n - \alpha - j)
\end{equation*}

Proposition 4.16 provides normed polynomial

\[ f_1, ..., f_k \in R_{n-1}[z_n] \]

which factorize

\[ f = \prod_{j=1}^{k} f_j \text{ satisfying } f_j(0, ..., 0, z_n) = z_n \alpha_j, \ j = 1, ..., k, \]

Hence for each \( j = 1, ..., k \), the polynomial \( f_j \) has degree = 1 and therefore the form

\[ f_j = z_n - \phi_j \text{ with } \phi_j \in R_{n-1} \]

Then \( V \subset U \) of \( c \) may be choosen as a neighbourhood if \( c \) where all functions \( f_j \in R_{n-1}, \ j = 1, ..., k \) are represented by holomorphic functions. \( \square \)
Proposition 4.19 (Nakayama lemma for local rings). If a finitely generated $R$-module $L$ satisfies

$$L = m \cdot L$$

then $L = 0$.

Proof. The proof is indirect. Let

$$\{f_1, \ldots, f_k\} \subset L$$

be a minimal set of generators of $L$, and assume $k \geq 1$. Then

$$f_k = \sum_{j=1}^{k} m_j \cdot f_j$$

for suitable coefficients $m_j \in m$, $j = 1, \ldots, k$. Hence

$$(1 - m_k) \cdot f_k = \sum_{j=1}^{k-1} m_j \cdot f_j$$

The element $1 - m_k \in R$ is a unit because $1 - m_k \notin m$. Hence

$$f_k = \sum_{j=1}^{k-1} \frac{m_j}{1 - m_k} \cdot f_j,$$

a contradiction to the minimality. \qed

Corollary 4.20 (Implications of the Nakayama lemma). If $M$ is a finitely generated $R$-module and $N \subset M$ a submodule with

$$M = m \cdot M + N$$

then

$$M = N.$$

Proof. By assumption

$$M/N = (m \cdot M + N)/N = m \cdot (M/N)$$

Hence Proposition 4.19 implies

$$M/N = 0,$$ i.e. $M = N$.

\qed
Proposition 4.21 (Krull Lemma). For each finitely generated submodule 
\[ M \subset R^n_p, \ p \in \mathbb{N}, \]
holds:
\[ M = \bigcap_{s \geq 1} (M + m^s \cdot R^n_p) \]

For a proof of Proposition 4.21 see [13, Anhang, §2, Satz 2 Folgerung].

From local homological algebra of finitely generated \( R^n \)-modules we need the following results, which will be later generalized to certain modules over the structure sheaf of a complex manifold.

Definition 4.22 (Homological dimension of finitely generated modules). Consider a finitely generated \( R \)-module \( M \).

1. The homological dimension \( \text{hd}_R M \) of \( M \) is defined as the minimal length \( d \) of a finite resolution of \( M \) by finitely generated free \( R \)-modules
\[ 0 \rightarrow R^d \xrightarrow{\phi_d} \ldots \rightarrow R^1 \xrightarrow{\phi_1} R^0 \xrightarrow{\phi_0} M \rightarrow 0 \]

2. Each such resolution of \( M \) of length \( = \text{hd}_R M \) is named a Hilbert resolution of \( M \).

3. The \( R \)-modules
\[ S_i := \text{im} [R^{i+1} \xrightarrow{\phi_{i+1}} R^i] \subset R^i, \ i \in \{0, \ldots, d\} \]
from a Hilbert resolution of \( M \) are named \( i \)-th syzygy modules of \( M \).

Proposition 4.23 (Homological dimension in exact sequences). Consider an exact sequence
\[ 0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0 \]
of \( R \)-modules with a free \( R \)-module \( F \) of finite rank, and assume \( \text{hd}_R M \geq 1 \). Then
\[ \text{hd}_R K = (\text{hd}_R M) - 1. \]

Note that in Proposition 4.23 both \( R \)-modules \( M \) and \( F \) are finitely generated.
For a proof of Proposition 4.23 see [13, Kap. III, §5, Satz 4 u. HS 5]. In order to apply HS 5 of the reference note the relation between homological dimension and profondeur of an R-module:

\[ \text{prof}_R M = \text{prof} R - \text{hd}_R M \text{ and } \text{prof}_R K = \text{prof} R - \text{hd}_R K \]

which implies

\[ \text{hd}_R M \geq 1 = \text{hd}_R F \iff \text{prof}_R M \leq (\text{prof}_R F) - 1. \]

Theorem 4.24 shows that each finitely generated R-module M has finite homological dimension.

**Theorem 4.24 (Hilbert’s syzygy theorem for finitely generated R-modules).** Each finitely generated R-module M has finite homological dimension

\[ \text{hd}_R M \leq n. \]


**4.2 Oka’s coherence theorem for the structure sheaf**

We now leave the field of sheaves of Abelian groups and focus on sheaves of rings and sheaves of modules over these of rings. The most important sheaf of rings on a complex manifold is the structure sheaf \( \mathcal{O} \).

**Definition 4.25 (Module sheaf).** Consider a topological space X and a sheaf \( \mathcal{R} \) of rings on X. A sheaf \( \mathcal{F} \) on X is an \( \mathcal{R} \)-module sheaf or just an \( \mathcal{R} \)-module if

- for each open \( U \subset X \) the set \( \mathcal{F}(U) \) is a module over the ring \( \mathcal{R}(U) \) due to a map

\[ \mathcal{R}(U) \times \mathcal{F}(U) \to \mathcal{F}(U) \]

- such that for each open \( V \subset U \) the induced diagram with vertical restrictions

\[
\begin{array}{ccc}
\mathcal{R}(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\
\downarrow & & \downarrow \\
\mathcal{R}(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V)
\end{array}
\]

commutes.
In complex analysis the most important examples of $\mathcal{R}$-modules are modules over the structure sheaf $\mathcal{O}$ of a complex manifold $X$, in particular coherent $\mathcal{O}$-modules, see Definition 4.26.

**Definition 4.26 (Coherence).** Consider a complex manifold $X$.

1. **Finite type:** An $\mathcal{O}$-module $\mathcal{F}$ is of finite type iff each $x \in X$ has a neighbourhood $U \subset X$ and finitely many sections
   \[ f_1, \ldots, f_k \in \mathcal{F}(U) \]
   such that for all $y \in U$ the canonical map on the level of stalks
   \[ \mathcal{O}_y^k \to \mathcal{F}_y, \quad (s_1,y, \ldots, s_k,y) \mapsto \sum_{j=1}^k s_{j,y} \cdot f_{j,y} \]
   is surjective.

2. **Sheaf of relations:** Consider an open set $Y \subset X$ and finitely many sections
   \[ f_1, \ldots, f_k \in \mathcal{F}(Y) \]
   of an $\mathcal{O}_Y$-module $\mathcal{F}$. Their sheaf of relations
   \[ \mathcal{R}(f_1, \ldots, f_k) \subset \mathcal{O}_Y^k \]
   is defined as the sheaf on $Y$
   \[ \mathcal{R}(f_1, \ldots, f_k)(U) := \left\{ (s_1, \ldots, s_k) \in \mathcal{O}_Y^k(U) : \sum_{j=1}^k s_j \cdot (f_j|U) = 0 \right\}, \quad U \subset Y \text{ open}, \]
   with the canonical restrictions.

3. **Coherent:** An $\mathcal{O}$-module $\mathcal{F}$ of finite type is **coherent** if it is also relation-finite, i.e. for each open subset $V \subset X$ and finitely many section
   \[ f_1, \ldots, f_k \in \mathcal{F}(V) \]
   their sheaf of relations $\mathcal{R}(f_1, \ldots, f_k)$ is an $\mathcal{O}$-module of finite type on $V$.

Remark 4.27 translates Definition 4.26 into a statement about exact sequences of $\mathcal{O}$-module sheaf morphisms over open subsets.

**Remark 4.27 (Coherent $\mathcal{O}$-module).** Consider a complex manifold $X$. 
1. An $\mathcal{O}$-module $\mathcal{F}$ is of finite type iff each $x \in X$ has an open neighbourhood $U \subset X$ and over $U$ an exact sequence of $\mathcal{O}$-module sheaf morphisms
\[ \mathcal{O}^k \xrightarrow{\alpha} \mathcal{F} \to 0 \]
for a suitable $k \in \mathbb{N}$.

2. An $\mathcal{O}$-module $\mathcal{F}$ of finite type is coherent iff each open $U \subset X$ and each sheaf morphisms of $\mathcal{O}$-modules over $U$
\[ \mathcal{O}^k \xrightarrow{\alpha} \mathcal{F} \]
satisfies: For each $x \in U$ exists an open neighbourhood $V$ of $x$ in $U$ and over $V$ an exact sequence of $\mathcal{O}$-module sheaf morphisms
\[ \mathcal{O}^m \xrightarrow{\beta} \mathcal{O}^k \xrightarrow{\alpha} \mathcal{F} \]
for a suitable $m \in \mathbb{N}$.

3. A submodule $\mathcal{G} \subset \mathcal{F}$ of a coherent $\mathcal{O}$-module $\mathcal{F}$ is coherent iff $\mathcal{G}$ is of finite type.

**Theorem 4.28 (Oka’s coherence theorem).** The structure sheaf $\mathcal{O}$ of a complex manifold $X$ is coherent.

**Proof.** Coherence is a local property- Hence it suffices to consider $X \subset \mathbb{C}$ an open set. The $\mathcal{O}$-module is finite type. Hence it remains to show its relation-finiteness:

For any open set $D \subset X$ and finitely many sections
\[ F_1, \ldots, F_k \in \mathcal{O}^m(D) \]
the sheaf of relations
\[ \mathcal{R}(f_1, \ldots, f_k) \]
on $D$ is of finite type. The proof will be given by induction on $(n, m)$ according to the following scheme:
\[ (n, \leq m) \implies (n, m + 1) \]
and
\[ (n - 1, \text{all } m \in \mathbb{N}) \implies (n, 1) \]
i) *Induction start $n = 0$:* We have $X = \mathbb{C}^0 = \{0\}$. Hence $\mathcal{O}_X$ is concentrated on the point $0 \in \{0\}$ with stalk the field $\mathbb{C}$.

ii) *Induction step $(n, \leq m) \implies (n, m + 1)$: The induction step increases the rank of the sheaf in the domain of a morphism*
\[ \mathcal{O}^m \to \mathcal{O}. \]

Consider sections \( F_1, \ldots, F_k \in \mathcal{O}^{m+1}(D) \). They define on \( D \) a sheaf morphism
\[ F : \mathcal{O}^k \to \mathcal{O}^{m+1}, \quad (s_1, \ldots, s_k) \mapsto \sum_{j=1}^k s_j \cdot F_j, \]

We use the exact sequence
\[ 0 \to \mathcal{O} \xrightarrow{i} \mathcal{O}^{m+1} \xrightarrow{p} \mathcal{O}^m \to 0 \]

with
\[ i : \mathcal{O} \xrightarrow{i} \mathcal{O}^{m+1}, \quad i(s) := (s, 0, \ldots, 0) \quad \text{and} \quad p(s_1, \ldots, s_{m+1}) = (s_2, \ldots, s_m) \]

and define
\[ F' := p \circ F : \mathcal{O}^k \to \mathcal{O}^m \]

obtaining the commutative square at the right-hand side of the following diagram

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{O} & \xrightarrow{i} & \mathcal{O}^{m+1} & \xrightarrow{p} & \mathcal{O}^m & \to & 0 \\
& & \downarrow{id} & & \downarrow{G \circ K} & & \downarrow{id} & & \\
& & \downarrow{K} & & \downarrow{G} & & \downarrow{F'} & & \\
& & \mathcal{O}^r & \to & \mathcal{O}^k & \xrightarrow{F} & \mathcal{O}^m & & \\
0 & \to & \mathcal{O} & \xrightarrow{i} & \mathcal{O}^{m+1} & \xrightarrow{p} & \mathcal{O}^m & \to & 0
\end{array}
\]

The induction assumption for the exponent \( \beta = m \) applies to the morphism \( F' \): Each given point \( a \in D \) has an open neighbourhood \( U \) in \( D \) and an \( \mathcal{O} \)-module morphism
\[ G : \mathcal{O}^r \to \mathcal{O}^k \]

which provides an exact sequence
\[ \mathcal{O}^r \xrightarrow{G} \mathcal{O}^k \xrightarrow{F'} \mathcal{O}^{m+1} \to 0 \]

The composition \( F \circ G \) satisfies
\[ p \circ (F \circ G) = 0. \]

Because the lower row of the above diagram is exact, one obtains an \( \mathcal{O} \)-module morphism
\[ H : \mathcal{O}^r \to \mathcal{O} \]
such that the left-hand square in the above diagram commutes. The induction assumption for the exponent \( = 1 \) applies to the morphism \( H \): For the given point \( a \in D \) and possibly after restricting the neighbourhood \( U \) there exists an \( \mathcal{O} \)-module morphism

\[
K : \mathcal{O}^s \to \mathcal{O}^r
\]

which provides an exact sequence

\[
\mathcal{O}^s \xrightarrow{K} \mathcal{O}^r \xrightarrow{H} \mathcal{O} \to 0
\]

The upper square in the upper left part of the diagram is commutative. A final diagram chasing proves the exactness of the sequence

\[
\mathcal{O}^s \xrightarrow{G \circ K} \mathcal{O}^k \xrightarrow{E} \mathcal{O}^{m+1}
\]

and terminates the induction step.

iii) Induction step \((n - 1, all m) \Rightarrow (n, 1)\): The induction step increases the dimension of the underlying domain of definition of the sheaves. One has to consider the relation sheaf of a morphism on an \( n \)-dimensional domain under the assumption that the relation sheaves of morphisms on \( n - 1 \)-dimensional domains are of finite type. The idea is to choose a coordinate transformation which makes all sections \( F_1, ..., F_k \) distinguished in \( z_n \) and replace them by Weierstrass polynomials. The Weierstrass division theorem reduces modulo \( F_1 \) all occurring statements about \( R_n \) to statements about \( R_{n-1}[z_n] \). For the reduction it is necessary to bound the degree of all occurring polynomials. Hence \( F_1 \) is chosen as a polynomial with maximal degree.

Simplifying the original sections: In case of an index \( j \in \{1, ..., k\} \) with \( F_j = 0 \) the sheaf of relations satisfies

\[
\mathcal{R}(F_1, ..., F_k) = \mathcal{R}(F_1, \ldots, \hat{F}_j, ..., F_k) \oplus \mathcal{O}, \text{ (skip } F_j \text{)}
\]

Hence we may assume \( F_j \neq 0 \) for all \( j = 1, ..., k \). Moreover:

- Because w.l.o.g \( a = 0 \) and \( F_j \in R_n \) is distinguished in \( z_n \), see Lemma 4.2, we may assume: For all \( j = 1, ..., k \) the holomorphic function \( F_j \in R_{n-1}[z_n] \) is polynomial and distinguished in \( z_a \).

- The Weierstrass preparation Theorem 4.6 provides open zero-neighbourhoods

\[
U' \subset \mathbb{C}^{n-1}, \ U_0 \subset \mathbb{C}
\]

and in \( U := U' \times U_0 \subset D \) representations

\[
F_j = u_j \cdot \Phi_j, \ j = 1, ..., k,
\]

with invertible holomorphic functions \( u_j \in \mathcal{O}(U) \) and Weierstrass polynomials
\( \Phi_j \in \mathcal{O}(U')[z_n]. \)

We obtain on \( U \) an isomorphism of \( \mathcal{O} \)-module morphisms

\[
\mathcal{R}(F_1, \ldots, F_k) \xrightarrow{\sim} \mathcal{R}(\Phi_1, \ldots, \Phi_k), \quad (f_1, \ldots, f_k) \mapsto e_1 \cdot f_1, \ldots, e_k \cdot f_k
\]

Hence w.l.o.g. the original functions from \( \mathcal{O}(D) \) are Weierstrass polynomials

\[
F_1, \ldots, F_k \in \mathcal{O}(U')[z_n], \quad j = 1, \ldots, k,
\]

with \( F_1 \) the polynomial of highest order, the degree denoted as \( d \).

After reducing the original sections to Weierstrass polynomials we now prove the induction step by reducing the original relations to polynomial relations. This will be done in two steps:

iii, 1) \textit{Reduction to germs of polynomial relations}: Consider an arbitrary but fixed point

\[
x = (x', x_n) \in U = U' \times U_0.
\]

We show that the stalk of the relation sheaf

\[
\mathcal{R}_x := \mathcal{R}(F_1, \ldots, F_k)_x
\]

contains a finite family of polynomial relations, which generate \( \mathcal{R}_x \) as \( \mathcal{O}_x \)-module. Moreover, the degree of all occurring polynomials is bounded by \( d \). The Weierstrass preparation theorem 4.6 splits

\[
F_1 = e \cdot P
\]

with a unit \( e \in \mathcal{O}_x \) and a Weierstrass polynomial

\[
P \in \mathbb{C}\{z' - x'\}[z_n - x_n]
\]

of order \( \leq d \). Lemma 4.8 shows that also

\[
e \in \mathbb{C}\{z' - x'\}[z_n - x_n]
\]

is a polynomial. Apparently, the \( \mathcal{O}(U) \)-module

\[
\mathcal{R}(F_1, \ldots, F_k)(U)
\]

contains the polynomial relations

\[
v_2 := (-F_2, F_1, 0, \ldots, 0)\\
\vdots\\
v_j = (-F_j, 0, \ldots, 0, F_1, 0, \ldots, 0), \quad (F_1 \text{ at position } j)
\]
\[ v_k = (-F_k, 0, \ldots, 0, F_1) \]

Consider now an arbitrary but fixed relation
\[ \phi = (\phi_1, \ldots, \phi_k) \in \mathcal{R}(F_1, \ldots, F_k) \subset \mathcal{O}_x^k \]

The Weierstrass division theorem, Theorem 4.9, reduces modulo \( F_1 \) its components as
\[ \phi_j = \alpha_j \cdot F_1 + r_j, \quad j = 1, \ldots, k, \]
with \( \alpha_j \in \mathcal{O}_x \) and rest \( r_j \in \mathcal{C}\{z' - x'\}[z_n - x_n] \) with \( \deg r_j < d \). We verify that the difference of relations
\[ \phi - \sum_{j=2}^{k} \alpha_j \cdot v_j \in \mathcal{R}_x \]
is polynomial: If the first component is denoted
\[ \beta := \phi_1 - \sum_{j=2}^{k} \alpha_j \cdot (-F_j) \in \mathcal{O}_x \]
then
\[ \phi - \sum_{j=2}^{k} \alpha_j \cdot v_j = (\beta, r_2, \ldots, r_k), \]
and
\[ \beta \cdot F_1 = -\sum_{j=2}^{k} r_j \cdot F_j \in \mathcal{C}\{z' - x'\}[z_n - x_n] \subset \mathcal{O}_x. \]
are polynomial. Lemma 4.8 applies to
\[ \beta \cdot F_1 = (\beta \cdot e) \cdot P \in \mathcal{O}_x \]
and shows that
\[ \beta \cdot e \in \mathcal{C}\{z' - x'\}[z_n - x_n] \]
is polynomial. Summing up:
\[ \phi = \left( \sum_{j=2}^{k} \alpha_j \cdot v_j \right) + e^{-1}(e \cdot \beta, e \cdot r_2, \ldots, e \cdot r_k) \in \mathcal{R}_x \]
represents \( \phi \) as an \( \mathcal{O}_x \)-linear combination of polynomial relations from \( \mathcal{R}_x \), because
- the relations \( v_j, \; j = 1, \ldots, k \),
- the element \( e \cdot \beta \).
• and the elements \( e \cdot r_j, \ j = 1, \ldots, k \), are polynomial. Each occurring polynomial has degree \( \leq d \) because 
\[
\text{deg}(e \cdot \beta) = \text{deg}(\beta \cdot F_1) - \text{deg} P < \text{deg} P + d - \text{deg} P = d
\]
and for \( j = 2, \ldots, k \) holds
\[
\text{deg}(e \cdot r_j) < \text{deg} e + \text{deg} P = \text{deg}(e \cdot P) = d
\]

iii,2) Finitely generated polynomial relations of bounded degree: Claim: There exists an open zero-neighbourhood \( V \subset U \) and finitely many polynomial relations 
\[
s_1, \ldots, s_l \in \mathcal{R}(F_1, \ldots, F_k)(V)
\]
such that for each point \( x \in V \) each germ of a polynomial relation of degree \( \leq d \)
\[
\sigma \in \mathcal{R}(F_1, \ldots, F_k)_x
\]
is an \( \mathcal{O}_x \)-linear combination of the germs \((s_1, x, \ldots, s_l, x)\).

In order to apply the induction hypothesis, which is valid on \( \mathbb{C}^{n-1} \), we derive from the given sections 
\[
F_1, \ldots, F_k \in \mathcal{O}(D), \ D \subset \mathbb{C}^n,
\]
a morphism between free sheaves on \( \mathbb{C}^{n-1} \): For each \( d \in \mathbb{C} \) consider the sheaf \( \mathcal{P}_d \) on \( \mathbb{C}^{n-1} \) defined as
\[
\mathcal{P}_d(V') := \left\{ \sum_{j=1}^d \psi_j \cdot z_j^{n} \in \mathcal{O}_{\mathbb{C}^{n-1}}[z_n] \right\}, \ V' \subset U,
\]
of polynomials of degree bounded by \( d \) with holomorphic coefficients from \( \mathcal{O}_{\mathbb{C}^{n-1}}(V') \). These sheaves are free and isomorphic to \( \mathcal{O}_{\mathbb{C}^{n-1}}^{d+1} \) via
\[
\mathcal{O}_{\mathbb{C}^{n-1}}^{d+1} \xrightarrow{\sim} \mathcal{P}_d, \ (\psi_0, \ldots, \psi_d) \mapsto \sum_{j=1}^k \psi_j \cdot z_n^j.
\]
The morphism on \( V \subset \mathbb{C}^n \)
\[
F : \mathcal{O}^k \to \mathcal{O}
\]
translates to a morphism of sheaves on \( \mathbb{C}^{n-1} \)
\[
F' : \mathcal{P}_d^k \to \mathcal{P}_{2d}, \ (\phi_1, \ldots, \phi_k) \mapsto \sum_{j=1}^k F_j \cdot \phi_j.
\]
To apply the induction hypothesis \((n-1, 2d+1)\) about \(ker F'\): There exists an open zero-neighbourhood

\[ V' \subset U' \subset \mathbb{C}^{n-1} \]

and finitely many sections

\[ s_1, \ldots, s_l \in ker F'(V') \]

which define on \(V'\) an exact sequence of \(\mathcal{O}_V\)-sheaves

\[ \mathcal{O}_{\mathbb{C}^{n-1}}^1 \to \mathcal{P}_{d}^{k} \to \mathcal{P}_{2d}^{k} \]

For suitable \(\varepsilon > 0\) and

\[ V := V' \times \{ \zeta_n \in \mathbb{C} : |\zeta_n| < \varepsilon \} \subset U \]

the migration from

\[ V' \subset \mathbb{C}^{n-1} \]

to \(V \subset \mathbb{C}^n\)

is provided by a canonical injection of functions

\[ \mathcal{P}_{d}^{k}(V') \to \mathcal{O}_{\mathbb{C}^{n}}^{k}(V) \]

For \(x = (x', x_n) \in V\) each polynomial relation of degree \(\leq d\)

\[ \sigma \in \mathcal{R}(F_1, \ldots, F_k)_x \]

can be considered an element \(\sigma \in ker F'_x\), hence can be represented due to part iii,1) in the form

\[ \sigma = \sum_{\lambda=1}^{l} g_{\lambda} \cdot s_{\lambda} \]

with coefficients

\[ g_{\lambda} \in \mathcal{O}_{\mathbb{C}^{n-1}, \nu} \subset \mathcal{O}_{\mathbb{C}^{n}, \nu} \]

The presentation finishes the proof of the claim from part iii,2).

The results of part iii,1) and part iii,2) finish the proof of the induction step

\[ (n-1, \forall m \in \mathbb{N}) \implies (n, 1). \]

\(\Box\)
4.3 Coherent $\mathcal{O}$-modules

**Proposition 4.29 (Coherence of the ideal sheaf of an analytic submanifold).** Consider an open subset $U \subset \mathbb{C}^n$. Each analytic submanifold $A \subset U$ has a coherent ideal sheaf

$$\mathcal{I}_A \subset \mathcal{O}_U$$

Here $\mathcal{I}_A$ is defined as the sheafification of the presheaf

$$V \mapsto \{ f \in \mathcal{O}_U(V) : f|A \cap V = 0 \}, \ V \subset U \text{ open}.$$  

**Proof.** Due to Oka’s Theorem 4.28 the structure sheaf $\mathcal{O}_U$ is coherent. Hence it remains to prove that the subsheaf $\mathcal{I}_A \subset \mathcal{O}_U$ is of finite type. Consider a given point $x \in U$.

- **Case $x \in A$:** According to Theorem 2.8 we may assume an open set $W \subset \mathbb{C}^n$ and $x = 0 \in W$ satisfying

  $$A = W \cap E_k \text{ with } E_k = \{(z_1, \ldots, z_n) \in W : z_{k+1} = \ldots = z_n = 0\}$$

  We claim that the functions

  $$z_{k+1}, \ldots, z_n \in \mathcal{I}_A(W)$$

  generate $\mathcal{I}_A|W$: Assume

  $$y \in W \text{ and } f \in (\mathcal{I}_A)_y.$$  

  i) If $y \in W \setminus E_k$ then exists index $j \in k + 1, \ldots, n$ with $z_j(y) \neq 0$. Hence

  $$z_{j,y} \neq 0 \in (\mathcal{O}_U)_y \text{ and } \frac{f}{z_{j,y}} \in (\mathcal{O}_U)_y$$

  which implies

  $$f = \frac{f}{z_{j,y}} \cdot z_{j,y}$$

  ii) If $y \in W \cap E_k$ then $y = (y_1, \ldots, y_k, 0, \ldots, 0)$. The function $f$ is defined by a convergent power series

  $$f(z) = \sum_{I=(i_1,\ldots,i_n)\in\mathbb{N}^n} c_I \cdot (z_1 - y_1)^{i_1} \cdot \ldots \cdot (z_k - y_k)^{i_k} \cdot z_{k+1}^{i_{k+1}} \cdot \ldots \cdot z_n^{i_n}$$

  The vanishing $f|E_k = 0$ implies that each summand of the power series contains a variable $z_j$ for at least one index $j \in k + 1, \ldots, n$. Hence in a neighbourhood of $y$

  $$f = \sum_{j=k+1}^{n} s_j \cdot z_j$$
4.3 Coherent \(\mathcal{O}\)-modules

with holomorphic coefficients \(s_j\).

- **Case** \(x \notin A\): Because \(A \subset U\) is closed we have \((\mathcal{I}_A)_x = (\mathcal{O}_U)_x\) which implies

\[
(\mathcal{I}_A)_y = (\mathcal{O}_U)_y
\]

for \(y\) in an open neighbourhood of \(x\). Hence the constant 1 generates \(\mathcal{I}_A\) in a neighbourhood of \(x\).

\[\Box\]

The result of Proposition 4.29 holds more general also for analytic subsets: The ideal sheaf \(\mathcal{I}_A\) of an analytic subset \(A\) in an open subset \(U \subset \mathbb{C}^n\) is a coherent \(\mathcal{O}_U\)-module, see [7, §16, Satz 1] and [15, Chap. IV, §2, Fund. Theor.].

Coherence of an \(\mathcal{R}\)-module allows to extend properties of a given stalk to all stalks in a neighbourhood.

**Proposition 4.30 (Stalks of a coherent \(\mathcal{O}\)-module).** Consider a coherent sheaf \(\mathcal{F}\) on a complex manifold \(X\). For each point

\[x \in X \text{ with } \mathcal{F}_x = 0\]

exists an open neighbourhood \(V\) of \(x\) with

\[\mathcal{F}_y = 0 \text{ for all } y \in V.\]

**Proof.** Because \(\mathcal{F}\) is of finite type there exists an open neighbourhood \(U\) of \(x\) and over \(U\) an exact sequence of \(\mathcal{O}|U\)-modules

\[\mathcal{O}^p \rightarrow \mathcal{F} \rightarrow 0\]

for a suitable \(p \in \mathbb{N}\). The canonical base \((e_i)_{i=1,...,p}\) of the \(\mathcal{O}(U)\)-module \(\mathcal{O}^p(U)\) provides a family of sections

\[f_1,...,f_p \in \mathcal{F}(U)\]

such that for each \(y \in U\) the germs

\[f_{1,y},...,f_{p,y} \in \mathcal{F}_y\]

generate the stalk \(\mathcal{F}_y\) as \(\mathcal{O}_y\)-module. The assumption

\[\mathcal{F}_x = 0\]

implies according to the definition of stalks: In a common neighbourhood \(V \subset U\) of \(x\) each section \(f_1,...,f_p\) restricts to zero, as a consequence its germ

\[f_{j,y} \in \mathcal{F}_y, \ y \in V,\]
vanishes. Hence $\mathcal{F}_y = 0$ for all $y \in V$. □

**Proposition 4.31 (Kernel and cokernel of morphisms between coherent $\mathcal{O}$-modules).**

Consider a complex manifold $X$ and a morphism

$$\alpha : \mathcal{F} \to \mathcal{G}$$

between two coherent $\mathcal{O}$-modules. Then the $\mathcal{O}$-modules on $X$

$$\ker \alpha \text{ and } \text{coker } \alpha$$

are also coherent.

**Proof.**

i) **Coherence of $\ker \alpha$:** The $\mathcal{O}$-module

$$\mathcal{K} := \ker \alpha \subset \mathcal{F}$$

is a subsheaf of the coherent sheaf $\mathcal{F}$. Hence it suffices to show that $\mathcal{K}$ is of finite type, i.e. each $x \in X$ has a neighbourhood $U_1$ and over $U_1$ a sheaf epimorphism

$$\mathcal{O}^m \xrightarrow{\beta} \mathcal{K}$$

We construct the following commutative diagram with an exact row:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \xrightarrow{\alpha} & \mathcal{F} & \xrightarrow{\alpha \circ \beta} & \mathcal{G} \\
\end{array}
$$

Coherence of $\mathcal{F}$ provides an open neighbourhood $U$ of $x$ with a sheaf epimorphism

$$\mathcal{O}^k \xrightarrow{\beta} \mathcal{K}.$$ 

The sheaf of relations

$$\mathcal{R} := \ker \left[ \mathcal{O}^k \xrightarrow{\alpha \circ \beta} \mathcal{G} \right] \subset \mathcal{O}^k$$

is of finite type because $\mathcal{G}$ is coherent. Hence over an open neighbourhood $U_2 \subset U_1$ of $x$ exists a sheaf morphism

$$\mathcal{O}^m \xrightarrow{\gamma} \mathcal{O}^k$$

with an exact sequence

$$\mathcal{O}^m \xrightarrow{\gamma} \mathcal{O}^k \xrightarrow{\alpha \circ \beta} \mathcal{G}$$
The exact sequence
\[ \mathcal{O}^m \xrightarrow{\beta \circ \gamma} \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \]
induces the exact sequence over \( U_2 \)
\[ \mathcal{O}^m \xrightarrow{\beta \circ \gamma} \mathcal{K} \xrightarrow{} 0 \]
which proves that \( \mathcal{K} \) is of finite type.

ii) **Coherence of coker \( \alpha \):** First, the induced sequence
\[ \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \text{coker } \alpha \xrightarrow{} 0 \]
is exact. Because \( \mathcal{G} \) is of finite type, also \( \text{coker } \alpha \) is of finite type. Secondly, we have to show that for each morphism over an open set \( U \subset X \)
\[ \mathcal{O}^p \xrightarrow{\gamma} \text{coker } \alpha \]
the sheaf of relations is of finite type:

First, the sheaf \( \mathcal{O}^p \) is a free \( \mathcal{O} \)-module. On the level of sections for each open \( V \subset X \) the \( \mathcal{O}(V) \)-module \( \mathcal{O}^p(V) \) is free. Denote by
\[ (e_i)_{i=1,...,p} \]
its canonical base. Due to the surjectivity of \( \beta \), for given \( x \in X \) exist on the level of stalks germs
\[ g_{i,x} \in \mathcal{G}_x, \ i = 1,...,p, \]
with
\[ \beta_x(g_{i,x}) = \gamma_x(e_{i,x}) \in (\text{coker } \alpha)_x \]
There exists a common open neighbourhood \( V \subset X \) of \( x \) and for \( i = 1,...,p \) sections
\[ g_i \in \mathcal{G}(V) \]
which represent the germ \( g_{i,x} \in \mathcal{G}_x \). Setting
\[ \psi(e_i) := g_i, \ i = 1,...,p, \]
defines a morphism of \( \mathcal{O} \)-modules over \( V \)
\[ \psi : \mathcal{O}^p \rightarrow \mathcal{G} \]
such that the following diagram of sheaf morphisms over \( V \) commutes
Secondly, by assumption $\mathcal{F}$ is of finite type. Hence each $x \in U$ has an open neighbourhood $W \subset V$ and over $W$ an epimorphism

$$\mathcal{O}^m \xrightarrow{\delta} \mathcal{F}$$

Combining the latter two results provides over $W$ the diagram of sheaf morphisms

$$
\begin{array}{ccccccc}
\mathcal{O}^m & \xrightarrow{i} & \mathcal{O}^m \oplus \mathcal{O}^p & \xrightarrow{\pi} & \mathcal{O}^p & \xrightarrow{} & 0 \\
\mathcal{F} & \xrightarrow{\alpha} & \mathcal{G} & \xleftarrow{\beta} & \text{coker } \alpha & \xleftarrow{} & 0 \\
\end{array}
$$

Here

$$\varepsilon : \mathcal{O}^m \oplus \mathcal{O}^p \rightarrow \mathcal{G}$$

is defined as

$$\varepsilon(a, b) := \alpha(\delta(a)) + \psi(b)$$

and

$$\mathcal{O}^m \xrightarrow{i} \mathcal{O}^m \oplus \mathcal{O}^p$$

and

$$\mathcal{O}^m \oplus \mathcal{O}^p \simeq \mathcal{O}^m \times \mathcal{O}^p \xrightarrow{\pi} \mathcal{O}^p$$

are the canonical injection and projection. The diagram is commutative with respect to its horizontal and vertical morphisms. Moreover by construction

$$\gamma = \beta \circ \psi$$

Because

$$\beta \circ \alpha = 0$$

the restriction

$$\pi|_{\ker \varepsilon} : \ker \varepsilon \rightarrow \ker \gamma$$

is well-defined. The restriction is surjective: Consider a point $x \in W$ and a germ $c \in \mathcal{O}^p_x$ with

$$0 = \gamma(c) = \beta(\psi(c)).$$

Then exist a germ $a \in \mathcal{F}_x$ satisfying

$$\psi(c) = \alpha(a),$$

and a germ
4.3 Coherent \( \mathcal{O} \)-modules

\[ d \in \mathcal{O}_x^m \]

satisfying
\[ \delta(d) = a. \]

Then \((-d, c) \in (\ker \varepsilon)_x\) because
\[ \varepsilon(-d, c) = -\alpha(\delta(a)) + \psi(c) = -\psi(c) + \psi(c) = 0 \]

and
\[ \pi(-d, c) = c, \]

which finishes the proof of the surjectivity of
\[ \ker \varepsilon \to \ker \gamma. \]

Because \( \ker \varepsilon \) is of finite type, also \( \ker \gamma \) is of finite type, which proves that the relation sheaf of \( \gamma \) is of finite type, and finishes the proof. \( \square \)

**Corollary 4.32 (Morphisms between coherent \( \mathcal{O} \)-modules).** Consider a morphism
\[ \phi : \mathcal{F} \to \mathcal{G} \]

between two coherent sheaves on a complex manifold \( X \). The subsets
\[ \text{Inj}(\phi) := \{ x \in X : \phi_x : \mathcal{F}_x \to \mathcal{G}_x \text{ injective} \} \]

and
\[ \text{Surj}(\phi) := \{ x \in X : \phi_x : \mathcal{F}_x \to \mathcal{G}_x \text{ surjective} \} \]

and
\[ \text{Iso}(\phi) := \{ x \in X : \phi_x : \mathcal{F}_x \to \mathcal{G}_x \text{ isomorphism} \} \]

are open subsets of \( X \).

**Proof.** According to Proposition 4.31 the \( \mathcal{O} \)-modules \( \ker \phi \) and \( \text{coker} \phi \) are coherent. The injectivity of \( \phi_x \) is equivalent to \( (\ker \phi)_x = 0 \) and the surjectivity of \( \phi_x \) is equivalent to \( (\text{coker} \phi)_x = 0 \). Hence the claim follows from Proposition 4.30. \( \square \)

**Proposition 4.33 (Coherence in short exact sequences).** Consider a complex manifold \( X \) and an exact sequence of \( \mathcal{O} \)-modules on \( X \)
\[ 0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0 \]

If two of the sheaves are coherent, then also the third sheaf is coherent.

**Proof.** The proof follows from Proposition 4.31:
i) Assume $\mathcal{G}$ and $\mathcal{H}$ coherent: Then

$$\mathcal{F} \simeq \ker \beta$$

ii) Assume $\mathcal{F}$ and $\mathcal{G}$ coherent: Then

$$\mathcal{H} \simeq \mathcal{G} / \ker \beta \simeq \mathcal{G} / \text{im} \alpha = \text{coker} \alpha$$

iii) Assume $\mathcal{F}$ and $\mathcal{H}$ coherent:

- **Finite type:** Alike to the proof from Proposition 4.31 we consider in a suitable neighbourhood of a given point $x \in X$ the diagram of sheaf morphisms with exact rows

$$
\begin{array}{cccc}
\mathcal{O}_x^m & \rightarrow & \mathcal{O}_x^m \oplus \mathcal{O}_x^p & \rightarrow & \mathcal{O}_x^p \\
\downarrow \delta & & \downarrow \varepsilon & & \downarrow \gamma \\
\mathcal{F} & \rightarrow & \mathcal{G} & \rightarrow & \mathcal{H} \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
\end{array}
$$

The existence of the epimorphisms $\delta$ and $\gamma$ follows because $\mathcal{F}$ and $\mathcal{H}$ are finitely generated. The morphism $i$ and $\pi$ are the canonical injection respectively projection. The diagram is commutative with respect to its horizontal and vertical morphisms. The morphism $\gamma$ induces a morphism $\psi$ with

$$\beta \circ \psi = \gamma$$

because $\mathcal{O}_x^p$ is a free sheaf. Eventually we define

$$\varepsilon : \mathcal{O}_x^m \oplus \mathcal{O}_x^p \rightarrow \mathcal{G}, \; \varepsilon(d,b) := \alpha(\delta(d)) + \psi(b)$$

The morphism $\varepsilon$ is surjective: For a given germ $g \in \mathcal{G}_x$ first choose a germ $b \in \mathcal{O}_x^p$ with

$$\beta(g) = \gamma(b)$$

Then

$$\beta(g - \psi(b)) = \beta(g) - \beta(\psi(b)) = \beta(g) - \gamma(b) = \gamma(b) - \gamma(b) = 0$$

Hence there exists a germ

$$d \in \mathcal{O}_x^m$$

such that $a := \delta(d)$ satisfies

$$\alpha(a) = g - \psi(b)$$

Then

$$\varepsilon(d,b) = \alpha(\delta(d)) + \psi(b) = \alpha(a) + \psi(b) = g - \psi(b) + \psi(b) = g,$$
which shows the surjectivity of $\varepsilon$ and proves that $\mathcal{G}$ is of finite type.

- **Relation-finite:** Consider a given morphism

$$g : O^k \to \mathcal{G}$$

over an open set $U \subset X$ and a given point $x \in U$. The morphism extends to a morphism

$$\beta \circ g : O^k \to \mathcal{H}.$$  

Then exists over a suitable open neighbourhood $V \subset U$ of $x$ a commutative diagram of sheaf morphisms over $V$ with exact rows

$$
\begin{array}{cccc}
O^m & \xrightarrow{\gamma} & O^k & \xrightarrow{\beta \circ g} \mathcal{H} \\
\downarrow{\psi} & & \downarrow{g} & \downarrow{id} \\
0 & \xrightarrow{\alpha} & \mathcal{F} & \xrightarrow{\beta} \mathcal{G} & \xrightarrow{\gamma} \mathcal{H} & \xrightarrow{\psi} 0
\end{array}
$$

Here the morphism $\gamma$ is induced because $\mathcal{H}$ is relation-finite. And the morphism $\psi$ is induced because

$$\beta \circ (g \circ \gamma) = 0 \text{ and } \ker \beta = \text{im} \alpha,$$

and because the sheaf $O^m|U$ is free. The restriction

$$\gamma|\ker \psi : \ker \psi \to \ker g$$

is well-defined and surjective. The coherence of $\mathcal{F}$ implies that $\ker \psi$ is of finite type. Hence also $\ker g$ is of finite type, which finishes the proof of the coherence of $\mathcal{G}$.

$\square$

As a consequence of Theorem 4.28 and Proposition 4.33, besides the structure sheaf $O$ of a complex manifold $X$ also finite direct sums $O^p$, $p \in \mathbb{N}$, and locally free $O$-modules of finite rank are coherent.

The concept of the direct image from Definition 2.25 carries over to cohomology groups of higher order.

**Definition 4.34 (Higher direct image).** Consider a continous map

$$f : X \to Y$$
between topological spaces and a sheaf \( \mathcal{F} \) on \( X \). For each \( q \geq 1 \) define the presheaf on \( Y \)

\[
V \mapsto H^q(f^{-1}(V), \mathcal{F}), \quad V \subset Y \text{ open},
\]

Its sheafification, denoted \( R^q f_* \mathcal{F} \), is named the \( q \)-th direct image sheaf of \( \mathcal{F} \); see [3, Chap. III, § 1].

**Remark 4.35 (Grauert’s coherence theorem).** Consider a holomorphic map

\[
f : X \to Y
\]

between complex manifolds and a coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) on \( X \). Grauert’s coherence theorem states: Under the additional topological assumption that \( f \) is a proper map, i.e. the inverse image of compact sets is compact, all direct image sheaves

\[
R^q f_* \mathcal{F}, \quad q \in \mathbb{N},
\]

are coherent \( \mathcal{O}_Y \)-modules.

Grauert’s coherence theorem is a highlight of complex analysis. Besides Grauert’s original proof there exist separate proofs by Forster-Knorr and Kiehl-Verdier. Grauert’s theorem is considered one of the deepest theorems of complex analysis, see also [3, Chap. III, §2].
Chapter 5
Cartan’s lemma for holomorphic matrices

5.1 Proof of Cartan’s lemma

Cartan’s lemma for holomorphic matrices is a result about the multiplicative splitting of invertible matrices of bounded holomorphic functions which are defined in a neighbourhood of the intersection of two adjacent product domains. It generalizes Proposition 2.34, which can be considered the particular case of holomorphic functions with values in $GL(1, \mathbb{C}) = \mathbb{C}^*$. 

In order to prove Cartan’s lemma about the splitting of matrices we have to generalize the analysis used so far. Until now we considered maps which depend on a finite set of complex variables and have their values in $\mathbb{C}^n$. These maps are assumed to be smooth or holomorphic. The present section considers maps with arguments depending on infinitely many parameters. More precisely, the argument varies in Banach spaces of bounded holomorphic functions. Also the image of the maps are Banach spaces. As a consequence we use some results from functional analysis.

Differentiability of such a map $f$ at a point in an open set is defined via the usual differential quotient, i.e. by approximating $f$ in a neighbourhood of the point by a linear map. Definition 5.3 formalizes a condition which is slightly stricter concerning the precision of the approximation.

Notation 5.1 (Banach norms). For an open set $U \subset \mathbb{C}^n$ denote by $\mathcal{B}(U)$ the Banach algebra of bounded holomorphic functions $f \in O(U)$ with the norm

$$
\|f\|_U := \sup \{|f(z)| : z \in U\}
$$

Consider two open sets $U_j \subset \mathbb{C}^n$, $j = 1, 2$ with intersection

$$
U_{12} := U_1 \cap U_2
$$

1. For $k \in \mathbb{N}$ and $j = 1, 2$ denote by

$$
E_j := M(k \times k, \mathcal{B}(U_j))
$$
the vector space of all $k \times k$-matrices with entries in $\mathcal{B}(U_j)$ and by
\[
E_j^* := GL(k, \mathcal{B}(U_j)) \subset E_j
\]
the subgroup of invertible matrices.

2. We consider the matrices $A \in E_j$ as linear maps between the Banach spaces $\mathcal{B}(U_j)^{\oplus k}$ with respect to the following norms: For
\[
f = (f_1, \ldots, f_k) \in \mathcal{B}(U_j)^{\oplus k}
\]
set
\[
\|f\| := \|f\|_{U_j} := \sup \{ \|f_\kappa\|_{U_j} : \kappa = 1, \ldots, k \}.
\]
We recall that a linear map $A \in E_j$, $j = 1, 2$, is continuous iff the operator norm
\[
\|A\| := \sup \{ \|Af\| : f \in \mathcal{B}(U_j)^{\oplus k} \text{ and } \|f\| \leq 1 \}
\]
is finite. For a pair of continuous linear maps $(A, B) \in E_1 \times E_2$ define the operator norm
\[
\|(A, B)\| := \sup \{ \|A\|, \|B\| \}
\]

We will derive Cartan’s lemma, Theorem 5.8, about the splitting of holomorphic matrices from the following results:

- Additive splitting of bounded holomorphic functions (Prop. 5.2)
- Openness of strictly differentiable maps between Banach spaces at points with surjective tangent map (Proposition 5.4)
- Approximation of holomorphic matrix functions on product domains by global holomorphic matrix functions (Cor. 5.7)

We first extend Cartan’s lemma for holomorphic functions, Proposition 2.34, to the corresponding statement for bounded holomorphic functions.

**Proposition 5.2 (Additive splitting of bounded holomorphic functions).** Consider two compact adjacent product domains
\[
Q_1, Q_2 \subset \mathbb{C}^n,
\]
an open set $U \subset \mathbb{C}^n$ with
\[
(Q_1 \cap Q_2) \subset U
\]
together with a bounded holomorphic function $f \in \mathcal{B}(U)$. Then exist open product domains

...
5.1 Proof of Cartan’s lemma

\[ U_1, U_2 \text{ with } Q_j \subset U_j, \ j = 1, 2, \text{ and } U_0 := (U_1 \cap U_2) \subset U, \]

together with a pair of bounded holomorphic functions

\[ (f_1, f_2) \in \mathcal{B}(U_1) \times \mathcal{B}(U_2) \]

satisfying on \( U_0 \)

\[ f = f_1 - f_2. \]

Proof. Under the assumptions of the proposition above, Proposition 2.34 provides a pair of holomorphic functions

\[ (f_1, f_2) \in \mathcal{O}(U_1) \times \mathcal{O}(U_2) \]

satisfying on \( U_0 \)

\[ f = f_1 - f_2. \]

We now restrict the solution to a pair of relatively compact, open product domains

\[ Q_j \subset U'_j \subset \subset U_j, \ j = 1, 2. \]

Then the restrictions are bounded

\[ f'_j := f_j|_{U'_j} \in \mathcal{B}(U'_j), \ j = 1, 2, \]

and satisfy on

\[ U'_0 := U'_1 \cap U'_2 \]

the splitting with bounded holomorphic functions

\[ f|_{U_0} = f'_1 - f'_2. \]

\[ \Box \]

Next we need a kind of implicit function theorem to conclude that a differentiable map with surjective tangent map at a point is locally an open map. The maps in question are maps between Banach spaces, specifically maps with arguments in an open subset of a Banach space of bounded holomorphic functions. We first define the concept of strict differentiability, see Definition 5.3. Then we prove a criterion for the local openness of strictly differentiable maps between Banach spaces, see Lemma 5.4.

Definition 5.3 (Strict differentiability). Consider two Banach spaces \( E \) and \( F \), an open set \( U \subset E \) and a point \( a \in U \). A map

\[ f : U \to F \]

is differentiable in the strict sense at the point \( a \) if there exists a linear continuous map
\[ \phi : E \to F \]

satisfying

\[
\lim_{\substack{u,v \to a \\
u \neq v}} \frac{\|f(u) - f(v) - \phi(u - v)\|}{\|u - v\|} = 0
\]

The tangent map

\[ \phi =: f'(a) \]

is named the *derivation* of \( f \) at \( a \).

The notion of *strict differentiability* is stricter than the notion of differentiability. Strict differentiability considers the limit with two arguments \( u \) and \( v \) varying independently around \( a \) when approximating \( f \) near \( a \) by a continuous linear map \( \phi \) with

\[ f(u) - f(v) = \phi(u - v) + o(\|u - v\|) \]

Lemma 5.4 proves the local openness criterion for strictly differentiable maps with surjective tangent map at the distinguished point.

**Proposition 5.4 (Local openness criterion).** Consider two Banach spaces \( E \) and \( F \), an open set \( U \subset E \), and a map

\[ f : U \to F, \]

which is strict differentiable at a point \( a \in U \) and has a surjective tangent map at \( a \)

\[ f'(a) : E \to F \]

Then \( f \) is open at \( a \), i.e. the image \( f(U) \subset F \) is a neighbourhood of \( f(a) \).

**Proof.**

1) *Openness of the linear tangent map:* W.l.o.g. we may assume

\[ a = 0, \ f(a) = 0. \]

According to Banach’s theorem the tangent map, the surjective linear continuous map,

\[ \phi := f'(a) : E \to F \]

is open, i.e. for each zero-neighbourhood \( V \subset E \) exists a zero-neighbourhood \( W \subset F \) with

\[ W \subset \phi(V). \]

When expressing this property by the Banach norms it suffices to consider only the specific zero-neighbourhood...
$V := \{ x \in E : \|x\|_E \leq 1 \}$

The condition on openness states: There exists $\epsilon > 0$ such that

$W := \{ y \in F : \|y\|_F \leq \epsilon \}$

satisfies

$W \subset \phi(V)$. 

In particular, for each $y \in F$ with $\|y\|_F = \epsilon$ exists

$x \in \phi^{-1}(y) \cap V$.

Then

$\|x\|_E \leq 1 = \frac{1}{\epsilon} \cdot \|y\|_F = C \cdot \|y\|_F, \quad C := \frac{1}{\epsilon}$

We may assume $C = 1$ after rescaling $\|y\|_F$ by the factor $1/\epsilon$. Then each $y \in F$ has an inverse image $x \in E$ with

$\|x\|_E \leq \|y\|_F$.

ii) Local openness of $f$ at $a = 0$: We derive from the openness of the tangent map $\phi$ at $a$ that the original map $f$ is open at $a$. Strict differentiability of $f$ at $a$ means for pairs $(u,v) \in E \times E$

$\|f(u) - f(v) - \phi(u - v)\|_F = o(\|u - v\|_E)$

In particular, there exists $\delta > 0$ such that for all pairs from the product of the closed $\delta$-neighbourhoods of the origin $(0,0) \in E \times E$

$(u,v) \in \mathcal{K}_{\delta}(0) \times \mathcal{K}_{\delta}(0) \subset E \times E$

holds

$\|f(u) - f(v) - \phi(u - v)\|_F \leq (1/2) \cdot \|u - v\|_E$

At this point, referring to pairs, we use that the differentiability of $f$ holds in the strict sense.

We choose a radius

$r \in \mathbb{R}^+_\ast$, with $0 < r \leq \delta$ and $\mathcal{K}_r(0) \subset U$

and claim

$V := \mathcal{K}_{r/2} \subset f(U)$.

For each given $y_0 \in V$ we have to find an element $\tilde{x} \in U$ with

$f(\tilde{x}) = y_0$.

By part i) there exists an element $x_0 \in E$ with
\[ \phi(x_0) = y_0 \text{ and } \|x_0\|_E \leq \|y_0\|_F \]

Because not necessarily \( f(x_0) = y_0 \) we construct a sequence \((x_n)_{n \in \mathbb{N}}\) in \( K_r(0) \) which converges against a solution \( \tilde{x} \) with

\[ f(\tilde{x}) = y_0. \]

The stepwise construction relies on the map

\[ g : U \to F, \quad g(x) := y_0 + \phi(x) - f(x) \]

The construction is by induction on \( n \in \mathbb{N} \). The induction step assumes the existence of the elements \( x_0, \ldots, x_n \) from the sequence and constructs the next element \( x_{n+1} \in E \) as an inverse image satisfying Equation (1)

\[ g(x_n) = \phi(x_{n+1}) \]

and Equation (2):

\[ \|x_{n+1} - x_n\|_E \leq (1/2)^n \cdot (r/4) \]

As a consequence, \( x_{n+1} \) satisfies also

\[ \|x_{n+1}\|_E \leq \|x_n\|_E + \|x_{n+1} - x_n\|_E \leq r \cdot \left( \frac{3}{4} + \frac{1}{4} \cdot \sum_{k=1}^{n-1} \left( \frac{1}{2} \right)^k \right) + \frac{r}{4} \cdot \left( \frac{1}{2} \right)^n \]

\[ \leq r \cdot \left( \frac{3}{4} + \frac{1}{4} \cdot \sum_{k=1}^{n} \left( \frac{1}{2} \right)^k \right) < r \]

- **Induction start** \( n = 0 \): Due to

\[ \|x_0\|_E \leq \|y_0\|_F \leq \frac{r}{2} \]

the estimate from part i), applied to the pair

\( (x_0, 0) \in \mathcal{K}_\delta(0) \times \mathcal{K}_\delta(0) \)

shows

\[ \|\phi(x_0) - f(x_0)\|_F \leq \frac{1}{2} \cdot \|x_0\|_E \leq \frac{r}{2} \cdot \frac{1}{2} \]

Hence taking the inverse image with respect to \( \phi \) provides an element \( z \in E \) satisfying

\[ \phi(z) = \phi(x_0) - f(x_0) \text{ and } \|z\|_E \leq \|\phi(x_0) - f(x_0)\|_F \leq \frac{r}{4} \]

Define

\[ x_1 := z + x_0 \]
5.1 Proof of Cartan’s lemma

The pair \((x_0, x_1)\) satisfies Equation (1)
\[
\phi(x_1) = \phi(z) + \phi(x_0) = (\phi(x_0) - f(x_0)) + \phi(x_0) = y_0 - f(x_0) + \phi(x_0) = g(x_0),
\]
and Equation (2)
\[
\|x_1 - x_0\|_E = \|z\|_E \leq \frac{r}{4}
\]

• Induction step \(n \rightarrow n + 1\): Assume the existence of the elements \(x_0, \ldots, x_n\). Considering inverse images with respect to \(\phi\) provides an element \(z \in E\) satisfying
\[
\phi(z) = g(x_n) - g(x_{n-1})
\]
The element \(z \in E\) improves the approximation of the sequence: Define
\[
x_{n+1} := z + x_n
\]
The pair \((x_n, x_{n+1})\) satisfies equation (1)
\[
\phi(x_{n+1}) = \phi(z + x_n) = \phi(z) + \phi(x_n) = (g(x_n) - g(x_{n-1})) + \phi(x_n) =
\]
using the induction assumption \(g(x_{n-1}) = \phi(x_n)\), and equation (2)
\[
\|x_{n+1} - x_n\|_E = \|z\|_E \leq \|g(x_n) - g(x_{n-1})\|_F = \|\phi(x_n) - \phi(x_{n-1}) - (f(x_n) - f(x_{n-1}))\|_F =
\]
\[
= \|f(x_n) - f(x_{n-1}) - (\phi(x_n) - \phi(x_{n-1}))\|_F \leq (1/2) \cdot \|x_n - x_{n-1}\|_E \leq
\]
\[
\leq (1/2) \cdot (1/2)^{n-1} \cdot \frac{r}{4} = (1/2)^n \cdot \frac{r}{4}
\]
using the openness of the tangent map from part i) and the induction assumption.

Due to the set of equations (2) the sequence \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(E\). Hence there exists the limit
\[
\bar{x} := \lim_{n \to \infty} x_n \in \mathcal{K}_r(0) \subset U
\]
In order to show \(f(\bar{x}) = y_0\) we first note
\[
f(\bar{x}) = \lim_{n \to \infty} f(x_n)
\]
due to the continuity of \(f\). Secondly we consider the equality due to equation (1)
\[
y_0 - f(x_n) = (y_0 + \phi(x_n) - f(x_n)) - \phi(x_n) = g(x_n) - \phi(x_n) =
\]
\[
= \phi(x_{n+1}) - \phi(x_n) = \phi(x_{n+1} - x_n)
\]
Hence
\[
\lim_{n \to \infty} \|y_0 - f(x_n)\|_F = \lim_{n \to \infty} \|\phi(x_{n+1} - x_n)\|_F = 0
\]
which completes the proof. □

**Proposition 5.5 (Holomorphic approximation: additive case).** For an open product domain $Q \subset \mathbb{C}$ the restriction

$$\mathcal{O}(\mathbb{C}) \to \mathcal{O}(Q), \ f \mapsto f|_Q,$$

has dense image with respect to the canonical Fréchet topologies.

Proposition 5.5 is a specific case of the Runge approximation theorem for the 1-dimensional case, see [24, Kap. 12, §2, Approximationssatz]. We give an elementary proof for product domains. It relies on Cauchy’s integral formula. First, the integral is approximated by a suitable Riemann sum. Secondly, the integrand gets evaluated at finitely many points. At each point a suitable Taylor approximation provides a global holomorphic approximation of the integrand.

**Proof.** Consider a given function $f \in \mathcal{O}(Q)$, a compact $K \subset Q$, and a number $\varepsilon > 0$. W.l.o.g. $K$ is a compact product domain itself.

i) **Approximation by Riemann sums:**

![Fig. 5.1 Uniform approximation over $K$ by global functions](image)

We choose a closed path

$$\gamma \subset Q \setminus K$$

around $K$ with positive orientation. For each point $z_0 \in K$ the Cauchy integral formula from complex analysis in the plane states

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{\xi - z_0} d\xi.$$

Approximating the integral by its Riemann sums provides a finite set of intermediate points $\eta_v \in \gamma, \ v = 1, \ldots, m$, such that
5.1 Proof of Cartan’s lemma

\[ f(z_0) = \frac{1}{2\pi i} \sum_{v=1}^{m-1} \frac{f(\eta_v)}{\eta_v - z_0} \cdot (\eta_{v+1} - \eta_v) < \varepsilon \]

The estimate holds for each refinement of intermediate points, and generalizes also to each point \( z \in K \) varying in a suitable neighbourhood of \( z_0 \): Due to the compactness of \( K \) there exist finitely many intermediate points \( \zeta_v \in \gamma, v = 1, \ldots, N \) and a family \( (c_1, \ldots, c_N) \in \mathbb{C}^N \) such that for all \( z \in K \)

\[ \sup_{z \in K} \left| f(z) - \sum_{v=1}^{N} \frac{c_v}{\zeta_v - z} \right| < \varepsilon/2 \]

ii) \textbf{Approximation by Taylor polynomials}: For each index \( v = 1, \ldots, N \) the point \( \zeta_v \in \gamma \subset Q \setminus K \) and \( K \) can be separated by a circle, i.e. there exists a center \( m_v \in \mathbb{C} \) and a radius \( r_v > 0 \) such that

\[ K \subset \Delta r_v(m_v) \text{ but } \zeta_v \notin \overline{\Delta r_v(m_v)}, \]

see Figure 5.1. Here we use that \( K \) is a compact product domain. For each \( v = 1, \ldots, N \) the summand

\[ \frac{c_v}{\zeta_v - z} \]

from the estimate of part i) is holomorphic in the disc \( \Delta r_v(m_v) \) when considered as a function of \( z \). Therefore the summand can be approximated by one of its Taylor polynomials \( g_v \in \mathcal{O}(\mathbb{C}) \) with precision

\[ \sup_{z \in K} \left| \frac{c_v}{\zeta_v - z} - g_v(z) \right| < \varepsilon \cdot \frac{2}{N} \]

As a consequence

\[ \sup_{z \in K} \left| f(z) - \sum_{v=1}^{N} g_v(z) \right| \leq \sup_{z \in K} \left| f(z) - \sum_{v=1}^{N} \frac{c_v}{\zeta_v - z} \right| + \sum_{v=1}^{N} \sup_{z \in K} \left| \frac{c_v}{\zeta_v - z} - g_v(z) \right| \leq \]

\[ \leq (\varepsilon/2) + N \cdot \varepsilon/(2N) = \varepsilon \]

Hence the global holomorphic function
\[ g := \sum_{\nu=1}^{N} g_{\nu} \in \mathcal{O}(\mathbb{C}) \]

approximates \( f \) with the given precision \( \varepsilon > 0 \) on \( K \). □

**Lemma 5.6 (Path connectedness of \( GL(k, \mathbb{C}) \)).** The topological space

\[ GL(k, \mathbb{C}) \subset \mathbb{C}^{k^2} \]

is path connected.

**Proof.** For each non-zero polynomial \( P(t) \in \mathbb{C}[t] \) its zero set

\[ \{ t \in \mathbb{C} : P(t) = 0 \} \]

is finite, hence the complement in \( \mathbb{C} \)

\[ \{ t \in \mathbb{C} : P(t) \neq 0 \} \]

is path connected. For a given matrix \( A \in GL(k, \mathbb{C}) \) the specific polynomial

\[ P(t) := \det(A + t \cdot (\mathbb{I} - A)) \]

satisfies

\[ P(0) = \det A \neq 0 \text{ and } P(1) = \det \mathbb{I} = 1 \neq 0, \]

and in particular \( P \neq 0 \). Hence there exists a path in \( \mathbb{C} \) from 0 to 1

\[ \gamma : [0, 1] \to \mathbb{C} \]

which avoids the zero set of \( P(t) \), i.e. for all \( t \in [0, 1] \)

\[ P(\gamma(t)) \neq 0. \]

Therefore the path

\[ [0, 1] \to GL(k, \mathbb{C}), t \mapsto A + \gamma(t) \cdot (\mathbb{I} - A), \]

is well-defined and joins \( A \) and \( \mathbb{I} \). As a consequence \( GL(k, \mathbb{C}) \) is path connected. □

**Corollary 5.7 (Holomorphic approximation: multiplicative case).** For an open product domain

\[ Q := \prod_{j=1}^{n} Q_j \subset \mathbb{C}^{n} \]
with product domains \( Q_j \subset \subset \mathbb{C} \), \( j = 1, \ldots, n \), the restriction map
\[
GL(k, \mathcal{O}(\mathbb{C}^n)) \to GL(k, \mathcal{O}(Q)), \ F \mapsto F|_Q,
\]
has dense image with respect to the canonical Fréchet topologies.

The proof derives the claim as a corollary of Proposition 5.5. The reduction relies on the exponential series of matrices and its logarithm
\[
\log(1 + X) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{X^n}{n}, \ X \in M(k \times k, \mathbb{C}) \text{ with } \|X\| < 1
\]
which mediates between the additive and the multiplicative context.

**Proof.**

i) **Approximation in the additive case:** By Riemann’s mapping theorem each product domain \( Q_j \), \( j = 1, \ldots, n \), is biholomorphic equivalent to the unit disc \( \Delta \subset \mathbb{C} \), i.e. there exists a biholomorphic map
\[
\phi_j : Q_j \xrightarrow{\sim} \Delta.
\]
Define the biholomorphic map
\[
\phi := \prod_{j=1}^{n} \phi_j : Q \xrightarrow{\sim} \Delta^n \subset \mathbb{C}^n.
\]
For each holomorphic function \( f \in \mathcal{O}(Q) \) and compact set \( K \subset Q \) the holomorphic function
\[
f \circ \phi^{-1} \in \mathcal{O}(\Delta^n)
\]
expands into a convergent Taylor series. Hence it can be approximated on the compact set \( \phi(K) \subset \Delta^n \) with arbitrary precision by a suitable Taylor polynomial
\[
(f \circ \phi^{-1})(z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n=0}^{N} c_{k_1, \ldots, k_n} \cdot z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}
\]
Proposition 5.5 - applied to the holomorphic functions \( \phi_j \), \( j = 1, \ldots, n \), and their powers - implies: The finite sum
\[
\sum_{k_1, \ldots, k_n=0}^{N} c_{k_1, \ldots, k_n} \cdot \phi^{k_1} \cdot \ldots \cdot \phi^{k_n} \in \mathcal{O}(Q)
\]
can be approximated on \( K \subset Q \) with arbitrary precision by a global holomorphic function from \( \mathcal{O}(\mathbb{C}^n) \), which provides an approximation of \( f \) over \( K \).

ii) **Approximation in the multiplicative case near the unit matrix:** Consider a given compact set \( K \subset Q \), w.l.o.g. \( K \) a compact product domain, and \( \varepsilon > 0 \). We choose an open product domain \( K' \subset \mathbb{C}^n \) with
\[ \mathbb{K} \subset \mathbb{K}' \subset \subset \mathbb{Q} \]

For each matrix function \( A \in GL(k, \mathcal{O}(Q)) \) with
\[ \| I - A \|_{K'} < 1 \]
the logarithm
\[ G := \log(A|K') := - \sum_{n=1}^{\infty} \frac{(1 - A|K')^n}{n} \in M(k \times k, \mathcal{O}(K')) \]
is well-defined, and satisfies on \( K' \)
\[ e^G = (A|K'), \]
see [31, Chap. 1.3]. Part i) applied to the matrix function \( G \in M(k \times k, \mathcal{O}(K')) \) provides a global matrix function
\[ \tilde{G} \in M(k \times k, \mathcal{O}(\mathbb{C}^n)) \]
which approximates \( G \) on \( K' \) by such precision, that
\[ e^{\tilde{G}} \] approximates \( A = e^G \) on \( K \)
with precision \( \varepsilon > 0 \). Here we use the continuity of the exponential series of matrices.

iii) Starlikeness of the product domain: The product domain \( Q \subset \mathbb{C}^n \) is starlike, w.l.o.g. \( 0 \in Q \). For each \( F \in GL(k, \mathcal{O}(Q)) \) we construct a continuous map
\[ \alpha : [0, 1] \to GL(k, \mathcal{O}(Q)) \]
with
\[ \alpha(0) = F \text{ and } \alpha(1) = I : \]
First, we define the continuous map
\[ [0, 1/2] \to GL(k, \mathcal{O}(Q)), t \mapsto F_t, \]
with
\[ F_t(z) := F((1 - 2t) \cdot z), z \in Q. \]
Secondly, Lemma 5.6 provides a path
\[ \beta : [1/2, 1] \to GL(k, \mathbb{C}) \]
satisfying
\[ \beta(1/2) = F(0) \text{ and } \beta(1) = I. \]
Combining both maps provides the continuous map

\[ \alpha : [0, 1] \to GL(k, O(Q)), \quad \alpha(t) := \begin{cases} F_t & t \in [0, 1/2] \\ \beta(t) & t \in [1/2, 1] \end{cases} \]

which in \( GL(k, O(Q)) \) joins \( F \) and the unit matrix \( 1 \).

iv) **Approximation in the multiplicative case: general case.** In order to approximate a general matrix function \( F \in GL(k, O(Q)) \) consider the neighbourhood

\[ U := \{ A \in GL(k, O(Q)) : \| 1 - A \|_{K'} < 1 \} \]

of \( 1 \) in \( GL(k, O(Q)) \) where the logarithm series from part ii) is defined. Note that \( F \) does not necessarily belong to \( U \). Part iii) provides a path

\[ \alpha : [0, 1] \to GL(k, O(Q)) \]

from \( F \) to \( 1 \), see Figure 5.2. The family \( (G \cdot U)_{G \in \alpha([0,1])} \) is an open covering of the compact set \( \alpha([0,1]) \). Hence there exists a finite set of matrices

\[ G_j \in \alpha([0,1]), \quad j = 0, ..., s, \quad \text{with } G_0 = F \text{ and } G_s = 1 \]

with products

\[ G_j \cdot G_{j+1}^{-1} \in U \text{ for } j = 0, ..., s - 1. \]

By construction

\[ F = \prod_{j=0}^{s-1} G_j \cdot G_{j+1}^{-1} \]

**Fig. 5.2** Product splitting of \( F \)

Part ii) provides for each \( j = 0, ..., s - 1 \) a global approximation of the factor

\[ G_j \cdot G_{j+1}^{-1} \in U \]
with the necessary precision on $K$ such that their product provides a global approximation of $F$ with the given precision $\varepsilon > 0$ on $K$. □

**Theorem 5.8 (Cartan’s lemma for holomorphic matrices).** Consider two adjacent product domains

$$Q_1, Q_2 \subset \mathcal{O}(\mathbb{C}^n),$$

an open set $U \subset \mathbb{C}^n$ with

$$(Q_1 \cap Q_2) \subset U$$

and a holomorphic matrix

$$A \in GL(k, \mathcal{O}(U)),$$

see Figure 5.3. Then for $j = 1, 2$ exist open neighbourhoods $U_j$ of $Q_j$ with

$$(U_1 \cap U_2) \subset \subset U$$

and two matrix functions

$$A_j \in GL(k, \mathcal{O}(U_j)), \ j = 1, 2,$$

satisfying on $U_1 \cap U_2$

$$A = A_1 \cdot A_2^{-1}. $$

---

**Fig. 5.3** Cartan’s lemma for invertible matrices
5.1 Proof of Cartan’s lemma

Proof. We choose \( U_j, j = 1, 2 \), as open product domains with

\[
U_0 := (U_1 \cap U_2) \subset \subset U.
\]

Due to the relative compactness the matrix function

\[
A \in GL(k, O(U))
\]

restricts to a bounded matrix function

\[
A|U_0 \in GL(k, B(U_0))
\]

For \( j = 0, 1, 2 \) consider the Banach spaces

\[
E_j := M(k \times k, B(U_j)), \quad j = 0, 1, 2
\]

and the open subsets

\[
E_j^* = GL(k, B(U_j)) \subset E_j.
\]

The idea of the proof is to reformulate the lemma as a claim about the surjectivity of the map

\[
f : E_1^* \times E_2^* \to E_0^*, \quad (A, B) \mapsto A|U_0 \cdot B^{-1}|U_0.
\]

i) Strict differentiability: The map \( f \) is strictly differentiable at the point \( a := (1, 1) \in E_1^* \times E_2^* \).

For the proof consider the linear map

\[
\phi : E_1 \times E_2 \to E_0, (A, B) \mapsto (A - B)|U_0
\]

For two arbitrary pairs \( (A, B), (C, D) \in E_1^* \times E_2^* \) factoring out implies

\[
f(A, B) - f(C, D) - \phi((A, B) - (C, D)) = f(A, B) - f(C, D) - ((A - B) - (C - D)) =
\]

\[
= A \cdot B^{-1} - C \cdot D^{-1} - ((A - C) - (B - D)) =
\]

\[
= (C - B) \cdot D^{-1} \cdot (D - B) \cdot B^{-1} + (A - C) \cdot (B^{-1} - 1) - (B - D)(D^{-1} - 1)
\]

The norm definition

\[
\|(A, B) - (C, D)\| = \|(A - C, B - D)\| = \sup\{\|A - C\|, \|B - D\|\}
\]

provides the estimate

\[
\lim_{(A, B)/(C, D) \to a} \frac{\|f(A, B) - f(C, D) - \phi((A, B) - (C, D))\|}{\|(A, B) - (C, D)\|} \leq
\]
\[
\lim_{(A,B),(C,D) \to a} \left( \frac{\|A - C\| \cdot \|B^{-1} - I\|}{\|A - C\|} + \frac{\|B - D\| \cdot \|D^{-1} - I\|}{\|B - D\|} + \frac{\|C - B\| \cdot \|D^{-1}\| \cdot \|B - D\| \cdot \|B^{-1}\|}{\|B - D\|} \right) = \\
\lim_{(A,B),(C,D) \to a} \left( \|B^{-1} - I\| + \|D^{-1} - I\| + \|C - B\| \cdot \|D^{-1}\| \cdot \|B^{-1}\| \right) = 0
\]

The result also holds for \(A = C\) or \(B = D\) as far as \((A,B) \neq (C,D)\). As a consequence \(f\) is strictly differentiable at \(a\) with tangent map

\[ f'(a) = \phi. \]

ii) Local openness of \(f\): Due to Proposition 5.2 the map \(\phi\) is surjective. Hence Lemma 5.4 implies that \(f\) is open at \(a\): There exists a neighbourhood \(V \subset E^*_0\) of \(f(a) = 1 \in E^*_0\) such that for each \(G \in V\) exists a pair

\[ (G_1, G_2) \in E^*_1 \times E^*_2 \]

with

\[ G = G_1|U_0 \cdot G_2^{-1}|U_0 \]

iii) Holomorphic approximation of matrix functions: Due to Corollary 5.7 the matrix function

\[ A^{-1} \in GL(k, \mathcal{O}(U)) \]

can be approximated on \(\overline{U}_0\) with arbitrary precision by global elements from \(GL(k, \mathcal{O}(\mathbb{C}^n))\): For the distinguished neighbourhood \(V\) of \(1 \in GL(k, \mathcal{O}(U_0))\) from part ii) exists

\[ B \in GL(k, \mathcal{O}(\mathbb{C}^n)) \]

with

\[ (A \cdot B)|U_0 \in V \]

The product splits due to part ii): There exist matrix functions

\[ A_1 \in GL(k, \mathcal{O}(U_1)) \text{ and } \tilde{A}_2 \in GL(k, \mathcal{O}(U_2)) \]

which satisfy on \(U_0\)

\[ A \cdot B = A_1 \cdot \tilde{A}_2^{-1} \]

Set

\[ A_2 := B \cdot \tilde{A}_2 \in GL(k, \mathcal{O}(U_2)). \]

Then on \(U_0\)

\[ A = A_1 \cdot (\tilde{A}_2^{-1} \cdot B^{-1}) = A_1 \cdot A_2^{-1} \]

which finishes the proof. \(\square\)
5.2 Hilbert’s syzygy theorem for coherent $\mathcal{O}$-modules

The present section proves for coherent sheaves the global form of Hilbert’s syzygy Theorem 4.24 after shrinking to a relatively-compact, open polydisc $\Delta$. As a consequence coherent sheaves are acyclic on $\Delta$.

**Proposition 5.9 (Acyclicity of $\mathcal{O}$-modules with a finite free resolution on a polydisc).** If an $\mathcal{O}$-module $\mathcal{F}$ on a polydisc $\Delta \subset \mathbb{C}^n$ has a finite resolution by free sheaves of finite rank

$$0 \to \mathcal{O}^d \to \mathcal{O}^{d-1} \to \cdots \to \mathcal{O}^0 \to \mathcal{F} \to 0,$$

then $\mathcal{F}$ is acyclic.

**Proof.** The proof is by induction on the length $d \in \mathbb{N}$ of the resolution.

i) *Induction start $d = 0$: *Then $\mathcal{F} \simeq \mathcal{O}^p$ and

$$H^j(\Delta, \mathcal{F}) = H^j(\Delta, \mathcal{O}^p)$$

The claim follows because the theorem holds for the structure sheaf $\mathcal{O}$ due to Corollary 3.23.

ii) *Induction step $d - 1 \mapsto d$: *We set

$$\mathcal{K} := \ker [\mathcal{O}^0 \to \mathcal{F}]$$

and obtain the free resolution of the coherent sheaf $\mathcal{K}$ of length $d - 1$

$$0 \to \mathcal{O}^d \to \cdots \to \mathcal{O}^1 \to \mathcal{K} \to 0$$

By induction assumption for all $j \geq 1$

$$H^j(\Delta, \mathcal{K}) = 0$$

The exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}^0 \to \mathcal{F} \to 0$$

provides for each $j \geq 1$ the section of the long exact cohomology sequence

$$0 = H^j(\Delta, \mathcal{O}^0) \to H^j(\Delta, \mathcal{F}) \to H^{j+1}(\Delta, \mathcal{K}) = 0,$$

which proves the induction step and finishes the proof of the proposition.  \( \square \)
Definition 5.10 (Locally free sheaf of finite rank). On a complex manifold $X$ an $\mathcal{O}$-module $\mathcal{L}$ is locally-free of finite rank, if for each point $x \in X$ exist an open neighbourhood $U \subset X$ and an isomorphism of $\mathcal{O}_U$-modules onto a free $\mathcal{O}$-module of finite rank

$$\mathcal{L}|U \xrightarrow{\cong} (\mathcal{O}|U)^k, \quad k \in \mathbb{N}.$$ 

If $k = 1$ then $\mathcal{L}$ is named an invertible sheaf.

Remark 5.11 (Invertible sheaves and line bundles). Consider a complex manifold $X$, an $\mathcal{O}$-module $\mathcal{L}$ and an open covering $\mathcal{U} := (U_j)_{j \in I}$ of $X$ together with a family of isomorphisms

$$\phi_j : \mathcal{L}|U_j \xrightarrow{\cong} (\mathcal{O}|U_j), \quad j \in I.$$ 

Then the family

$$(\phi_j|U_{ij} \circ \phi_i^{-1}|U_{ij})_{i,j} \in \mathcal{Z}^1(\mathcal{U}, \mathcal{O}^\times), \quad U_{ij} := U_i \cap U_j,$$

defines a holomorphic line bundle $\xi$ on $X$ with $\mathcal{L}$ the sheaf of holomorphic sections in $\xi$. In the reverse direction, each holomorphic line bundle $\xi$ on $X$ defines a 1-cocycle with values in $\mathcal{O}^\times$ over a suitable open covering of $X$. Two holomorphic line bundles on $X$ are isomorphic iff their 1-cocycles are cohomologous: The isomorphy classes of holomorphic line bundles on $X$ correspond bijectively to the elements of the cohomology group $H^1(X, \mathcal{O}^\times)$.

Theorem 5.12 (Locally free sheaves after shrinking to a relatively compact polydisc). Consider an open set $U \subset \mathbb{C}^n$ and a relatively compact polydisc

$$\Delta \subset \subset U.$$ 

Then each locally free $\mathcal{O}_U$-module $\mathcal{L}$ of finite rank is free on $\Delta$.

Proof. We may assume $U$ connected. Denote by $k \in \mathbb{N}$ the rank of $\mathcal{L}$.

i) Transformation to product domains: We choose an open polydisc $\Delta' \subset U$ with

$$\Delta \subset \subset \Delta' \subset U.$$ 

Riemann’s mapping theorem provides an open product domain

$$Q' \subset \mathbb{C}^n$$

and a biholomorphic map

$$f : \Delta' \xrightarrow{\cong} Q'.$$

Consider on $Q'$ the locally free sheaf
5.2 Hilbert’s syzygy theorem for coherent $\mathcal{O}$-modules

$$\mathcal{F} := f_\ast (\mathcal{L} | \Delta')$$

There exists a compact product set $\hat{Q} \subset \mathbb{C}^n$ satisfying

$$f(\Delta) \subset \hat{Q} \subset \subset Q',$$

see the following figure.

\[\begin{array}{ccc}
\Delta & & \Delta' \\
\downarrow f|\Delta & & \downarrow f \\
\hat{f}(\Delta) & \subset \hat{Q} \subset \subset Q' \end{array}\]

We show that $\mathcal{F}$ is free in an open neighbourhood of $\hat{Q}$.

ii) Cousin’s principle of induction: To apply Proposition 2.33 we consider the following function

$$A : \mathcal{D}(Q') \rightarrow \{TRUE, FALSE\},$$

which is defined for each product domain $Q \in \mathcal{D}(Q')$ as

$$A(Q) := \begin{cases} 
TRUE & \text{if } \mathcal{F} \simeq \mathcal{O}^k \text{ in an open neighbourhood of } Q \\
FALSE & \text{otherwise} 
\end{cases}$$

The function $A$ satisfies the assumptions of Proposition 2.33:

- Because $\mathcal{F}$ is locally free, each point $x \in Q'$ has an open neighbourhood $V$ such that $A(Q) = TRUE$ for all product domains $Q \in \mathcal{D}(V)$.

- Consider two adjacent product domains

$$Q_1, Q_2 \in \mathcal{D}(Q')$$

satisfying

$$A(Q_1) = A(Q_2) = TRUE.$$

For $j = 1, 2$ exist by assumption open neighbourhoods $U'_j$ of $Q_j$ such that $\mathcal{F}$ is free on $U'_j$: There exist isomorphisms over $U'_j$

$$\phi_j : \mathcal{O}^k \cong \mathcal{F}$$

Over $U'_1 \cap U'_2$ the transition function

$$\phi := \phi_1^{-1} \circ \phi_2 : \mathcal{O}^k \cong \mathcal{O}^k$$

is represented by a holomorphic matrix function.
\[ \tilde{\phi} \in GL(k, \mathcal{O}(U_1' \cap U_2')) \]

Cartan’s lemma for holomorphic matrix functions, Theorem 5.8, provides for \( j = 1, 2 \) open neighbourhoods \( U_j \) of \( Q_j \) with

\[ Q_j \subset U_j \subset U_j' \]

and matrix functions \( \tilde{\Phi}_j \in GL(k, \mathcal{O})(U_j) \) which split \( \tilde{\phi} \) on \( U_1 \cap U_2 \) as

\[ \tilde{\phi} = \tilde{\Phi}_1 \circ \tilde{\Phi}_2^{-1}. \]

The corresponding isomorphisms on \( U_j \)

\[ \phi_j \circ \tilde{\Phi}_j : \mathcal{O} \to \mathcal{F} \]

satisfy on \( U_1 \cap U_2 \)

\[ \phi_1 \circ \tilde{\Phi}_1 = \phi_2 \circ \tilde{\Phi}_2 \]

because

\[ \phi_1^{-1} \circ \phi_2 = \tilde{\Phi}_1 \circ \tilde{\Phi}_2^{-1}. \]

Hence the family

\[ (\phi_j \circ \tilde{\Phi}_j)_{j=1,2} \]

defines on the open neighbourhood \( U_1 \cup U_2 \) of \( Q_1 \cup Q_2 \) an isomorphism

\[ \mathcal{O} \simeq \mathcal{F}, \]

i.e. \( A(Q_1 \cup Q_2) = TRUE \).

Proposition 2.33 implies

\[ A(Q) = TRUE \]

for each \( Q \in \mathcal{D}(Q') \). In particular

\[ A(\hat{Q}) = TRUE. \]

Hence the restriction \( \mathcal{F}|f(\Delta) \) is free, which implies that \( \mathcal{L} \) is free because

\[ f|\Delta : \Delta \simeq f(\Delta) \]

is a holomorphic isomorphism. \( \square \)

**Definition 5.13 (Homological dimension of coherent \( \mathcal{O} \)-modules).** Consider a complex manifold \( X \) and a coherent \( \mathcal{O} \)-module \( \mathcal{F} \). The **homological dimension** of \( \mathcal{F} \) is defined as the supremum of the homological dimension of its stalks in the sense of Definition 4.22

\[ \text{hd}_{\mathcal{O}} \mathcal{F} := \sup \{ \text{hd}_{\mathcal{O}_x} \mathcal{F}_x : x \in X \}. \]
5.2 Hilbert’s syzygy theorem for coherent $\mathcal{O}$-modules

Theorem 4.24 implies that on an $n$-dimensional complex manifold each coherent $\mathcal{O}$-module $\mathcal{F}$ has finite homological dimension

$$hd_{\mathcal{O}}\mathcal{F} \leq n.$$ 

Theorem 5.14 (Hilbert’s syzygy theorem for a coherent $\mathcal{O}$-module after shrinking to a polydisc). Consider an open set $U \subset \mathbb{C}^n$ and a relatively compact, open polydisc

$$\Delta \subset \subset U.$$ 

Each coherent $\mathcal{O}$-module $\mathcal{F}$ on $U$ with homological dimension

$$d := hd_{\mathcal{O}}\mathcal{F}$$

has over $\Delta$ a resolution of length $d$ by free $\mathcal{O}$-modules of finite rank

$$0 \rightarrow \mathcal{O}^{k_d} \rightarrow \mathcal{O}^{k_{d-1}} \rightarrow \ldots \rightarrow \mathcal{O}^{k_0} \rightarrow \mathcal{F} \rightarrow 0.$$ 

Proof. The claim is proved by induction on $d$.

i) Induction start for $d = 0$: Due to Theorem 5.12 it suffices to prove that $\mathcal{F}$ is locally free on $\Delta$. Consider an arbitrary point $x \in U$. Because

$$hd_{\mathcal{O}_x}\mathcal{F}_x = 0$$

there exists a $k \in \mathbb{N}$ and an isomorphism of stalks

$$\alpha_x : \mathcal{O}_x^k \cong \mathcal{F}_x$$

We denote by

$$(e_j \in \mathcal{O}^k(U))_{j=1,\ldots,k}$$

the canonical $\mathcal{O}(U)$-basis of $\mathcal{O}^k(U)$ and consider the germs

$$f_{j,x} := \alpha_x(e_{j,x}) \in \mathcal{F}_x$$

There exists a neighbourhood $V \subset U$ of $x$ and sections

$$f_j \in \mathcal{F}(V), \ j = 1,\ldots,k,$$

representing $f_{j,x}$. Those sections define an $\mathcal{O}$-module morphisms over $V$

$$\alpha : \mathcal{O}^k|V \rightarrow \mathcal{F}|V, \ \alpha(e_j) := f_j, \ j = 1,\ldots,k,$$

on sections, which extends the primary isomorphism on the stalks at $x$

$$\alpha_x : \mathcal{O}_x^k \cong \mathcal{F}_x.$$
The coherence of $\mathcal{F}$ implies due to Corollary 4.32 the existence of a neighbourhood $W \subset V$ of $x$ such that the restriction
\[ \alpha|_W : \mathcal{O}^k|_W \to \mathcal{F}|_W \]
is an isomorphism of sheaves, i.e. it induces an isomorphism on the stalks for all points $y \in W$.

ii) Preparing the induction step: For the induction step in part iii) we need an argument to carry over the surjectivity of a sheaf morphism over a polydisc $Y$
\[ \alpha : \mathcal{O}^k \to \mathcal{F}, \]
i.e. the surjectivity on the level of stalks, to the surjectivity of the induced morphism of sections over $Y$
\[ \alpha_Y : \mathcal{O}(Y)^k \to \mathcal{F}(Y). \]
The argument will be the vanishing of a first cohomology group:

Assume: The claim of the theorem holds for $d-1$. Consider a relatively compact, open polydisc $Y \subset \subset U$ and over $Y$ an epimorphism of sheaves
\[ \alpha : \mathcal{O}^k \to \mathcal{F}. \]
If $hd_{\mathcal{O}_U, \mathcal{F}} = d$ then the $\mathcal{O}_U$-module
\[ \mathcal{K} := ker \left[ \mathcal{O}^k \to \mathcal{F} \right] \]
fits into the short exact sequence
\[ 0 \to \mathcal{K} \to \mathcal{O}^k \xrightarrow{\alpha} \mathcal{F} \to 0 \]
and has homological dimension
\[ hd_{\mathcal{O}_U, \mathcal{K}} < d \]
as a consequence of Proposition 4.23. Hence by induction assumption $\mathcal{K}$ has a free resolution on $Y$. Then Proposition 5.9 implies the vanishing
\[ H^1(Y, \mathcal{K}) = 0 \]
The segment from the long exact cohomology sequence
\[ \mathcal{O}^k(Y) \xrightarrow{\alpha_Y} \mathcal{F}(Y) \to H^1(Y, \mathcal{K}) = 0 \]
implies the surjectivity of the morphism of sections
5.2 Hilbert’s syzygy theorem for coherent $\mathcal{O}$-modules

$$\alpha_Y: \mathcal{O}^k(Y) \rightarrow \mathcal{F}(Y).$$

iii) *Induction step* $d - 1 \rightarrow d$: Consider a coherent $\mathcal{O}_U$-module $\mathcal{F}$ with

$$\text{hd}_{\mathcal{O}_U}\mathcal{F} = d \geq 1$$

We choose a polydisc $\Delta'$ with

$$\Delta \subset \subset \Delta' \subset \subset U$$

and construct an $\mathcal{O}$-module epimorphism over $\Delta'$

$$\mathcal{O}^k|\Delta' \rightarrow \mathcal{F}|\Delta' \rightarrow 0 :$$

The construction uses Cousin’s principle of induction. We consider the function

$$A : \mathcal{D}(U) \rightarrow \{\text{TRUE}, \text{FALSE}\}$$

which is defined for $Q \in \mathcal{D}(U)$ as

$$A(Q) := \begin{cases} 
\text{TRUE} & \text{There exists an epimorphism } \mathcal{O}^k \rightarrow \mathcal{F} \text{ in an open neighbourhood of } Q \\
\text{FALSE} & \text{otherwise}
\end{cases}$$

The function $A$ satisfies the assumptions of Proposition 2.33:

- The coherence of $\mathcal{F}$ implies that each point $x \in U$ has an open neighbourhood $V$ and for each $Q \in \mathcal{D}(V)$ an epimorphism over $V$

$$\mathcal{O}^k|V \rightarrow \mathcal{F}|V.$$  

- For $j = 1, 2$ consider two compact adjacent product domains $Q_j \in \mathcal{D}(U)$ with

$$A(Q_j) = \text{TRUE}.$$  

Hence for $j = 1, 2$ exist open neighbourhoods $U_j \supset Q_j$ with epimorphisms of sheaves over $U_j$

$$\mathcal{O}^k_j \xrightarrow{\alpha_j} \mathcal{F}.$$  

On the level of sections we denote for $j = 1$ by

$$f_1 := (f_{1,1}, \ldots, f_{1,k_1}) \in \mathcal{F}^{k_1}(U_1)$$

the family of images under $\alpha_1$ of the elements of the canonical base of $\mathcal{O}^{k_1}(U_1)$, and analogously for $j = 2$ the family

$$f_2 := (f_{2,1}, \ldots, f_{2,k_2}) \in \mathcal{F}^{k_2}(U_2)$$

For each
$x \in U_1 \cap U_2$

the germs of the components of $f_1$ as well as the germs of the components of $f_2$
generate the stalk $\mathcal{F}_x$ over $\mathcal{O}_x$. Part ii) implies the existence of an open polydisc

$$V \subset \subset (U_1 \cap U_2)$$

such that for $j = 1, 2$ each sheaf morphism $\alpha_j$ induces an epimorphism of sections

$$\alpha_{j,V} : \mathcal{O}^k(V) \to \mathcal{F}(V)$$

We obtain two holomorphic matrix functions

$$A_1 \in M(k_2 \times k_1, \mathcal{O}(V)) \text{ and } A_2 \in M(k_1 \times k_2, \mathcal{O}(V))$$

satisfying

$$f_2 \cdot A_1 = f_1 \text{ and } f_1 \cdot A_2 = f_2.$$  

The two matrix equations combine to one single equation of $(k_1 + k_2)$-tuples

$$(0, f_2) = (f_1, 0) \cdot C$$

with the matrix function

$$C := \begin{pmatrix} 1 - A_2 \cdot A_1 & A_2 \\ -A_1 & 1 \end{pmatrix} \in M((k_1 + k_2) \times (k_1 + k_2), \mathcal{O}(V)).$$

The matrix is invertible

$$C \in GL(k_1 + k_2, \mathcal{O}(V))$$

with inverse

$$C^{-1} = \begin{pmatrix} 1 & -A_2 \\ A_1 & 1 - A_1 \cdot A_2 \end{pmatrix}$$

Cartan’s lemma, Theorem 5.8, applies to the matrix function $C$ and provides for $j = 1, 2$
two open neighbourhoods

$$V_j \subset U_j$$

of $Q_j$ with $V_1 \cap V_2 \subset V$

and corresponding matrix functions

$$C_j \in GL(k_1 + k_2, \mathcal{O}(V_j)), \; j = 1, 2,$$

which satisfy on $V_1 \cap V_2$

$$C = \begin{pmatrix} 1 - A_2 \cdot A_1 & A_2 \\ -A_1 & 1 \end{pmatrix} = C_1 \cdot C_2^{-1}.$$
5.2 Hilbert’s syzygy theorem for coherent \( \mathcal{O} \)-modules

Hence over \( V_1 \cap V_2 \)

\[
(0, f_2) = (f_1, 0) \cdot C = (f_1, 0) \cdot C_1 \cdot C_2^{-1} \implies (0, f_2) \cdot C_2 = (f_1, 0) \cdot C_1
\]
or

\[
f_2 \cdot C_2 = f_1 \cdot C_1'
\]

with each matrix \( C_j', \ j = 1, 2 \), derived from two blocks of the corresponding matrix \( C_j \). On the union \( V_1 \cup V_2 \) both families

\[
f_1 \cdot C_1' \text{ and } f_2 \cdot C_2'
\]

combine to a family of sections

\[
F := (F_1, ..., F_{k_1+k_2}) \in \mathcal{F}^{k_1+k_2}(V_1 \cup V_2)
\]
satisfying

\[
(0, f_2) \cdot C_2 = F \text{ over } V_2 \text{ and } (f_1, 0) \cdot C_1 = F \text{ over } V_1
\]

Each matrix function \( C_j, \ j = 1, 2 \) is invertible. Hence the germs of \( F \) generate the stalk \( \mathcal{F}_x \) for each point \( x \in V_1 \cup V_2 \). The family provides over \( V_1 \cup V_2 \) the epimorphism of sheaves

\[
\mathcal{O}^{k_1+k_2} \xrightarrow{F} \mathcal{F}.
\]

Hence

\[
A(Q_1 \cup Q_2) = TRUE.
\]

Cousin’s principle of induction ensures that each product domain \( Q \in \mathcal{D}(U) \) has an epimorphism of \( \mathcal{O} \)-modules

\[
\mathcal{O}^k \xrightarrow{F} \mathcal{F}
\]

over a suitable open neighbourhood of \( Q \). The \( \mathcal{O} \)-module

\[
\mathcal{K} := ker [\mathcal{O}^k \rightarrow \mathcal{F}]
\]
satisfies

\[
hd \mathcal{K} = (hd \mathcal{F}) - 1
\]
due to Proposition 4.23. Hence the induction assumption applied to \( \mathcal{K} \) finishes the induction step and terminates the proof of the theorem. \( \square \)

Corollary 5.15 shows the consequences of Hilbert’s syzygy theorem for the cohomology of coherent sheaves on polydiscs. The corollary will play an important role in Chapter 6 for the proof of Theorem B.

**Corollary 5.15 (Acyclicity of coherent sheaves after shrinking to a relatively compact, open polydisc).** Consider a relatively compact polydisc

\[
\Delta \subset \subset \mathbb{C}^n
\]
and a coherent sheaf $\mathcal{F}$ defined in an open neighbourhood of $\overline{X}$. Then the restriction $\mathcal{F}|_{\Delta}$ is acyclic.

Proof. The corollary follows from Theorem 5.14 and Proposition 5.9. □

5.3 Fréchet topology in the context of cohomology

Definition 5.16 (Fréchet sheaf). A sheaf of complex vector spaces $\mathcal{F}$ on a topological space $X$ is a Fréchet sheaf if for each open $U \subset X$ the vector space of sections $\mathcal{F}(U)$ is a Fréchet space and the restrictions

$$\mathcal{F}(U) \to \mathcal{F}(V), \quad V \subset U \text{ open},$$

are continuous linear maps.

The structure sheaf $\mathcal{O}$ of a complex manifold $X$ is a Fréchet sheaf when providing the vector spaces $\mathcal{O}(U), \quad U \subset X \text{ open}$, with the topology of compact convergence. The result follows from Proposition 1.19.

Proposition 5.17 (Coherent sheaves as Fréchet sheaves). On a complex manifold $X$ each coherent $\mathcal{O}$-module $\mathcal{F}$ is a Fréchet sheaf.

Proof. i) Distinguished basis of the topology: We choose a basis $\mathcal{B}$ of the topology of $X$ by open sets $U \subset X$ with the following property: For each $U \in \mathcal{B}$ exists an open set

$$U' \subset X \text{ with } U \subset U'$$

and a biholomorphic map

$$\alpha : U' \to \Delta'$$

onto a polydisc $\Delta'$ such that

$$\Delta := \alpha(U) \subset \subset \Delta'$$

is a relatively-compact, concentric polydisc. The family $\mathcal{B}$ can be obtained by choosing first a locally finite covering $(U'_i)_{i \in I}$ by basis elements, and then selecting the family $(U_i)_{i \in I}$ as a suitable shrinking, see Remark 2.3.

Theorem 5.14 provides over $U$ an epimorphism

$$\mathcal{O}^k \to \mathcal{F} \to 0.$$
The kernel
\[ \mathcal{K} := \ker \phi \]
is coherent according to Proposition 4.31. Corollary 5.15 implies
\[ H^1(U, \mathcal{F}) = 0. \]
The exact sequence of vector spaces
\[ 0 \to \mathcal{K}(U) \to \mathcal{O}^k(U) \to \mathcal{F}(U) \to 0 \]
shows the algebraic isomorphy of vector spaces
\[ \mathcal{F}(U) \simeq \mathcal{O}^k(U)/\mathcal{K}(U). \]

ii) *Closedness of* \( \mathcal{K}(U) \subset \mathcal{O}^k(U) \): One has to show: For a sequence \( (f_\nu)_{\nu \in \mathbb{N}} \) of \( k \)-tuples of sections
\[ f_\nu \in \mathcal{K}(U) \subset \mathcal{O}^k(U), \ \nu \in \mathbb{N}, \]
with limit
\[ f := \lim_{\nu \to \infty} f_\nu \in \mathcal{O}^k(U) \]
holds
\[ f \in \mathcal{K}(U). \]
For a section \( f \in \mathcal{O}^k(U) \) holds the equivalence
\[ f \in \mathcal{K}(U) \iff f_x \in \mathcal{K}_x \text{ for all } x \in U \]
because \( \mathcal{K} \) is a sheaf. For given \( x \in U \) set
\[ R := \mathcal{O}_x, \ m \in R \text{ the maximal ideal}, \ M := \mathcal{K}_x \]
Due to the Krull lemma, see Proposition 4.21,
\[ M = \bigcap_{s \in \mathbb{N}^*} (M + m^s \cdot R^k) \]
Hence it suffices to show: For each \( s \in \mathbb{N}^* \)
\[ f_x \in M + m^s \cdot R^k \]
For given \( s \in \mathbb{N}^* \) consider the residue map
\[ \pi_s : R^k \to R^k/m^s \cdot R^k \]
with
\[ R^k/m^s \cdot R^k \simeq (P_{s-1})^k \text{ with } P_{s-1} \subset \mathbb{C}[z_1, \ldots, z_n] \]
the vector space of all polynomials of degree \( \leq s - 1 \). The vector space is finite-dimensional. Therefore it has a unique structure of a Hausdorff topological vector space, and its subspace

\[
\pi_s(M) \subset (P_{s-1})^k
\]

is closed. The Cauchy integral formula, Corollary 1.7, implies: For arbitrary but fixed \( I \in \mathbb{N}^k \) and \( j \in \{1, \ldots, k\} \) the sequence of \( I \)-th coefficients of the power series of the \( j \)-th components of the sequence \( (f_{\nu,x})_\nu \) are convergent towards the \( I \)-th coefficient of the \( j \)-th component of \( f_x \). Hence

\[
\pi_s(f_x) = \lim_{\nu \to \infty} \pi_s(f_{\nu,x}) \in \pi_s(M)
\]

which implies

\[
f_x \in M + m^s \cdot \mathbb{R}^k
\]

Because \( s \in \mathbb{N}^+ \) was arbitrary, the result finishes the proof of the claim \( f_x \in M \). As a consequence

\[
K(U) \subset \mathcal{O}^k(U)
\]

is closed, and the quotient

\[
\mathcal{F}(U) = \mathcal{O}^k(U) / K(U)
\]

provided with the quotient topology becomes a Fréchet space.

iii) **Independence of the topology**: We have to show that the topology on \( \mathcal{F}(U) \) does not depend on the choice of a resolution of \( \mathcal{F} \), i.e. on the choice of generators \( f_1, \ldots, f_k \in \mathcal{F}(U) \). Hence we assume a second choice \( f_{k+1}, \ldots, f_m \in \mathcal{F}(U) \) of generators. We show that the two families

\[
(f_1, \ldots, f_k) \text{ and } (f_1, \ldots, f_k, f_{k+1}, \ldots, f_m)
\]

induce the same topology: The injection

\[
\mathcal{O}^k(U) \hookrightarrow \mathcal{O}^m(U), \ e_j \mapsto e_j, \ j = 1, \ldots, k,
\]

induces the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{K}(U) & \longrightarrow & \mathcal{O}^k(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \text{id} & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{K}'(U) & \longrightarrow & \mathcal{O}^m(U) & \longrightarrow & \mathcal{F}(U)' & \longrightarrow & 0
\end{array}
\]

Here \( \mathcal{F}(U)' \) denotes the quotient topology on the vector space \( \mathcal{F}(U) \) induced from the lower row. The map
5.3 Fréchet topology in the context of cohomology

\[ \text{id} : \mathcal{F}(U) \to \mathcal{F}(U)' \]

is continuous and surjective between Fréchet spaces, hence a homeomorphism due to Remark 1.18.

iv) **Continuity of the restriction maps:** Consider two open sets

\[ U, V \in \mathcal{B} \text{ with } V \subset U \]

and let

\[ \phi : \mathcal{F}(U) \to \mathcal{F}(V) \]

be the restriction. We obtain the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{K}(U) & \longrightarrow & \mathcal{O}^k(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & 0 \\
& & \downarrow{\phi_1} & & \downarrow{\phi_2} & & \downarrow{\phi_3} & \\
0 & \longrightarrow & \mathcal{K}(V) & \longrightarrow & \mathcal{O}^k(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & 0
\end{array}
\]

The map \( \phi_2 \) is continuous because \( \mathcal{O} \) is a Fréchet sheaf due to Proposition 1.19. Hence also the restriction \( \phi_1 \) is continuous. Diagram chasing and the definition of the quotient topology on \( \mathcal{F}(U) \) implies the continuity of \( \phi_3 \).

v) **General open sets:** An arbitrary open set \( U \) can be represented as

\[ U = \bigcup_{i \in I} U_i \]

with a countable index set \( I \) and open sets

\[ U_i \in \mathcal{B}, \ i \in I. \]

The canonical injection of vector spaces

\[ \lambda : \mathcal{F}(U) \hookrightarrow \prod_{i \in I} \mathcal{F}(U_i), \ f \mapsto (f|U_i)_i \]

has a closed image: The idea is to represent the image as a kernel. For each given pair \( (i, j) \in I \times I \) the open intersection

\[ U_{ij} := U_i \cap U_j \]

can be represented as

\[ U_{ij} = \bigcup_{U_{ij}, v \in \mathcal{B}} U_{ij, v}, \ U_{ij, v} := U_{ij} \cap U_v \]
The continuous map
\[ \kappa : \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j,v} \mathcal{F}(U_{ij,v}), \quad (f_i)_i \mapsto ((f_i - f_j)(U_{ij,v})_{i,j,v}, \]

satisfies
\[ \ker \kappa = \text{im} \lambda. \]

Hence
\[ \mathcal{F}(U) = \text{im} \lambda = \ker \kappa \subset \prod_{i \in I} \mathcal{F}(U_i) \]

is a closed subspace, and therefore a Fréchet space itself with respect to the induced topology. \( \square \)

**Corollary 5.18 (Fréchet topology on cochains and cocycles).** Consider a complex manifold \( X \) and a coherent \( \mathcal{O} \)-module \( \mathcal{F} \). For each countable open covering \( \mathcal{U} = (U_i)_{i \in I} \) of \( X \) and each \( q \in \mathbb{N} \) the group of Čech cochains
\[ C^q(\mathcal{U}, \mathcal{F}) \]

are Fréchet spaces and the coboundary maps are continuous. The subgroup of cocycles
\[ Z^q(\mathcal{U}, \mathcal{F}) \subset C^q(\mathcal{U}, \mathcal{F}) \]

is a closed subspace, hence a Fréchet space too.

Note that the subspace of coboundaries
\[ B^q(\mathcal{U}, \mathcal{F}) \subset C^q(\mathcal{U}, \mathcal{F}) \]

is not necessarily closed. As a consequence the quotient \( H^q(\mathcal{U}, \mathcal{F}) \) is not necessarily a Fréchet space; see [14, Kap. V, §6.5 Bem.]
Chapter 6
Theorem B and Theorem A on Stein manifolds

The present chapter defines the concept of a Stein manifold, Definition 6.10 and proves the main result on Stein manifolds, Theorem B, see Theorem 6.22, as well as some of its fundamental consequences. The theorem will be proved in the following steps:

• Theor. 4.28 (Oka’s coherence theorem)
• Theor. 6.13 (Embedding analytic polyhedra of a Stein manifold)
• Theor. 5.14 (Hilbert’s syzygy theorem)
• Prop. 6.16 (Theorem B for analytic polyhedra)
• Prop. 6.12 (Exhaustion by analytic polyhedra)
• Theor. 6.21 (Runge approximation for coherent sheaves)
• Prop. 1.26 (Mittag-Leffler principle for projective limes)
• Theor. 3.19 (Leray’s theorem)

The following diagram shows always for a given result the sufficient combination of prerequisites.

Theor. 4.28  Theor. 6.13  Theor. 5.14  Prop. 6.12  Theor. 6.21  Prop. 1.26  Theor. 3.19

Prop. 6.16

Theor. 6.22
6.1 Holomorphic convexity and analytic polyhedra

We start our investigation with two fundamental concepts from complex analysis:

- **Domain of holomorphy**: Characterize those domains $G \subset \mathbb{C}^n$ which are a domain of holomorphy, i.e. the domain of a holomorphic function which does not extend across any boundary point. During the first half of the 20th century this problem was the main challenge for complex analysis in several variables.

- **Holomorphic approximation**: Find conditions for a pair $(X, Y)$ with a complex manifold $X$ and an open $Y \subset X$ which ensure: The canonical restriction of Fréchet spaces
  $$\mathcal{O}(X) \rightarrow \mathcal{O}(Y)$$
  has dense image.

We first investigate domains of holomorphy. We consider holomorphic convexity as a tool, to test a domain for being a domain of holomorphy.

**Definition 6.1 (Domain of holomorphy and holomorphic convexity).**

1. A domain $G \subset \mathbb{C}^n$ is a **domain of holomorphy** if there exists a holomorphic function $f \in \mathcal{O}(G)$ which has no holomorphic extension over any boundary point $x \in \partial G$.

Consider a complex manifold $X$.

2. The **holomorphically convex hull** of a compact set $K \subset X$ is the set
   $$\hat{K} := \hat{K}_X := \bigcap_{f \in \mathcal{O}(X)} \{ x \in X : |f(x)| \leq \|f\|_K \}.$$

3. The manifold $X$ is **holomorphically convex** if each compact set $K \subset X$ has a compact holomorphically convex hull $\hat{K}$.

4. An open subset $Y \subset X$ is **relatively-holomorphically convex with respect to $X$** if for each compact $K \subset Y$ the relatively-holomorphically convex hull
   $$\hat{K}_{X,Y} := \bigcap_{f \in \mathcal{O}(X)} \{ y \in Y : |f(y)| \leq \|f\|_K \}$$
   is compact.

For a relatively-holomorphically convex open subset $Y \subset X$ one builds the holomorphically convex hull $\hat{K}_{X,Y}$ only with those global functions from $\mathcal{O}(Y)$ which are restrictions of holomorphic functions on $X$. 
Remark 6.2 (Holomorphically convex).

1. The holomorphic convex hull $\hat{K}$ of a compact set $K \subset X$ is the subset of $X$ such for each global holomorphic function $f \in \mathcal{O}(X)$ the modulus $|f|$ on $\hat{K}$ is bounded by its maximum on $K$.

2. Due to the continuity of holomorphic functions, for a compact $K \subset X$ the holomorphically convex hull $\hat{K} \subset X$ is closed. The fact that in locally compact topological spaces each closed subset of a compact set is compact itself implies: If an open $Y \subset X$ is relatively-holomorphically convex with respect to $X$ then $Y$ is even holomorphically convex.

Proposition 6.3 proves a useful criterion for holomorphic convexity.

**Proposition 6.3 (Holomorphic convexity).** A complex manifold $X$ is holomorphically convex if it satisfies the following property: For each discrete infinite closed subset $D \subset X$ exists a holomorphic function $f \in \mathcal{O}(X)$ which is unbounded on $D$, i.e.

$$\sup \{|f(x)| : x \in D\} = \infty.$$ 

**Proof.** We have to show that under the assumptions of the proposition holds

$K \subset X$ compact $\implies \hat{K} \subset X$ compact.

Compactness of $K \subset X$ implies the finiteness

$$\|f\|_K < \infty$$

To verify compactness of $\hat{K}$ it is enough to show that $\hat{K}$ is sequentially compact, because $X$ has a second countable topology. Therefore we prove that $\hat{K}$ does not contain a closed, infinite subset which is discrete. The proof is by contradiction: For a given discrete infinite closed subset $D \subset X$ exists by assumption a holomorphic function $f \in \mathcal{O}(X)$ with

$$\sup \{x \in D : |f(x)|\} = \infty > \|f\|_K$$

Hence for a suitable element $x_0 \in D$

$$|f(x_0)| > \|f\|_K$$

and therefore

$$x_0 \notin \{x \in X : |f(x)| \leq \|f\|_K\}$$

In particular

$$x_0 \notin \bigcap_{g \in \mathcal{O}(X)} \{x \in X : |g(x)| \leq \|g\|_K\} = \hat{K}.$$

$\square$
The sufficient condition for holomorphic convexity from Proposition 6.3 is also necessary: For each discrete infinite subset $D \subset X$ of a holomorphically complex manifold $X$ exists a holomorphic function $f \in \mathcal{O}(X)$ which is unbounded on $D$, see [14, Kap. IV, §2, Satz 12] and for the affine case [16, Satz 5.6].

The existence of domains in $\mathbb{C}^n$, $n \geq 2$, which are not holomorphically convex, distinguishes complex analysis in several variables from the 1-dimensional theory.

**Example 6.4 (Holomorphic convexity).**

1. Each domain $X \subset \mathbb{C}$ is holomorphically convex.

   For an indirect proof assume the existence of a compact set $K \subset X$ with $\hat{K}$ not compact. The set $\hat{K}$ is closed. There exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $\hat{K}$ with pointwise distinct elements without accumulation point.

   - If the sequence $(x_n)_{n \in \mathbb{N}}$ is unbounded then the holomorphic function
     
     $$ f \in \mathcal{O}(X) \text{ defined as } f(z) := z, \ z \in X,$$

     is unbounded, a contradiction to
     
     $$ \infty = \sup \{ |f(x_n)| : n \in \mathbb{N} \} \leq \sup \{ |f(x)| : x \in \hat{K} \} \leq \|f\|_K < \infty.$$

   - If the sequence $(x_n)_{n \in \mathbb{N}}$ is bounded then it has an accumulation point $a \in \partial X$ due to the Heine-Borel theorem. The openness of $X$ implies $a \notin X$. The holomorphic function
     
     $$ f \in \mathcal{O}(X) \text{ defined as } f(z) := \frac{1}{z-a}, \ z \in X,$$

     is unbounded which implies an analogous contradiction.

2. Each polydisc $\Delta \subset \mathbb{C}^n$ is holomorphically convex: Denote by

     $$ r = (r_1, \ldots, r_n) $$

the polyradius of $\Delta$. We apply Proposition 6.3: Consider a discrete infinite closed sequence $(a_v)_{v \in \mathbb{N}}$ in $\Delta$. W.l.o.g. $(a_v)_{v \in \mathbb{N}}$ is convergent towards a point

$$ a = (a_1, \ldots, a_n) \in \overline{\Delta} $$

because $\overline{\Delta}$ is compact. Because the sequence forms a discrete set and is closed in $\Delta$, the limit satisfies $a \notin \Delta$. Hence for a suitable $j \in \{1, \ldots, n\}$ holds $a_j = r_j$. Then

$$ f \in \mathcal{O}(\Delta), \ f(z_1, \ldots, z_n) := \frac{1}{z_j - r_j} $$
is unbounded on $\Delta$.

3. The Hartogs figure from Figure 1.3

$$H := U \times S \cup G \times R \subset \mathbb{C}^2$$

with its radii $r_1 < r_2$ is not a domain of holomorphy due to Theorem 1.14: Each holomorphic function $f \in \mathcal{O}(H)$ extends to a holomorphic function

$$\tilde{f} \in \mathcal{O}(G \times S).$$

Note: The Hartogs figure $H \subset \mathbb{C}$ is not holomorphically convex: Using the annotation from Figure 1.3 choose $a \in G \setminus U$ and $\rho > 0$ satisfying

$$r_1 < \rho < r_2,$$

and consider the compact subset of $H$

$$K := H \cap \{(a, w) \in \mathbb{C}^2 : |w| = \rho\} \simeq \partial(\Delta(\rho)),$$

which is biholomorphic to the boundary of the circle of radius $= \rho$.

Theorem 1.14 implies

$$\mathcal{O}(H) = \mathcal{O}(G \times S)$$

Hence the restriction

$$\tilde{f}(a, -) : \Delta(r_2) \to \mathbb{C}$$

is a well-defined holomorphic function, satisfying

$$\|f(a, -)\|_{r_1 < |w| \leq \rho} = \|\tilde{f}(a, -)\|_{\Delta(\rho)} \leq \|\tilde{f}(a, -)\|_{\partial(\Delta(\rho))} = \|f\|_K$$

due to the maximum principle of complex analysis of one variable. Choose a sequence $(w_\nu)_{\nu \in \mathbb{N}}$ of elements $w_\nu \in \Delta(\rho)$, $\nu \in \mathbb{N}$, with

$$\lim_{\nu \to \infty} w_\nu = r_1$$

Then for all $\nu \in \mathbb{N}$

$$(a, w_\nu) \in \hat{K} \text{ but } \lim_{\nu \to \infty} (a, w_\nu) = (a, r_1) \notin H$$

in particular

$$\lim_{\nu \to \infty} (a, w_\nu) \notin \hat{K},$$

which proves that $\hat{K}$ is not compact. Hence $H$ is not holomorphically convex.

The interest for holomorphically convex domains $X \subset \mathbb{C}^n$ results from the fact that these domains are exactly the domains of holomorphy in $\mathbb{C}^n$. 


Remark 6.5 (Domain of holomorphy).

1. For a domain $X \subset \mathbb{C}^n$ are equivalent:
   - The set $X$ is a domain of holomorphy.
   - The set $X$ is holomorphically convex.
   - The set $X$ satisfies the sufficient condition from Proposition 6.3.
   - The set $X$ is Hartogs-pseudoconvex, see Figure 6.1 for a counter example.

\[\text{Fig. 6.1 Hartogs figure } H \subset X \text{ which hinders } X \text{ being Hartogs-pseudoconvex.}\]

2. A domain $X \subset \mathbb{C}^n$ with a boundary $\partial X$, locally defined by a $C^2$-function $\phi$, is a domain of holomorphy iff the Levi form $L_x \phi$ is positive semi-definite at each boundary point $x \in \partial X$, see Figure 6.2.

\[\text{Fig. 6.2 Local description of the boundary}\]
6.1 Holomorphic convexity and analytic polyhedra

Concerning the proof of Remark 6.5: For part 1) see [16, Satz 6.2] and also [10, Sect. 2.1]. For part 2) cf. [10, Theor. 2.1.4].

Complex polydiscs are distinguished open subsets of the affine space $\mathbb{C}^n$. According to Corollary 3.23 the structure sheaf of a polydisc $\Delta$ is acyclic. On a general complex manifold $X$ the concept of a polydisc is not well-defined because the concept refers to the coordinate functions of $\mathbb{C}^n$.

We will see in Section 6.2: On a Stein manifold there are enough global holomorphic functions to replace the concept of a polydisc by the concept of an analytic polyhedron, see also [20, 12.10 Exampl.]. An analytic polyhedron is a global concept. Its definition is independent from any choice of coordinates. Analytic polyhedra turn out as the main tool to prove Theorem B on Stein manifolds. We will show:

- Proposition 6.12: Each Stein manifold can be exhausted by a sequence of analytic polyhedra.
- Theorem 6.13: Analytic polyhedra of a Stein manifold embed as analytic submanifolds into polydiscs. Hence the vanishing result from the local theory for polydiscs, Corollary 5.15, carries over to analytic polyhedra.
- Theorem 6.21: For relatively-holomorphically convex subdomains $Y$ of a Stein manifold $X$ holds an approximation theorem even for sections of coherent sheaves.

**Definition 6.6 (Analytic polyhedron).** Consider a complex manifold $X$. A relatively compact, open subset

$$P \subset \subset X$$

is an analytic polyhedron in $X$ if there exist an open neighbourhood $U$ of $P$ and finitely many holomorphic functions

$$f_1, \ldots, f_k \in \mathcal{O}(X)$$

such that

$$P = \bigcap_{j=1}^{k} \left\{ x \in U : |f_j(x)| < 1 \right\}$$

Note in Definition 6.6 that the functions $f_j \in \mathcal{O}(X)$ are defined on the whole manifold $X$. Being an analytic polyhedron is a relative concept. Whether a set is an analytic polyhedron depends on the choice of the ambient manifold to consider.
Example 6.7 (Analytic polyhedron).

1. Each relatively compact, open polydisc

$$\Delta \subset \subset \mathbb{C}^n$$

is an analytic polyhedron in $\mathbb{C}^n$: If

$$\Delta = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < r_j \text{ for } j = 1, \ldots, n\}$$

then consider

$$f_j(z) := \frac{z_j}{r_j} \in \mathcal{O}(\mathbb{C}^n), \; j = 1, \ldots, n$$

As a consequence

$$\Delta = \bigcap_{j=1}^n \{z \in \mathbb{C}^n : |f_j(z)| < 1\},$$

one may choose $U = \mathbb{C}^n$.

2. The 1-dimensional annulus

$$P := \{z \in \mathbb{C} : 1 < |z| < 2\}$$

is an analytic polyhedron in $\mathbb{C}^1$ being the intersection of two analytic polyhedra in $\mathbb{C}^1$

$$P = \{z \in \mathbb{C}^1 : |z| < 2\} \cap \{z \in \mathbb{C}^1 : |1/z| < 1\}$$

But $P$ is not an analytic polyhedron in $\mathbb{C}$ due to the maximum principle.

Proposition 6.8 (Analytic polyhedra are relatively-holomorphically convex).

Consider a complex manifold $X$ and an analytic polyhedron $P \subset \subset X$. Then $P$ is relatively-holomorphically convex with respect to $X$.

Proof. By definition exists an open set $U \subset X$ and finitely many holomorphic functions

$$f_j \in \mathcal{O}(X), \; j = 1, \ldots, k,$$

such that

$$P = \{x \in U : |f_j(x)| < 1 \text{ for all } j = 1, \ldots, k\} \subset \subset U$$

Consider a given compact $K \subset P$. Because $K$ is compact and $P$ is open we have

$$K \cap \partial P = \emptyset.$$

i) If there exists an index $j \in \{1, \ldots, k\}$ with

$$\|f_j\|_K = 1,$$
then compactness of $K$ implies the existence of a point $x \in K \subset P$ with

$$|f_j(x)| = 1$$

Hence

$$x \in \partial P \cap K = \emptyset,$$

a contradiction. As a consequence for all $j = 1, \ldots, k$

$$\|f_j\|_K < 1$$

ii) We have to show: For each compact $K \subset P$ the hull

$$L := \hat{K}_{X,P} := \bigcap_{f \in \mathcal{O}(X)} \{ x \in P : |f(x)| \leq \|f\|_K \}$$

is compact. It suffices to show that each sequence $(x_\nu)_{\nu \in \mathbb{N}}$ in $L$ has a subsequence, which is convergent towards a limit $x_0 \in L$.

The relative compactness

$$P \subset\subset U$$

implies the compactness of $\mathcal{P}$ and a forteriori the compactness of

$$\mathcal{T} \subset \mathcal{P}$$

Hence there exists a subsequence $(x_{\nu_k})_{k \in \mathbb{N}}$ in $L \subset P$ which is convergent towards a limit

$$x_0 \in \mathcal{T} \subset U.$$

Part i) implies for all $j = 1, \ldots, k$ in the limit

$$|f_j(x_0)| \leq \|f_j\|_K < 1,$$

hence $x_0 \in P$.

Moreover for all $f \in \mathcal{O}(X)$ in the limit

$$|f(x_0)| \leq \|f\|_K$$

As a consequence $x_0 \in L$. $\Box$

**Proposition 6.9 (Neighbourhood basis of analytic polyhedra).** Consider a complex manifold $X$.

1. Holomorphically convex: If $X$ is holomorphically convex, $K \subset X$ compact and $U \subset X$ an open neighbourhood of $\hat{K}$, then exists an analytic polyhedron $P$ in $X$ with

$$\hat{K} \subset P \subset\subset U.$$
2. Relatively-holomorphically convex: If \( Y \subseteq X \) is open and relatively-holomorphically convex with respect to \( X \), \( K \subseteq Y \) compact and \( U \subseteq Y \) an open neighbourhood of \( \hat{K} \), then exist an analytic polyhedron \( P \) in \( Y \), defined in \( U \) by finitely many holomorphic functions from \( \mathcal{O}(X) \), which satisfies

\[
\hat{K} \subset P \subset \subset U.
\]

Proof. i) The set \( \hat{K} \) is compact because \( X \) is holomorphically convex. The manifold \( X \) is locally compact. Hence we may assume \( U \subset \subset X \) relatively compact.

\[
\hat{K} \subset U \text{ and } U \text{ open } \implies \hat{K} \cap \partial U = \emptyset.
\]

There exists for each \( x \in \partial U \) a holomorphic function \( f_x \in \mathcal{O}(X) \) with

\[
|f_x(x)| > \|f_x\|_K; \text{ w.l.o.g. } |f_x(x)| > 1 \text{ and } \|f_x\|_K < 1
\]

Then also

\[
|f_x(y)| > 1
\]

for all \( y \in V_x \), a suitable neighbourhood of \( x \), see Figure 6.3. The relative compactness of \( U \) implies the compactness of \( \partial U \). Hence there exist finitely many open sets \( V_{x_1}, \ldots, V_{x_k} \) with

\[
\partial U \subset \bigcup_{j=1,\ldots,k} V_{x_j}
\]

- Define

\[
P := \{x \in U : |f_{x_j}(x)| < 1 \text{ for } j = 1, \ldots, k\}
\]

Then

\[
P \subset \bigcup_{j=1,\ldots,k} V_{x_j}
\]

Hence \( P \subset U \) compact

- Moreover

\[
x \in \hat{K} \subset U \implies \text{For each } j = 1, \ldots, k : |f_{x_j}(x)| \leq \|f_{x_j}\|_K < 1.
\]

Hence \( \hat{K} \subset P \).

- Eventually

\[
P \subset \{x \in U : |f_{x_j}(x)| \leq 1 \text{ for } j = 1, \ldots, k\} \text{ and } \{x \in \partial U : |f_{x_j}(x)| \leq 1 \text{ for } j = 1, \ldots, k\} = \emptyset,
\]

Hence \( P \subset U \).

As a consequence \( P \) is an analytic polyhedron with

\[
\hat{K} \subset P \subset \subset U.
\]
6.2 Stein manifolds and Theorem B

Definition 6.10 is the original definition, see [12, Einleitg.].

**Definition 6.10 (Stein manifold).** A Stein manifold is a complex manifold $X$ being

1. holomorphically convex,

2. and holomorphically separable,
   
i.e. for each pair of points $x \neq y \in X$ exists a global holomorphic function $f \in \mathcal{O}(X)$ with $f(x) \neq f(y),$

3. and locally uniformizable by global holomorphic functions,
   
i.e. for each $x \in X$ exist an open neighbourhood $U \subset X$ of $x$ and global holomorphic functions

   $f_1, \ldots, f_n \in \mathcal{O}(X)$

   such that the restriction

   $(f_1|_U, \ldots, f_n|_U) : U \to \mathbb{C}^n$

   is a chart of $X$ around $x.$
Each of the requirements from Definition 6.10 refers to *global* holomorphic functions resp. requires the existence of certain global holomorphic functions on $X$. Each subdomain $X \subset \mathbb{C}^n$ is holomorphically separable and locally uniformizable. But if $n \geq 2$ then $X$ is not necessarily holomorphically convex, see Example 6.4. Hence the characteristic property of an open Stein submanifold $X \subset \mathbb{C}^n$ is its holomorphic convexity.

**Remark 6.11 (Stein manifold).** The conditions from Definition 6.10 which characterize a Stein manifold can be weakened considerably.

1. A complex manifold $X$ is *holomorphically spreadable* if each point $x \in X$ has an open neighbourhood $U$ and finitely many global holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(X)$ such that
   \[ \{ z \in U : f_j(z) = f_j(x) \text{ for } j = 1, \ldots, k \} = \{ x \} \]
   Apparently, if $X$ is locally uniformizable then $X$ is also holomorphically spreadable.

2. Due to a result of Grauert for a complex manifold holds:
   \[ \text{Holomorphically convex + holomorphic spreadable } \implies \text{Stein manifold} \]
   Moreover, the fact that $X$ is holomorphically spreadable implies that the topology of $X$ is second countable, see [12, Satz 5 und 8] and also [20, 51. A.3].

The following diagram shows the fundamental implications between those concepts.

---

Proposition 6.12 sharpens Proposition 6.9 in case the ambient manifold $X$ is holomorphically convex.
Proposition 6.12 (Exhaustion by analytic polyhedra). Each holomorphically convex complex manifold \( X \), notably each Stein manifold, has an exhaustion by a sequence of analytic polyhedra, i.e. there exists a sequence \((P_n)_{n \in \mathbb{N}}\) of analytic polyhedra in \( X \) satisfying
\[
P_n \subset \subset P_{n+1}, \; n \in \mathbb{N}, \text{ with } X = \bigcup_{n \in \mathbb{N}} P_n
\]

Proof. Because \( X \) is locally compact with second countable topology there exists an exhaustion of \( X \) by an ascending sequence of compact subsets \((K_\nu)_{\nu \in \mathbb{N}}\) with
\[
K_\nu \subset K_{\nu+1}, \; \nu \in \mathbb{N}, \text{ and } X = \bigcup_{\nu \in \mathbb{N}} K_\nu
\]
The assumption that \( X \) is holomorphically convex implies: For each compact \( K \subset X \) its holomorphically convex hull \( \hat{K} \subset X \) is also compact.

We construct the required sequence \((P_\nu)_{\nu \in \mathbb{N}}\) of analytic polyhedra by induction:
Due to compactness of \( \hat{K}_0 \) Proposition 6.9 provides an analytic polyhedron \( P_0 \) in \( X \) with
\[
\hat{K}_0 \subset P_0 \subset \subset X.
\]
The set
\[
L_0 := K_1 \cup \overline{P_0}
\]
is compact, hence its convex hull \( \hat{L}_0 \) is compact too. As a consequence there exists an analytic polyhedron \( P_1 \) in \( X \) satisfying
\[
L_0 \subset P_1 \subset \subset X.
\]
The construction continues with the compact set
\[
L_1 := K_2 \cup \overline{P_1}
\]
and proceeds step by step in an analogous way. \( \square \)

Theorem 6.13 (Closed embedding of analytic polyhedra of a Stein manifold). Each analytic polyhedron \( P \subset \subset X \) in a Stein manifold \( X \) embeds by global holomorphic functions as analytic submanifold of a polydisc, i.e. there exist finitely many holomorphic functions
\[
\phi_1, \ldots, \phi_m \in \mathcal{O}(X)
\]
such that the restriction
\[ \phi := (\phi_1, \ldots, \phi_m) : P \to \Delta \subset \mathbb{C}^m \]

is a closed embedding.

**Proof.** By definition of the analytic polyhedron there exist an open set \( U \subset X \) and holomorphic functions 
\[ f_1, \ldots, f_r \in \mathcal{O}(X) \]
such that 
\[ P = \{ z \in U : |f_j(z)| < 1 \text{ for } j = 1, \ldots, r \} \subset U \]
Because \( \overline{P} \) is compact, we may assume after shrinking that also \( \overline{U} \) is compact.

The proof will extend charts (part i)) to a global injective immersion (part ii)), a proper map (part iii)) and finally to a closed embedding of \( P \).

i) **Immersion due to local uniformizability**: Set \( n := \dim X \). The local uniformization of the Stein manifold \( X \) by global holomorphic functions, see Definition 6.10, part 3, implies that each point \( a \in \overline{P} \) has an open neighbourhood \( U_a \) and an \( n \)-tuple of holomorphic functions 
\[ g_a := (g_{a,1}, \ldots, g_{a,n}) \in \mathcal{O}(X)^{\oplus n} \]
such that the restriction 
\[ g_a|U_a : U_a \to \mathbb{C}^n \]
is a chart of \( U_a \) around \( a \). Compactness of \( \overline{P} \) implies that \( \overline{P} \) is covered by finitely such open neighbourhoods 
\[ U_{a_1}, \ldots, U_{a_k} \]
with corresponding \( n \)-tuples \( g_{a_1}, \ldots, g_{a_k} \in \mathcal{O}(X)^{\oplus n} \).

Denote by 
\[ G := (g_1, \ldots, g_s) \in \mathcal{O}(X)^{\oplus s}, \ s = k \cdot n, \]
the family of all component functions of these \( n \)-tuples. The map 
\[ G|P : P \to \mathbb{C}^s \]
is an immersion of the analytic polyhedron \( P \).

ii) **Injectivity due to holomorphic separability**: Consider the diagonal 
\[ \text{Diag} := \{ (x, y) \in \overline{P} \times \overline{P} : x = y \} \]
The union of product sets 
\[ W := (\overline{P} \times \overline{P}) \cap \bigcup_{j=1,\ldots,k} (U_{a_j} \times U_{a_j}) \]
is an open neighbourhood of \( \text{Diag} \).
6.2 Stein manifolds and Theorem \( B \)

For a pair \((x, y) \in \overline{P} \times \overline{P}\)

we distinguish two cases:

- If \((x, y) \in W\)

  then exists an index \( j \in \{1, \ldots, k\} \) with \((x, y) \in U_{a_j} \times U_{a_j}\).

  If \( x \neq y \) then part i) provides a component function \( g \in \mathcal{O}(X) \) of \( G \) with

  \[ g(x) \neq g(y) \]

- If \((x, y) \in (\overline{P} \times \overline{P}) \setminus W\)

  then \( x \neq y \) and due to the holomorphic separability of \( X \) there exists \( h = h_{(x,y)} \in \mathcal{O}(X) \) with

  \[ h(x) \neq h(y). \]

Due to continuity the function \( h \) separates also points which form a pair in a neighbourhood \( W_{(x,y)} \) of \((x, y)\). The set

\[ (\overline{P} \times \overline{P}) \setminus W \]

is a closed subset of the compact

\[ \overline{P} \times \overline{P} \]
and therefore compact itself. As a compact set it is covered by finitely many global $W_1, ..., W_t$ with separating functions $h_1, ..., h_t \in \mathcal{O}(X)$.

By combining the families $G = (g_1, ..., g_s)$ and $H := (h_1, ..., h_t)$ we obtain a finite family $(G, H) \in \mathcal{O}(X)^{s+t}$ which provides an injective map

$$(G, H) \mid \mathcal{P} : \mathcal{P} \to \mathbb{C}^{s+t}.$$  

We may normalize the components such that

$$\|g_j\|_\mathcal{P} < 1, \ j = 1, ..., s$$ and $$\|h_j\|_\mathcal{P} < 1, \ j = 1, ..., t.$$  

The map

$$(G, H) \mid \mathcal{P} : \mathcal{P} \to \Delta \subset \mathbb{C}^{s+t}$$  

is an injective immersion.

iii) **Closed embedding due to relative compactness**: To obtain a proper map we now add the family of defining functions of $P$

$$F := (f_1, ..., f_r) \in \mathcal{O}(X)^{\oplus r}$$

to the map from part ii). Set

$$m := r + s + t$$

and consider the polydisc

$$\Delta \subset \mathbb{C}^m$$

and the map

$$\Phi : P \to \Delta$$

with

$$\Phi(x) := (F, G, H)(x) = (f_1(x), ..., f_r(x), g_1(x), ..., g_s(x), h_1(x), ..., h_t(x)).$$

In order to show the properness of $\Phi$ consider a compact $K \subset \Delta$. For a suitable radius $r < 1$

$$K \subset \Delta_r := \{w = (w_1, ..., w_m) \in \Delta : |w_v| \leq r, \ v = 1, ..., m\}$$

Then $\Phi^{-1}(K) \subset \Phi^{-1}(\Delta_r)$ is closed. Moreover

$$\Phi^{-1}(\Delta_r) \subset \{x \in P : |f_v(x)| \leq r, \ v = 1, ..., r\},$$

and

$$\{x \in P : |f_v(x)| \leq r, \ v = 1, ..., r\} = \{x \in \mathcal{P} : |f_v(x)| \leq r, \ v = 1, ..., r\}$$

because each point
6.2 Stein manifolds and Theorem B

\[ x \in P \setminus P \text{ with } |f_j(x)| \leq r \text{ for } j = 1, \ldots, r \]

had to satisfy \( |f_j(x)| = 1 \) for at least one index \( j \in \{1, \ldots, r\} \), which is impossible.

iv) Closed embedding: Due to part i) - iii) the map

\[ \Phi : P \to \Delta \subset \mathbb{C}^m \]

is an injective immersion, which is a proper map. Hence \( \Phi \) is a closed embedding.

\( \square \)

Lemma 6.14 (Coherence of the direct image under closed embeddings). Consider a complex manifold \( X \), an open set \( U \subset \mathbb{C}^n \), and a closed holomorphic embedding

\[ j : X \to U \]

For each coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) the direct image \( j_* \mathcal{F} \) is a coherent \( \mathcal{O}_U \)-module.

Proof. Denote by \( A := j(X) \subset U \) the analytic submanifold.

i) Structure sheaf: First we consider the specific case of the structure sheaf \( \mathcal{F} = \mathcal{O}_X \). We have

\[ j_* \mathcal{O}_X = \mathcal{O}_U / \mathcal{I}_A \]

with \( \mathcal{I}_A \subset \mathcal{O}_U \) the ideal sheaf of \( A \subset U \). The sheaf \( \mathcal{I}_A \subset \mathcal{O}_U \) is coherent due to Proposition 4.29. Hence

\[ \mathcal{O}_A = \mathcal{O}_U / \mathcal{I}_A = \text{coker} \left[ \mathcal{I}_A \to \mathcal{O}_U \right] \]

is coherent due to Proposition 4.31.

ii) General case: A general coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) has locally in \( X \) a resolution by free sheaves of finite rank

\[ \mathcal{O}_X^p \to \mathcal{O}_X^p \to \mathcal{F} \to 0 \]

The induced sequence of sheaf morphisms

\[ j_* \mathcal{O}_X^p \to j_* \mathcal{O}_X^p \to j_* \mathcal{F} \to 0 \]

is also exact, because \( j_* \mathcal{F} \) has the stalks

\[ (j_* \mathcal{F})_y = \begin{cases} \mathcal{F}_x & \text{if } y = j(x), \ x \in X \\ 0 & \text{otherwise} \end{cases} \]

Due to part i) the \( \mathcal{O}_U \)-modules \( j_* \mathcal{O}_X^p \) and \( j_* \mathcal{O}_X^p \) are coherent. Hence Proposition 4.31 implies the coherence of
\[ j_* \mathcal{F} = \text{coker} \left[ j_* \mathcal{O}^n_X \to j_* \mathcal{O}^p_X \right] \]
\[ \square \]

Apparently Lemma 6.14 is a simple example of Grauert’s general coherence theorem, see Remark 4.35.

**Definition 6.15 (Concentric polyhedra).** Two analytic polyhedra

\[ P \subset P' \]

of a given complex manifold are concentric if there exists a closed embedding

\[ j : P' \to \Delta' \]

into a polydisc \( \Delta' \subset \mathbb{C}^n \), and a concentric polydisc \( \Delta \subset \subset \Delta' \) such that the restriction

\[ j|_P : P \to \Delta' \]

embeds \( P \) as an analytic submanifold of \( \Delta \).

Note: Two concentric polyhedra satisfy \( P \subset \subset P' \).

**Proposition 6.16 (Theorem B for concentric analytic polyhedra in Stein manifolds).** Consider a pair \( P \subset \subset P' \) of concentric analytical polyhedra of a Stein manifold \( X \). For each coherent \( \mathcal{O}_P \)-module \( \mathcal{F} \) the restriction \( \mathcal{F}|_P \) is acyclic.

The proof reduces the claim to the analogue for polydiscs in Corollary 5.15, which follows from Hilbert’s syzygy theorem for coherent sheaves on polydiscs, Theorem 5.14.

**Proof.** i) \textit{Consequences of Hilbert’s syzygy theorem:} Due to Definition 6.15 there exists an embedding

\[ j : P' \to \Delta' \]

with \( A := j(P') \) an analytic submanifold of a polydisc \( \Delta' \) and

\[ P = j^{-1}(\Delta) \subset \subset P' \]

for a concentric polydisc

\[ \Delta \subset \subset \Delta'. \]

Lemma 6.14 implies that \( j_* \mathcal{F} \) is a coherent \( \mathcal{O}_{\Delta'} \)-module. Corollary 5.15 of Hilbert’s syzygy theorem after shrinking to a polydisc implies that
6.2 Stein manifolds and Theorem B

\( j_\ast \mathcal{F} \mid \Delta \)

is acyclic, i.e.

\[ H^q(\Delta, j_\ast \mathcal{F}) = 0 \text{ for all } q \geq 1. \]

ii) **Comparing cohomology by comparing coverings:** We claim

\[ H^q(P, \mathcal{F}) = H^q(\Delta, j_\ast \mathcal{F}). \]

On the right-hand side of the equation: For each open covering \( \mathcal{U} \) of \( \Delta \) holds by definition of the direct image

\[ H^q(\Delta, j_\ast \mathcal{F}) = \lim_{\mathcal{U}} H^q(\mathcal{U}, j_\ast \mathcal{F}) \]

We have

\[ H^q(\Delta, j_\ast \mathcal{F}) = \lim_{\mathcal{U}} H^q(\mathcal{U}, j_\ast \mathcal{F}) \]

There is a bijective correspondence between open coverings \( \mathcal{U} \) of \( \Delta \) for the right-hand side and open coverings \( \mathcal{V} \) of \( P \) for the left-hand side: Each open covering \( \mathcal{V} = (U_i)_{i \in I} \) of \( \Delta \) defines the open covering

\[ \mathcal{V} := j^{-1}(\mathcal{U}) \]

of \( P \). For the opposite direction consider an open covering

\[ \mathcal{V} = (V_j)_{j \in J} \]

of \( P \). For each \( j \in J \) the set \( j(V_j) \subset j(\Delta) \) is open. Hence there exist an open \( U_j \subset \Delta \) with

\[ j(U_j) = U_j \cap j(P). \]

Because \( j(P) \subset \Delta \) is closed, the additional set

\[ U := \Delta \setminus j(P) \subset \Delta \]

is also open. Then

\[ \mathcal{U} := (U_j)_{j \in J} \cup \{ U \} \]

is an open covering of \( \Delta \) with

\[ j^{-1}(\mathcal{U}) = \mathcal{V}. \]

Hence the vanishing of \( H^q(\Delta, j_\ast \mathcal{F}) \) from part i) implies: For all \( q \geq 1 \)

\[ 0 = \lim_{\mathcal{U}} H^q(\mathcal{U}, j_\ast \mathcal{F}) = \lim_{\mathcal{U}} H^q(j^{-1}(\mathcal{U}), \mathcal{F}) = \lim_{\mathcal{V}} H^q(\mathcal{V}, \mathcal{F}) = H^q(P, \mathcal{F}) \]

\( \square \)
Proposition 6.16 provides a local step to obtain Theorem B on a Stein manifold X. In order to extend the solution to all of X we prove a result about holomorphic approximation by global functions and sections. Therefore we introduce the concept of Runge pairs.

**Definition 6.17 (Runge pair).** A Runge pair \((X, Y)\) is a complex manifold \(X\) and an open \(Y \subset X\) such that the restriction map
\[
\mathcal{O}(X) \to \mathcal{O}(Y), \ f \mapsto f|Y,
\]
has dense image with respect to the Fréchet topologies on both vector spaces.

We recall the 1-dimensional situation in \(\mathbb{C}\).

**Remark 6.18 (Runge pair).**

1. Each holomorphic function on the disc \(\Delta \subset \mathbb{C}\) is the compact limit of a sequence of holomorphic functions on \(\mathbb{C}\), e.g., of its Taylor polynomials. Hence the canonical restriction
\[
\mathcal{O}(\mathbb{C}) \to \mathcal{O}(\Delta), \ f \mapsto f|\Delta,
\]
has dense image with respect to the Fréchet topology of compact convergence on each of the two spaces. As a consequence the pair \((\mathbb{C}, \Delta)\) is a Runge pair. Note: Taylor approximation also applies to holomorphic functions in higher-dimensional polydiscs \(\Delta^n\). Hence each pair \((\mathbb{C}^n, \Delta^n)\) is a Runge pair.

2. For a general domain \(G \subset \mathbb{C}\) the pair \((\mathbb{C}, G)\) is not necessarily a Runge pair. A counterexample is the domain \(G := \mathbb{C}^*\): Consider
\[
f : G \to \mathbb{C}, \ f(z) := \frac{1}{z}
\]
and assume
\[
f = \lim_{n \to \infty} f_n, \ f_n \in \mathcal{O}(\mathbb{C}).
\]
The residue theorem and the compact convergence imply
\[
2\pi i = \int_{|z|=1} f(z) \, dz = \int_{|z|=1} \left( \lim_{n \to \infty} f_n(z) \right) \, dz = \lim_{n \to \infty} \int_{|z|=1} f_n(z) \, dz = 0
\]
The contradiction shows that \((\mathbb{C}, \mathbb{C}^*)\) is not a Runge pair.

3. For a domain \(G \subset \mathbb{C}\) holds the Runge approximation theorem: \((\mathbb{C}, G)\) is a Runge pair iff the complement \(\mathbb{C} \setminus G\) has no relatively compact holes, i.e. iff
\[
G = h_C(G) := G \cup \bigcup_{C \subset \mathbb{C} \setminus G} \text{component,} \text{ the Runge hull of } G \text{ with respect to } \mathbb{C}.
\]
6.2 Stein manifolds and Theorem B

See [24, Kap. 12, §2, Approximationssatz] and note that each entire function on \( \mathbb{C} \) is the compact limit of polynomials. For a proof of the more general case of an open Riemann surface see [32, Chap. 14.2 and Theor. 14.14].

**Theorem 6.19 (Runge approximation for the structure sheaf).** Consider a Stein manifold \( X \) and an open, relatively-holomorphically convex subset \( Y \subset X \). Then \( (X,Y) \) is a Runge pair.

**Proof.** Consider a holomorphic function \( f \in \mathcal{O}(Y) \) and a compact \( K \subset Y \). Because \( Y \) is relatively-holomorphically convex with respect to \( X \), the hull

\[
\hat{K}_{X,Y} \subset Y
\]

is compact. Hence we may assume \( K = \hat{K}_{X,Y} \). Proposition 6.9 provides an analytic polyhedron \( P \subset Y \)

- which is defined in an open set \( U \subset Y \) with \( P \subset U \)
- by finitely many holomorphic functions \( f_j, j = 1, \ldots, k \), not only from \( \mathcal{O}(Y) \) but even from \( \mathcal{O}(X) \), as

\[
P := \{ y \in U : |f_j(y)| < 1 \text{ for all } j = 1, \ldots, k \}
\]
- and satisfies

\[
K \subset P \subset U
\]

i) **Reduction to a polydisc:** Because \( X \) is a Stein manifold, Theorem 6.13 provides a closed affine embedding of \( P \): There exists finitely many holomorphic functions \( \phi_1, \ldots, \phi_n \in \mathcal{O}(X) \) such that

\[
\phi := (\phi_1|_P, \ldots, \phi_n|_P) : P \to \Delta
\]

embeds \( P \) into the unit polydisc \( \Delta \subset \mathbb{C}^n \) as an analytic submanifold \( A \subset \Delta \). Consider the holomorphic function on \( A \)

\[
g := f|_P \circ \phi^{-1} \in \mathcal{O}(A).
\]

Then

\[
f|_P = g \circ \phi.
\]

ii) **Theorem B for polydiscs:** For each given polydisk

\[
\Delta' \subset \subset \Delta
\]
we extend the restriction \( g|A \cap \Delta' \) to a holomorphic function on \( \Delta' \), i.e. we construct a holomorphic function

\[
G \in \mathcal{O}(\Delta') \text{ with } G|A \cap \Delta' = g|A \cap \Delta'.
\]

The ideal sheaf \( \mathcal{I} \subset \mathcal{O}_A \) of the submanifold \( A \subset \Delta \) fits into the exact sequence of sheaves on \( \Delta \)

\[
0 \to \mathcal{I}_A \to \mathcal{O}_\Delta \to \mathcal{O}_A \to 0
\]

which defines the structure sheaf \( \mathcal{O}_A \). Proposition 4.29 implies that \( \mathcal{I}_A \) is coherent, and Corollary 5.15 implies for the restriction

\[
H^1(\Delta', \mathcal{I}_A|\Delta') = 0.
\]

The segment of the long exact cohomology sequence

\[
H^0(\Delta', \mathcal{O}_{\Delta'}) \to H^0(\Delta', \mathcal{O}_A|\Delta') \to H^1(\Delta', \mathcal{I}_A|\Delta') = 0
\]

implies the surjectivity of the restriction

\[
\mathcal{O}(\Delta') \to \mathcal{O}_A(A \cap \Delta')
\]

and the existence of a holomorphic function \( G \in \mathcal{O}(\Delta') \) with

\[
G|A \cap \Delta' = g|A \cap \Delta'.
\]

iii) **Holomorphic approximation by Taylor polynomials**: We choose a polydisc \( \Delta' \)

For given precision \( \varepsilon > 0 \) we choose a finite segment \( F \) of the \( n \)-dimensional Taylor expansion \( F \) of \( G \) on \( \Delta' \)

\[
F(z) = \sum_{|I| \leq k} a_I \cdot z^I
\]

such that

\[
\|G - F\|_{\Phi(K)} \leq \varepsilon
\]

For \( x \in K \)

\[
f(x) = g(\phi(x)) = G(\phi(x))
\]

hence

\[
\|f - (F \circ \phi)\|_K = \|G \circ \Phi - F \circ \Phi\|_K \leq \varepsilon
\]

with the global holomorphic approximation

\[
F \circ \phi \in \mathcal{O}(X).
\]

One has
\[(F \circ \phi)(x) = F(\phi(x)) = \sum_{i_1, \ldots, i_n \leq k} a_{i_1 \ldots i_n} \cdot \phi_1(x)^{i_1} \cdot \ldots \cdot \phi_n(x)^{i_n}\]

Lemma 6.20 prepares the proof of Theorem 6.21. The theorem generalizes Theorem 6.19 to sections of coherent sheaves.

**Lemma 6.20 (Runge approximation along analytic polyedra).** Consider a Stein manifold \(X\) and two pairs \((P, P')\) and \((Q, Q')\) of concentric analytic polyhedra in \(X\) with \(P' \subset Q\).

For each coherent \(\mathcal{O}\)-module \(\mathcal{F}\) on \(X\) the restriction of Fréchet spaces \(\mathcal{F}(Q) \to \mathcal{F}(P)\) has dense image.

The idea is to apply Hilbert’s syzygy theorem, Theorem 5.14, for a pair of analytic polyhedra, which embed into the concentric pair \((\Delta, \Delta')\) of polydiscs. Note that \(P \subset Q\) is not required to be a pair of concentric polyhedra.

**Proof.** i) Resolving \(\mathcal{F}\) over \(Q\): Theorem 6.13 provides a closed embedding \(\phi : Q' \to \Delta'\).

The direct image \(\phi_*\mathcal{F}\) is a coherent \(\mathcal{O}_{\Delta'}\)-module due to Lemma 6.14. Theorem 5.14 provides over the shrinked polydisc \(\Delta \subset \Delta'\) with \(\phi^{-1}(\Delta) = Q\) an epimorphism \(\mathcal{O}_{\Delta}^k \to \phi_*\mathcal{F} \to 0\),

i.e. there exist sections \(f_1, \ldots, f_k \in (\phi_*\mathcal{F})(\Delta) = \mathcal{F}(\phi^{-1}(\Delta)) = \mathcal{F}(Q)\)

which generate for each \(y \in \Delta\) the stalk

\[(\phi_*\mathcal{F})_y = \mathcal{F}_y\]

The latter satisfies for each \(y = \phi(x), x \in Q,\)

\[(\phi_*\mathcal{F})_y = \mathcal{F}_x\]
The sections define an epimorphism of $\mathcal{O}$-modules over $Q$

$$\mathcal{O}^k_Q \xrightarrow{\beta} \mathcal{F}, \ e_j \mapsto f_j, \ j = 1,...,k,$$

with $(e_1,...,e_j) \in \mathcal{O}^n(Q)$ the canonical $\mathcal{O}(Q)$-base of $\mathcal{O}^n(Q)$. The epimorphism induces a short exact sequence of sheaves

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}^k_Q \xrightarrow{\beta} \mathcal{F} \rightarrow 0$$

with the coherent $\mathcal{O}_Q$-module

$$\mathcal{K} := \ker \beta.$$

ii) **Approximation of holomorphic functions**: The vanishing

$$H^1(P, \mathcal{K}) = 0$$

due to Theorem B, Proposition 6.16, applied to the concentric polydiscs

$$P \subset \subset P'$$

provides the epimorphism of sections

$$\mathcal{O}^k(P) \xrightarrow{\beta_P} \mathcal{F}(P)$$

A given section $f \in \mathcal{F}(P)$ can be represented as

$$f = \sum_{j=1}^{k} \alpha_j \cdot f_j$$

with holomorphic coefficients $\alpha_1,...,\alpha_k \in \mathcal{O}(P)$. Due to Theorem 6.19 each holomorphic function

$$\alpha_j, \ j = 1,...,k,$$

can be approximated with arbitrary precision on each compact subset of $P$ by global holomorphic functions from $\mathcal{O}(X)$, in particular by holomorphic functions on $Q$. The module multiplication is continuous. Hence $f \in \mathcal{F}(P)$ can be approximated with arbitrary precision by sections from $\mathcal{F}(Q)$. \qed

**Theorem 6.21 (Runge approximation for coherent sheaves).** Consider a Stein manifold $X$ and a relatively-holomorphically convex open $Y \subset X$. For each coherent sheaf $\mathcal{F}$ on $X$ the canonical restriction

$$\mathcal{F}(X) \rightarrow \mathcal{F}(Y), \ f \mapsto f|_Y,$$

has dense image with respect to the canonical Fréchet topologies. In particular, $(X,Y)$ is a Runge pair.
Proof. i) Exhausting $Y$ by analytic polyhedra $P_i$: Due to Remark 6.2 the subset $Y \subset X$ is holomorphically convex. Proposition 6.12 provides an exhaustion $(P_i)_{i \in \mathbb{N}}$ of $Y$ by analytic polyhedra in $Y$. W.l.o.g. for each index $i \in \mathbb{N}$ exists a concentric polyhedron $P'_i$ in $Y$ with

$$P_i \subset \subset P'_i.$$  

Due to Proposition 6.9 we may assume that these polyhedra are defined by holomorphic functions from $\mathcal{O}(X)$.

ii) Restriction $\mathcal{F}(X) \to \mathcal{F}(P_i)$: We claim for each arbitrary, but fixed $i \in \mathbb{N}$: The restriction

$$\rho^X_{P_i} : \mathcal{F}(X) \to \mathcal{F}(P_i)$$  

has dense image with respect to the canonical Fréchet topologies.

The Stein manifold $X$ is holomorphically convex. Proposition 6.12 provides an exhaustion of $X$ by analytic polyhedra in $X$. W.l.o.g. the exhaustion starts with a neighbourhood of the relatively compact $P'_i \subset \subset X$: Choose a sequence of pairs of concentric analytic polyhedra in $X$

$$(Q_0, Q'_0)$$

satisfying

$$P'_i \subset Q_0, \ Q'_v \subset Q_{v+1}, \ X = \bigcup_{v \in \mathbb{N}} Q_v.$$  

Lemma 6.20 ensures: All restrictions

$$\mathcal{F}(Q_0) \to \mathcal{F}(P_i) \text{ and } \mathcal{F}(Q_{v+1}) \to \mathcal{F}(Q_v), \ \forall v \in \mathbb{N},$$

have dense image.

For the given index $i \in \mathbb{N}$ consider an arbitrary but fixed section

$$f_i \in \mathcal{F}(P_i)$$

and a fixed closed neighbourhood

$$N_i \subset \mathcal{F}(P_i) \text{ of } f_i$$

in the Fréchet space $\mathcal{F}(P_i)$. If

$$\rho_v := \rho^Q_{P_i} : \mathcal{F}(Q_v) \to \mathcal{F}(P_i)$$

denotes the restriction, then the inverse images

$$M_v := \rho_v^{-1}(N_i) \subset \mathcal{F}(Q_v) \ \forall v \in \mathbb{N},$$
are complete metric spaces. The family of continuous morphisms with dense image

$$\cdots \to M_3 \to M_2 \to M_1 \to M_0$$

satisfies the assumptions of the Mittag-Leffler principle of exhaustion: Proposition 1.26 provides a family of compatible sections \((F_{\nu} \in \mathcal{F}(Q_{\nu}))_{\nu \in \mathbb{N}},\) i.e.

$$F_{\nu+1}|Q_{\nu} = F_{\nu}, \ \nu \in \mathbb{N}.$$  

Hence the family defines a section

$$F \in \mathcal{F}(X)$$

satisfying for all \(\nu \in \mathbb{N}\)

$$p^{X}_{Q_{\nu}}(F) = F_{\nu}.$$  

As a consequence \(F \in \mathcal{F}(X)\) approximates \(f_i \in \mathcal{F}(P_i)\) with the precision determined by the neighbourhood \(N_i\).

iii) Restriction \(\mathcal{F}(X) \to \mathcal{F}(Y)\): We prove that the restriction

$$\rho^X_Y : \mathcal{F}(X) \to \mathcal{F}(Y)$$

has dense image with respect to the canonical Fréchet topologies: For given

$$f \in \mathcal{F}(Y)$$

and zero-neighbourhood \(N \subset \mathcal{F}(Y)\) one has to find a global section \(F \in \mathcal{F}(X)\) such \(F|Y \in \mathcal{F}(Y)\) approximates \(f\) up to \(N\).

The proof relies on the following observation: Approximation with precision \(N \subset \mathcal{F}(Y)\) can be verified by considering a single index \(i_0\) and approximation of \(f|P_{i_0} \in \mathcal{F}(P_{i_0})\) up to a certain zero-neighbourhood \(N_{i_0} \subset \mathcal{F}(P_{i_0})\). Because

$$\mathcal{F}(Y) \subset \prod_{i \in \mathbb{N}} \mathcal{F}(P_i)$$

is a Fréchet subspace and open zero-neighbourhoods in the product

$$\prod_{i \in \mathbb{N}} \mathcal{F}(P_i)$$

are determined by open zero-neighbourhoods from finitely many factors \(\mathcal{F}(P_i)\). Because \((P_i)_{i \in \mathbb{N}}\) is an exhaustion of \(Y\), those finitely many zero-neighbourhoods result from a zero neighbourhood \(N_{i_0}\) in a single factor \(\mathcal{F}(P_{i_0})\).

Stated in a formal way: The Fréchet space \(\mathcal{F}(Y)\) injects via the continuous map

$$\alpha : \mathcal{F}(Y) \to \prod_{j=0}^{\infty} \mathcal{F}(P_j), \ g \mapsto (g|P_i)_{i \in \mathbb{N}},$$
as a closed subspace $K$ into a product of Fréchet spaces. The open mapping theorem for Fréchet spaces implies that

$$\alpha : \mathcal{F}(Y) \to K$$

is a homeomorphism. The given zero-neighborhood $N$ maps homeomorphic to a zero-neighborhood

$$\Psi := \alpha(N) \subset K.$$ Here $\Psi$ restricts only finitely many factors. There exists an index $i \in \mathbb{N}$ such that $\Psi$ contains an open zero-neighborhood $\Psi' \subset \Psi$ of the form

$$\Psi' = \Phi \times \prod_{j=0}^{\infty} \mathcal{F}(P_j)$$

with

$$\Phi \subset \prod_{j=0}^{i} N_j, \; N_j \subset \mathcal{F}(P_j) \text{ open zero-neighborhood.}$$

Because there are only finitely many factors to consider, one may assume that for $0 \leq j < i$ each restriction

$$\rho_{P_j}^P : \mathcal{F}(P_i) \to \mathcal{F}(P_j)$$

satisfies $\rho_{P_j}^P(N_i) \subset N_j$

- if necessary replace $N_i$ by

$$N_i \cap \bigcap_{j=0}^{i-1} (\rho_{P_j}^P)^{-1}(N_j)$$

Consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}(Y) & \xrightarrow{\beta} & \prod_{j=0}^{i} \mathcal{F}(P_j) \\
\rho := \rho_{P_i} & \downarrow & \\
\mathcal{F}(P_i) & \xrightarrow{\gamma} & \\
\end{array}$$

with

$$\beta := [\mathcal{F}(Y) \xrightarrow{\alpha} \prod_{j=0}^{\infty} \mathcal{F}(P_j) \to \prod_{j=0}^{i} \mathcal{F}(P_j)]$$

the composition of $\alpha$ with the canonical projection, and
\( \gamma : \mathcal{F}(P_i) \to \prod_{j=0}^{i} \mathcal{F}(P_j), \ g \mapsto (g|_{P_0}, \ldots, g|_{P_{i-1}}, g|_{P_i}) \).

The composition
\[ \gamma \circ \rho = \beta \]

implies
\[ \alpha^{-1}(\Psi') = \beta^{-1}(\Phi) = \rho^{-1}(\gamma^{-1}(\Phi)) \]

The open set
\[ M := \gamma^{-1}(\Phi) \subset \mathcal{F}(P_i) \]
is a zero-neighbourhood satisfying
\[ \rho^{-1}(M) = \alpha^{-1}(\Psi') \subset \alpha^{-1}(\Psi) = N, \]
i.e. for each section \( g \in \mathcal{F}(Y) \) holds
\[ g|_{P_i} \in M \subset \mathcal{F}(P_i) \implies g \in N \subset \mathcal{F}(Y) \]

Due to part ii) there exists \( F \in \mathcal{F}(X) \) with restriction \( F|_{P_i} \) approximating \( f|_{P_i} \) up to \( M \). As a consequence, the restriction
\[ F|_{Y} \in \mathcal{F}(Y) \]
approximates \( f \in \mathcal{F}(Y) \) up to \( N \). \( \square \)

**Theorem 6.22 (Theorem B for Stein manifolds).** On a Stein manifold \((X, \mathcal{O})\) each coherent \( \mathcal{O} \)-module \( \mathcal{F} \) is acyclic.

**Proof.** 1) Čech cohomology with respect to a Leray covering: Proposition 6.12 provides an exhaustion of \( X \)
\[ (P'_n)_{n \in \mathbb{N}} \]
by relatively compact analytic polyhedra. For each given \( n \in \mathbb{N} \) exists due to Theorem 6.13 a closed embedding
\[ \phi : P' := P'_n \to \Delta' \]
into a polydisc \( \Delta' \subset \mathbb{C}^m \). The components of \( \phi \) are the restrictions of global holomorphic functions on \( X \). There exists a concentric polydisc \( \Delta \subset \Delta' \) with
\[ \phi(P'_{n-1}) \subset \Delta \subset \Delta', \ P'_{-1} := \emptyset. \]

Set
\[ P_n := \phi^{-1}(\Delta). \]

Then
6.2 Stein manifolds and Theorem B

\[ P_n \subset P'_n \]

is a pair of concentric analytic polyhedra satisfying

\[ P'_{n-1} \subset P_n \subset P'_n \]

Therefore also the family

\[ \mathcal{P} := (P_n)_{n \in \mathbb{N}} \]

is an exhaustion of \( X \) by analytic polyhedra. For each \( n \in \mathbb{N} \) the pair of concentric polyhedra

\[ P_n \subset P'_n \]

satisfies the assumption of Proposition 6.16. Hence the restriction \( \mathcal{F}|P_n \) is acyclic. Because the intersection of finitely many analytic polyhedra is again an analytic polyhedron, the family \( \mathcal{P} \) is a Leray covering for \( \mathcal{F} \): Leray’s Theorem 3.19 implies that the cohomology of \( \mathcal{F} \) on \( X \) can be computed as Čech cohomology with respect to \( \mathcal{P} \):

\[ H^\bullet(X, \mathcal{F}) = H^\bullet(\mathcal{P}, \mathcal{F}). \]

ii) Splitting of cocycles, local solution: To prove the theorem we have to construct for each index \( j \geq 1 \) and for each cocycle

\[ \xi = (\xi_n)_{n \in \mathbb{N}} \in Z^j(\mathcal{P}, \mathcal{F}) \]

a splitting cochain

\[ \eta \in C^{j-1}(\mathcal{P}, \mathcal{F}) \] with \( \xi = \delta \eta \).

The present step constructs a local splitting, i.e. for given \( n \in \mathbb{N} \) a splitting of the family \( \xi|P_n \) over the open subset \( P_n \subset X \): The finite covering

\[ \mathcal{P}^n := (P_i)_{i \leq n} \]

of \( P_n \) by analytic polyhedra is a Leray covering for the sheaf

\[ \mathcal{F}_n := \mathcal{F}|P_n \]  

Leray’s theorem shows

\[ H^\bullet(P_n, \mathcal{F}_n) = H^\bullet(\mathcal{P}^n, \mathcal{F}_n), \]

in particular for all \( j \geq 1 \) due to Proposition 6.16

\[ 0 = H^1(P_n, \mathcal{F}_n) = H^1(\mathcal{P}^n, \mathcal{F}_n). \]

Due to

\[ H^j(\mathcal{P}^n, \mathcal{F}_n) = 0 \]

the affine space of splitting cochains is not empty
Theorem B and Theorem A on Stein manifolds

\[ M_n := \{ \eta \in C^{j-1}(\mathscr{P}^n, \mathscr{F}_n) : \delta \eta = \xi | P_n \neq \emptyset \} \]

iii) Mittag-Leffler principle and global solution: Due to part ii) for each \( n \in \mathbb{N} \) we may choose an element \( \eta_n \in M_n \). Then the affine space \( M_n \) has the form

\[ M_n = \eta_n + Z^{j-1}(\mathscr{P}^n, \mathscr{F}_n) \]

Due to Corollary 5.18 the vector space of cocycles

\[ Z^{j-1}(\mathscr{P}^n, \mathscr{F}_n) = \ker [\delta : C^{j-1}(\mathscr{P}^n, \mathscr{F}_n) \to C^j(\mathscr{P}^n, \mathscr{F}_n)] \]

is a Fréchet space. The Fréchet topology on \( Z^{j-1}(\mathscr{P}^n, \mathscr{F}_n) \) induces a Fréchet topology on the spaces \( M_n \) of local solutions. The Fréchet topology is independent of the choice of \( \eta_n \) because switching between two translates is a topological isomorphism of the Fréchet space \( C^{j-1}(\mathscr{P}^n, \mathscr{F}_n) \).

For each \( n \in \mathbb{N} \) the restriction

\[ M_{n+1} \to M_n \]

is linear and continuous and has dense image: Continuity follows from Proposition 5.17. To prove that the restriction has dense image, we may assume for a given pair \( (n, n+1) \) that

\[ \eta_n = \eta_{n+1}|U_n \]

Then we have to show: For all \( j \geq 1 \) the restriction

\[ Z^{j-1}(\mathscr{P}^{n+1}, \mathscr{F}_{n+1}) \to Z^{j-1}(\mathscr{P}^n, \mathscr{F}_n) \]

has dense image.

• \( j = 1 \): The horizontal maps of the following diagram are the canonical restrictions. The vertical maps result from the representation as affine spaces.

\[
\begin{array}{ccc}
Z^0(\mathscr{P}^{n+1}, \mathscr{F}_{n+1}) & \xrightarrow{\simeq} & Z^0(\mathscr{P}^n, \mathscr{F}_n) \\
\simeq & & \simeq \\
\mathcal{F}(P_{n+1}) & \longrightarrow & \mathcal{F}(P_n)
\end{array}
\]

The diagram commutes due to the choice of the pair \( (\eta_n, \eta_{n+1}) \). Due to Theorem 6.21 the restriction

\[ \mathcal{F}(X) \to \mathcal{F}(P_n) \]

has dense image. The following diagram of canonical restrictions is commutative.
Hence also the restriction
\[ \mathcal{F}(P_{n+1}) \to \mathcal{F}(P_n) \]
has dense image.

- \( j \geq 2 \): We show that the restriction
\[ Z^{j-1}(\mathcal{P}^{n+1}, \mathcal{F}_{n+1}) \to Z^{j-1}(\mathcal{P}^n, \mathcal{F}_n) \]
is even surjective: The vanishing
\[ H^{j-1}(\mathcal{P}^n, \mathcal{F}_n) = 0 \]
implies for a given cocycle \( \zeta \in Z^{j-1}(\mathcal{P}^n, \mathcal{F}_n) \) the existence of a cochain
\[ \chi \in C^{j-2}(\mathcal{P}^n, \mathcal{F}_n) \]
with \( \delta \chi = \zeta \).

The cochain \( \chi \) extends by zero on the additional covering element \( P_{n+1} \) of \( \mathcal{P}^{n+1} \) to a cochain
\[ \tilde{\chi} \in C^{j-2}(\mathcal{P}^{n+1}, \mathcal{F}_{n+1}) \]
with \( \tilde{\chi}|_{\mathcal{P}^n} = \chi \).

Hence the coboundary, notably cocycle
\[ \tilde{\zeta} := \delta \tilde{\chi} \in Z^{j-1}(\mathcal{P}^{n+1}, \mathcal{F}_{n+1}) \]
restricts as
\[ \tilde{\zeta}|_{\mathcal{P}^n} = \delta \tilde{\chi}|_{\mathcal{P}^n} = \delta \chi = \zeta. \]

Applying to the family of restrictions
\[ (M_{n+1} \to M_n)_{n \in \mathbb{N}} \]
the Mittag-Leffler principle, see Proposition 1.26, provides a compatible family of elements
\[ \eta = (\tilde{\eta}_n \in M_n)_{n \in \mathbb{N}} \]
which split \( \zeta \), hereby finishing the proof. \( \square \)

The proof of Theorem 6.22, part iii) makes use of the following trick: Extending a cochain by zero provides again a cochain, while the same does not hold for
extending cocycles. In part i) of the proof each analytic polyhedron $P_n', \ n \in \mathbb{N}$, provides an open neighbourhood of the closure of the polyhedron $P_n$, which is necessary to apply Proposition 6.16.

The global result for Stein manifolds from Theorem 6.22 has been proved along the local results Corollary 5.15 for polydiscs and Proposition 6.16 for concentric analytic polyhedra.

### 6.3 Theorem A and further applications

The first fundamental result, Theorem A on Stein manifolds, is an easy consequence of Theorem B and of the Nakayama Lemma from Corollary 4.20.

**Theorem 6.23 (Theorem A on Stein manifolds).** *On a Stein manifold $X$ with structure sheaf $\mathcal{O}$ each coherent $\mathcal{O}$-module sheaf $\mathcal{F}$ is globally generated, i.e. for each point $x \in X$ exist finitely many global sections

$$f_1, \ldots, f_k \in \mathcal{F}(X)$$

such that their germs

$$f_{1,x}, \ldots, f_{k,x} \in \mathcal{F}_x$$

generate the stalk $\mathcal{F}_x$ as $\mathcal{O}_x$-module.*

**Proof.** Denote by

$$\mathcal{I} := \mathcal{I}(x) \subset \mathcal{O}$$

the ideal sheaf of a given point $x \in X$:

$$\mathcal{I}(U) := \{ f \in \mathcal{O}(U) : f(x) = 0 \text{ if } x \in U \}, \ U \subset X \text{ open}.$$

The ideal sheaf $\mathcal{I}$ has the stalks

$$\mathcal{I}_y = \begin{cases} \mathcal{m}_x & \text{if } y = x \\ \mathcal{O}_y & \text{if } y \neq x \end{cases}$$

Here

$$\mathcal{m}_x \subset \mathcal{O}_x$$

is the maximal ideal of the local ring of convergent power series. The ideal sheaf $\mathcal{I}$ is coherent due to Proposition 4.29. The subsheaf

$$\mathcal{I} \cdot \mathcal{F} \subset \mathcal{F}$$

is finitely generated, hence a coherent submodule of the coherent $\mathcal{O}$-module $\mathcal{F}$. It fits into the exact sequence of sheaves.
The quotient sheaf has the stalks

\[(\mathcal{F} / \mathcal{I} \cdot \mathcal{F})_y = \begin{cases} \mathcal{F}_x / m_x \mathcal{F}_x & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \]

which implies

\[H^0(X, \mathcal{F} / \mathcal{I} \cdot \mathcal{F}) \cong \mathcal{F}_x / m_x \cdot \mathcal{F}_x\]

Theorem B, see Theorem 6.22, implies

\[H^1(X, \mathcal{I} \cdot \mathcal{F}) = 0\]

Hence the long exact cohomology sequence has the segment

\[H^0(X, \mathcal{F}) \rightarrow \mathcal{F}_x / m_x \cdot \mathcal{F}_x \rightarrow 0\]

which shows the surjectivity of the evaluation

\[\mathcal{F}(X) = H^0(X, \mathcal{F}) \rightarrow \mathcal{F}_x / m_x \cdot \mathcal{F}_x\]

One applies the Nakayama lemma, Corollary 4.20, with

\[M := \mathcal{F}_x, \ N := \text{span}_{\mathcal{O}_x} \langle f_{1,x}, \ldots, f_{k,x} \rangle\]

for a finite family \((f_j)_i\) of elements from \(\mathcal{F}(X)\) such that the classes \((\overline{f}_{j,x})_i\) generate \(\mathcal{F}_x / m_x \cdot \mathcal{F}_x\). The lemma implies that the family \((f_{j,x})_i\) generates the stalk \(\mathcal{F}_x\) as \(\mathcal{O}_x\)-module. \(\square\)

Note. In the end of the proof of Theorem 6.23 the quotient

\[\mathcal{F}_x / (m_x \cdot \mathcal{F}_x) = \mathcal{F}_x \otimes_{\mathcal{O}_x} (\mathcal{O}_x / m_x)\]

is a finite-dimensional vector space over the field \(\mathbb{C} \cong \mathcal{O}_x / m_x\). The Nakayama lemma lifts the surjectivity from the level of finite-dimensional complex vector spaces to generators at the level of finitely generated \(\mathcal{O}_x\)-modules.

**Theorem 6.24 (Characterization of Stein manifolds).** For a complex manifold \(X\) are equivalent:

i) The manifold \(X\) is a Stein manifold according to Definition 6.10.

ii) Each coherent \(\mathcal{O}_X\)-module is acyclic.

iii) Each coherent ideal sheaf \(\mathcal{I} \subset \mathcal{O}_X\) satisfies \(H^1(X, \mathcal{I}) = 0\).
iv) The section-functor $\Gamma(X, -)$ is exact on short exact sequences of coherent $\mathcal{O}_X$-modules.

**Proof.** We prove the implications according to the following diagram

\[\begin{array}{c}
\Rightarrow \quad \text{i) } X \text{ Stein } \leftarrow \\
\downarrow \quad ii) \mathcal{F} \text{ acyclic } \\
\uparrow \quad \text{iii) } H^1(X, \mathcal{F}) = 0 \quad \text{ iv) } \Gamma(X, -) \text{ exact} \\
\end{array}\]

i) $\implies$ ii): Theorem B, see Theorem 6.22.

ii) $\implies$ iii): trivial

ii) $\implies$ iv): The long exact cohomology sequence from Theorem 3.12.

iii) $\implies$ i): We verify the three conditions from Definition 6.10:

- **Holomorphic separability:** For each pair of distinct points $x \neq y \in X$ consider the zero-dimensional analytic submanifold

  \[A = \{x, y\} \subset X.\]

  Its ideal sheaf $\mathcal{I}_A \subset \mathcal{O}_X$ is coherent due to Proposition 4.29. By assumption

  \[H^1(X, \mathcal{I}_A) = 0.\]

  Hence the short exact sequence

  \[0 \to \mathcal{I}_A \to \mathcal{O}_X \to \mathcal{O}_A \to 0\]

induces the epimorphism

\[H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_A)\]

Due to the isomorphy

\[H^0(X, \mathcal{O}_A) \cong \mathbb{C} \oplus \mathbb{C}, \quad f \mapsto (f(x), f(y)),\]

the surjectivity

\[H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_A)\]
implies that $X$ is holomorphically separable.

- **Holomorphic convexity**: For an indirect proof assume that $X$ is not holomorphically convex. Proposition 6.3 provides the existence of an infinite closed discrete set $A \subset X$, such that each holomorphic function $f \in \mathcal{O}(X)$ stays bounded on $A$. W.l.o.g. $A$ is countable. Due to its discreteness the closed set $A \subset X$ is an analytic submanifold of $X$. Hence its ideal sheaf $\mathcal{I}_A \subset \mathcal{O}_X$ is coherent due to Proposition 4.29. By assumption

$$H^1(X, \mathcal{I}_A) = 0$$

Hence the short exact sequence

$$0 \to \mathcal{I}_A \to \mathcal{O}_X \to \mathcal{O}_A \to 0$$

induces the epimorphism

$$H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_A) = \mathbb{C}^N.$$ 

As a consequence, there exists a holomorphic function $f \in \mathcal{O}(X)$ which is unbounded on $A$, a contradiction.

- **Local uniformizability**: Consider a given point $a \in X$ and choose in a suitable neighbourhood $U$ of $a$ a chart of $X$

$$f = (f_1, \ldots, f_n) : U \to V \subset \mathbb{C}^n, \quad n = \dim X.$$ 

Then

$$A := \{a\} \subset X$$

is an analytic submanifold of $X$. Denote by $\mathcal{I}_A$ its ideal sheaf. It is coherent, and also the product $\mathcal{I}_A^2 \subset \mathcal{O}_X$ is a coherent ideal sheaf. By assumption

$$H^1(X, \mathcal{I}_A^2) = 0$$

Hence the short exact sequence

$$0 \to \mathcal{I}_A^2 \to \mathcal{O}_X \to \mathcal{O}_X / \mathcal{I}_A^2 \to 0$$

induces the epimorphism

$$H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X / \mathcal{I}_A^2).$$

As a consequence, for each $j = 1, \ldots, n$ exists a holomorphic function $F_j \in \mathcal{O}(X)$ with

$$F_{j,a} = \overline{f}_j \in H^0(X, \mathcal{O}_X / \mathcal{I}_A^2) \cong \mathbb{C}\{z_1, \ldots, z_n\} / \mathfrak{m}^2.$$ 

Due to Proposition 2.6 the map
\( F := (F_1, \ldots, F_n) : X \to \mathbb{C}^n \)

is a local isomorphism at \( a \). Hence \( X \) is locally uniformizable.

iv) \( \Rightarrow \) i): Alike to the previous implication iii) \( \Rightarrow \) i), replacing the assumption

\[ H^1(X, \mathcal{S}_A) = 0 \text{ resp. } H^1(X, \mathcal{S}_A^2) = 0 \]

by the exactness of \( \Gamma(X, -) \) provides the same epimorphisms on the level of sections, which prove the three requirements for \( X \) being a Stein manifold. \( \square \)

A second application of Theorem B proves that each additive Cousin problem on a Stein manifold \( X \) has a solution. Possible obstructions against a solution of a multiplicative Cousin problem are of a topological - not complex analytic - nature.

**Corollary 6.25 (Cousin problems on Stein manifolds).** Consider a Stein manifold \( X \).
1. Each additive Cousin problem on \( X \) is solvable.
2. If \( H^2(X, \mathbb{Z}) = 0 \) then each multiplicative Cousin problem on \( X \) is solvable.

**Proof.** Theorem 6.22 shows that the criteria from Corollary 3.24 are satisfied. \( \square \)

A further application of Theorem B is a far reaching generalization of the Kugelsatz from by Corollary 1.15.

**Corollary 6.26 (General Kugelsatz).** Consider a domain \( G \subset \mathbb{C}^n, n \geq 2 \), and a compact set \( K \subset G \) with \( G \setminus K \) connected. Then each holomorphic function \( f \in \mathcal{O}(G \setminus K) \) extends uniquely to a holomorphic function \( \tilde{f} \in \mathcal{O}(G) \).

**Proof.** i) Translation to a problem in cohomology: Consider the covering of \( \mathbb{C}^n \) by two open sets

\[ \mathcal{U} := (U_1 := G, U_2 := \mathbb{C}^n \setminus K) \]

Then

\[ U_1 \cap U_2 = G \setminus K \]

each holomorphic function \( f \in \mathcal{O}(G \setminus K) \) determines a 1-cocyle from \( Z^1(\mathcal{U}, \mathcal{O}) \).

Because \( \mathbb{C}^n \) is a Stein manifold we have

\[ H^1(\mathbb{C}^n, \mathcal{O}) = 0 \]

Due to Corollary 3.8

\[ H^1(\mathbb{C}^n, \mathcal{O}) = 0 \Rightarrow H^1(\mathcal{U}, \mathcal{O}) = 0, \]
which provides a cochain \((f_1, f_2) \in C^0(\mathcal{U}, \mathcal{O})\) satisfying
\[
f = f_2 - f_1
\]

ii) Applying the classical Kugelsatz: The function \(f_1\) is defined on all of \(G\). Hence the aim is to extend
\[
g := f_2 \in \mathcal{O}(\mathbb{C}^n \setminus \mathcal{K})
\]
to a holomorphic function \(\tilde{g} \in \mathcal{O}(\mathbb{C}^n)\) and then to restrict as \(\tilde{g}|_G\). Due to compactness of \(\mathcal{K}\) we may choose an open polydisc \(\Delta \subset \mathbb{C}^n\) with
\[
\mathcal{K} \subset \overline{\Delta}
\]
and with a point
\[
x_0 \in \mathcal{K} \cap \partial \Delta
\]
Hence there exists a non-empty open
\[
U \subset (G \setminus \overline{\Delta}) \subset (G \setminus \mathcal{K}),
\]
see Figure 6.5.

\[\text{Fig. 6.5 } U \subset (G \setminus \overline{\Delta}) \subset (G \setminus \mathcal{K})\]
The classical Kugelsatz, Corollary 1.15, applies for the pair of polydiscs $(\Delta, \tilde{\Delta})$ with
\[ \Delta \subset \subset \tilde{\Delta} \subset \subset \mathbb{C}^n \]
and $U \subset \tilde{\Delta}$.

It extends $g|\tilde{\Delta} \setminus \Delta$ to a holomorphic function on $\tilde{\Delta}$, which then combines with $g \in \mathcal{O}(\mathbb{C}^n \setminus K)$ to an extension $\tilde{g} \in \mathcal{O}(\mathbb{C}^n)$ of $g$. Equality on $U$
\[ \tilde{g}|U = g|U \]
and connectedness of $G \setminus K$

imply
\[ \tilde{g}|G = g|G \]

Hence
\[ f = (f_1 - f_2)|(G \setminus K) = (f_1 - \tilde{g})|(G \setminus K) \]
is the restriction of the holomorphic function
\[ f_1 - (\tilde{g}|G) \in \mathcal{O}(G) \].
\[ \square \]

**Proposition 6.27 (Analytic submanifolds of Stein manifolds).** Consider a Stein manifold $X$ and an analytic submanifold of $Y \subset X$ with ideal sheaf $\mathcal{I}_Y \subset \mathcal{I}_X$. If
\[ I(Y) \subset \mathcal{O}(X) \]
denotes the ideal of all global holomorphic functions on $X$ which vanish on $Y$, then
\[ Y = \{ x \in X : f(x) = 0 \text{ for all } f \in I(Y) \} \]
i.e. $Y$ is the “variety” of the ideal $I(Y)$.

**Proof.** We only show the non-trivial inclusion. Claim: If there exists $x \notin Y$ then exists a function $f \in \mathcal{O}(X)$ satisfying
\[ f(x) \neq 0 \text{ and } f|Y = 0 \]
Then
\[ I(Y) = \mathcal{I}_Y(X) = \mathcal{H}^0(X, \mathcal{I}_Y) \]
If $x \in X \setminus Y$ then $\mathcal{I}_{Y,x} = \mathcal{O}_{X,x}$. Theorem A, see Theorem 6.23, provides a global function $f \in \mathcal{H}^0(X, \mathcal{I}_A)$ such that the germ $f_x$ generates $\mathcal{O}_x$ as $\mathcal{O}_x$-module. In particular $f(x) \neq 0$. \[ \square \]

**Proposition 6.28 (Global extension of functions from analytic submanifolds of Stein manifolds).** Consider a Stein manifold $X$ and an analytic submanifold $Y \subset X$. For each holomorphic function $f \in \mathcal{O}_Y(Y)$ exists a holomorphic function
6.3 Theorem A and further applications

\[ F \in \mathcal{O}_X(X) \text{ with } F|_Y = f. \]

**Proof.** The short exact sequence

\[ 0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \]

induces the exact sequence

\[ H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{I}_Y) = 0, \]

see Theorem 6.24. \(\square\)

**Theorem 6.29 (Hilbert's Nullstellensatz).** Consider a Stein manifold \(X\) and finitely many global holomorphic functions \(f_1, \ldots, f_k \in \mathcal{O}(X)\) without a common zero \(x \in X\). Then exist holomorphic functions \(\alpha_1, \ldots, \alpha_k \in \mathcal{O}(X)\) with

\[ \sum_{j=1}^k \alpha_j \cdot f_j = 1 \]

**Proof.** Consider the free \(\mathcal{O}\)-module \(\mathcal{O}^k\) and its canoncial basis \((e_j)_{j=1}^k\) of global sections. The induced morphism of \(\mathcal{O}\)-modules

\[ \alpha : \mathcal{O}^k \to \mathcal{O}, \quad e_j \mapsto f_j, \quad j = 1, \ldots, k, \]

is surjective due to the assumption that the functions \(f_1, \ldots, f_k\) have no common zero. The kernel

\[ \mathcal{K} := \ker [\alpha : \mathcal{O}^k \to \mathcal{O}] \]

is coherent. It fits into the exact sequence

\[ 0 \to \mathcal{K} \to \mathcal{O}^k \xrightarrow{\alpha} \mathcal{O} \to 0 \]

Theorem 6.22 implies \(H^1(X, \mathcal{K}) = 0\). Hence the sequence

\[ H^0(X, \mathcal{O}^k) \xrightarrow{\alpha_\ast} H^0(X, \mathcal{O}) \to 0 \]

is exact, which proves the claim. \(\square\)

**Remark 6.30 (Hilbert's Nullstellensatz).**

1. In Algebraic Geometry Hilbert's Nullstellensatz (weak version) states: Consider an algebraically closed field \(k\). Each proper ideal \(I \subset k[X_1, \ldots, X_n]\) of polynomials has a non-empty variety

\[ \text{Var}(I) := \{ x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in I \} \neq \emptyset \]
The Nullstellensatz compares an algebraic property with a geometric property.

2. The weak form is equivalent to the statement: For any family

\[ f_1, \ldots, f_k \in k[X_1, \ldots, X_n] \]

of polynomials without a common zero exist polynomials \( \alpha_1, \ldots, \alpha_k \in k[X_1, \ldots, X_n] \) satisfying

\[ \sum_{j=1}^{k} \alpha_j \cdot f_j = 1 \]

Proof: Set \( I := \langle f_1, \ldots, f_k \rangle \).

1) \( \implies \) 2): If \( (f_1, \ldots, f_k) \) have no common zero, the ideal

\( I := \langle f_1, \ldots, f_k \rangle \subset k[X_1, \ldots, X_n] \)

satisfies \( V(I) = \emptyset \). Part 1) implies:

\[ I = k[X_1, \ldots, X_n], \quad \text{in particular } 1 \in I. \]

2) \( \implies \) 1): Proof by contraposition. Assume \( V(I) = \emptyset \). Then \( (f_1, \ldots, f_k) \) have no common zero. Part 2) implies a representation

\[ 1 = \sum_{j=1}^{k} \alpha_j \cdot f_j \]

Hence \( 1 \in I \) which implies \( I = k[x_1, \ldots, X_n] \).

3. Hence Theorem 6.29 is the analogue of the weak Nullstellensatz for ideals of global holomorphic functions on Stein manifolds.

**Theorem 6.31 (Holomorphic de Rham theorem).** Consider a Stein manifold \( X \) and denote by

\[ Rh_{\omega}^j(X) := \frac{\ker [\Omega^j(X) \xrightarrow{d} \Omega^{j+1}(X)]}{\text{im} [\Omega^{j-1}(X) \xrightarrow{d} \Omega^j(X)]}, \quad j \in \mathbb{N}, \]

the holomorphic de Rahm groups of \( X \). Then for all \( j \in \mathbb{N} \)

\[ Rh_{\omega}^j(X) \simeq H^j(X, \mathbb{C}) \]

Proof. Poincaré’s Lemma for the \( d \)-operator for differential forms on an \( n \)-dimensional complex manifold \( X \) shows the exactness of the following sequence of sheaf morphisms
6.3 Theorem A and further applications

$$0 \to \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n \to 0$$

Each $\mathcal{O}$-module is locally free of finite rank due to the local representation of differential forms in coordinates. In particular, each $\Omega^j, j \in \mathbb{N}$, is coherent. Theorem 6.22 shows $H^1(X, \Omega^j) = 0$, $j \in \mathbb{N}$, and the abstract de Rahm theorem, Theorem 3.16, proves the claim. □

**Corollary 6.32 (Cohomology of Stein manifolds).** For an $n$-dimensional Stein manifold $X$ holds for all $j > n$

$$H^j(X, \mathbb{C}) = 0$$

**Proof.** For $j > n$ holds $\Omega^j = 0$. Hence Theorem 6.31 proves the claim. □
7.1 Stein manifolds

Remark 7.1 (Embedding of Riemann surfaces). Consider a Riemann surface $X$.
1. For compact $X$ there exists a projective embedding $\phi : X \hookrightarrow \mathbb{P}^3$

   The proof uses the existence of a very ample line bundle $\mathcal{L} \in \text{Pic}(X)$ which defines an embedding

   $\phi_{\mathcal{L}} : X \to \mathbb{P}^n$

   The existence of $\mathcal{L}$ is implied by the Riemann-Roch theorem and makes use of Serre duality, cf. [32, Chap. 11]. In case $n > 3$ it is possible to project the image $\phi_{\mathcal{L}}(X) \subset \mathbb{P}^n$ injectively into $\mathbb{P}^3 \subset \mathbb{P}^n$, see [18, Chap. IV, Cor. 3.6].

2. An open Riemann surface is a Stein manifold, see [32, Chap. 15]. Due to Theorem 7.2 it embeds as analytic submanifold into an affine space $\mathbb{C}^3$.

Theorem 7.2 (Affine embedding of Stein manifolds). Each $n$-dimensional Stein manifold $X$ embeds as analytic submanifold into an affine space $X \hookrightarrow \mathbb{C}^m$.

For $n \geq 2$ one may choose $m = 2n$, while for $n = 1$ the choice $m = 3$ is possible.

The theorem is due to Remmert, see [20, Theor. 57.4].

Complex analysis on a Stein manifold $X$ with structure sheaf $\mathcal{O}$ often deals with the global holomorphic functions on $X$, i.e. with the elements of the $\mathbb{C}$-algebra.
Provided with the Fréchet topology of compact convergence, \( \mathcal{O}(X) \) becomes a topological \( \mathbb{C} \)-algebra. For a Stein manifold \( X \) the relation between the manifold and the topological algebra is very close: It is formalized by the \textit{spectrum} of \( \mathcal{O}(X) \).

\textbf{Definition 7.3 (Spectrum of a topological \( \mathbb{C} \)-algebra).} Consider a topological \( \mathbb{C} \)-algebra \( A \) with unity.

1. A \textit{character} of \( A \) is a continuous morphism of \( \mathbb{C} \)-algebras with unit
   \[
   \chi : A \to \mathbb{C}
   \]

2. The \textit{spectrum} of \( A \) is the set of all characters of \( A \)
   \[
   S(A) := \{ \chi : \text{character of } A \}.
   \]

3. Each element \( f \in A \) defines on \( S(A) \) the evaluation map
   \[
   \hat{f} : S(A) \to \mathbb{C}, \chi \mapsto \hat{f}(\chi) := \chi(f).
   \]

One provides the set \( S(A) \) with the corresponding \textit{initial topology}, i.e. with the coarsest topology such that

\[
(\hat{f} : S(A) \to \mathbb{C})_{f \in A}
\]

becomes a family of continuous maps.

Note that \( S(A) \subset \mathbb{C}^A \) is a subspace of a product of the Hausdorff spaces \( \mathbb{C} \), hence the spectrum \( S(A) \) is a Hausdorff space itself.

\textbf{Theorem 7.4 (Character theorem for Stein manifolds).} Consider a complex manifold \( X \) and denote by
\[
A := \Gamma(X, \mathcal{O})
\]
the topological \( \mathbb{C} \)-algebra of its global holomorphic functions provided with the canonical Fréchet topology. Then are equivalent:

1. The manifold \( X \) is a Stein manifold.

2. The canonical map
   \[
   \varepsilon : X \to S(A), \ x \mapsto \varepsilon_x \in S(A),
   \]
   defined as the point character
   \[
   \varepsilon_x : A \to \mathbb{C}, \ \varepsilon_x(f) := f(x),
   \]
   is a homeomorphism.
For a proof of Theorem 7.4 see [2, Anhang Kap. VI, Satz 7] and [20, Theor. 57.3].

The injectivity of $\varepsilon$ is equivalent to $X$ being holomorphically separable. The difficult part is to show the surjectivity of $\varepsilon$, i.e. to prove for a Stein manifold $X$ that each character of $\Gamma(X, \mathcal{O})$ is a point character.

A further topic deals with duality theorems on Stein manifolds. The prototype of a duality theorem is Serre’s theorem on a compact Riemann surface:

**Proposition 7.5 (Duality on compact Riemann surfaces).** On a compact Riemann surface $X$ denote by

$$\omega := \Omega$$

the sheaf of holomorphic differential forms. For each line bundle on $X$ with invertible sheaf $\mathcal{L}$ of holomorphic sections the bilinear composition of finite-dimensional complex vector spaces

$$H^j(X, \mathcal{L}) \times H^{1-j}(X, \mathcal{L}^\vee \otimes \mathcal{O}_\omega) \to H^1(X, \omega) \cong \mathbb{C}, \ j = 0, 1,$$

is a pairing. In particular,

$$H^j(X, \mathcal{L})^\vee \cong H^{1-j}(X, \mathcal{L}^\vee \otimes \mathcal{O}_\omega)$$

Here

$$\mathcal{L}^\vee := \mathcal{H}om_\mathcal{O}(\mathcal{L}, \mathcal{O})$$

denotes the dual sheaf of $\mathcal{L}$, corresponding to the holomorphic sections of the dual line bundle of $L$.

For a proof see [32, Chap. 9].

Proposition 7.5 generalizes to compact manifolds of dimension $n \geq 1$. The dualizing sheaf $\omega$ becomes the sheaf of holomorphic differential forms of highest degree

$$\omega := \Omega^n.$$

If one replaces the locally free sheaf $\mathcal{L}$ by an arbitrary coherent sheaf $\mathcal{F}$ then one has to take into account also the $\text{Ext}$-groups $\text{Ext}^j(\mathcal{F}, \omega)$.

Moreover, for a non-compact manifold $X$ the cohomology groups with values in a coherent $\mathcal{O}$-module $\mathcal{F}$

$$H^q(X, \mathcal{F})$$

are no longer finite-dimensional in general. As a consequence, to obtain duality results one has to introduce topologies and to consider topological duals and to consider cohomology with compact support

$$H_c^q(X, \mathcal{F}).$$
Proposition 7.6 (Cohomology with compact support). Consider an \( n \)-dimensional Stein manifold \( X \) and a coherent \( \mathcal{O} \)-module \( \mathcal{F} \). Then

\[
H^q_c(X, \mathcal{F}) = 0 \text{ for } q > n
\]

For the proof see [3, Chap. I, Lem. 2.5]. One uses

\[
H^q_c(X, \mathcal{F}) = \lim_{k \to \infty} H^q_c(P_k, \mathcal{F}|P_k)
\]

for an exhaustion of \( X \) by a sequence of analytic polyhedra \( P_k \subset X \). Over each analytic polyhedron \( P \subset X \) one has a finite resolution of \( \mathcal{F} \) by free \( \mathcal{O} \)-modules of finite rank. The proof of the claim follows from the Dolbeault resolution of \( \mathcal{O} \)

\[
0 \to \mathcal{O} \to \mathcal{E}^{0,0} \xrightarrow{d''} \mathcal{E}^{0,1} \xrightarrow{d''} \cdots \xrightarrow{d''} \mathcal{E}^{0,n} \to 0
\]

by tensoring over \( \mathcal{O} \) with \( \mathcal{F} \).

For a proof of Proposition 7.6 see [3, Chap. I, Lem. 2.5].

Theorem 7.7 is a first example from duality theory of coherent sheaves on non-compact complex manifolds.

Theorem 7.7 (Duality on Stein manifolds). Consider an \( n \)-dimensional Stein manifold \( X \) with structure sheaf \( \mathcal{O} \) and denote by

\[
\omega := \Omega^n
\]

the sheaf of holomorphic differential forms on \( X \) of highest order. For each coherent \( \mathcal{O} \)-module \( \mathcal{F} \) exists a Fréchet topology on the Ext-groups

\[
\text{Ext}_\mathcal{O}^j(\mathcal{F}, \omega), \ j \in \mathbb{N},
\]

such that their dual space of linear continuous functionals is algebraically isomorphic to the cohomology groups of \( \mathcal{F} \) with compact support:

\[
\text{Ext}_\mathcal{O}^{n-j}(\mathcal{F}, \omega)' \cong H^j_c(X, \mathcal{F})
\]

For a proof of Theorem 7.7 see [3, Chap. I, Theor. 2.1].

Theorem 7.8 shows: Being a Stein manifold \( X \) has remarkable consequences for the homology of \( X \).
7.2 Stein theory with singularities

**Theorem 7.8 (Stein manifold as CW-complex).** Each \( n \)-dimensional Stein manifold \( X \) is homotopy equivalent to a CW-complex of real dimension \( \leq n \). A consequence is the vanishing of higher homology groups:

\[
H_q(X, \mathbb{Z}) = 0 \text{ if } q > n.
\]

Moreover, the homology group \( H_n(X, \mathbb{Z}) \) of highest dimension is free.

The theorem is due to Andreotti and Frankel, see also [21] and [10, Cor. 3.12.2].

### 7.2 Stein theory with singularities

The concept of complex manifold generalizes to the concept of a complex space. A complex space is more general than a complex manifold in two aspects:

- The local models of a complex space build not necessarily on open subsets in \( \mathbb{C}^n \), but on analytic subsets of open subsets in \( \mathbb{C}^n \). Analytic sets may have singularities, the local rings \( \mathcal{O}_{A,x}, x \in A \), are quotients of rings of convergent power series.

- The holomorphic functions of a local model are not necessarily determined by their values on the analytic set, but may be nilpotent. Nilpotent holomorphic functions may be zero on open sets without defining the zero germs.

For a short introduction to complex spaces, written in the spirit of local models, see the first pages of [5].

**Definition 7.9 (Complex space).**

1. The local model of a complex space is a pair \( (A, \mathcal{O}_A) \) with an analytic set

\[
A \subset V, \ V \subset \mathbb{C}^n \text{ open},
\]

and on \( A \) a sheaf of rings

\[
\mathcal{O}_A := \mathcal{O}_V / \mathcal{I}_A
\]

with a coherent sheaf of ideals

\[
\mathcal{I}_A \subset \mathcal{O}_V \text{ with } \text{supp } \mathcal{O}_A \subset A,
\]

i.e. the stalks satisfy

\[
\mathcal{O}_{A,x} = 0 \text{ for all } x \in V \setminus A.
\]

The sheaf \( \mathcal{I}_A \) is named the ideal sheaf of the local model, while \( \mathcal{O}_A \) is named its structure sheaf.
2. A complex space is a pair \((X, \mathcal{O}_X)\) with a Hausdorff space \(X\) and a sheaf of rings \(\mathcal{O}_X\) on \(X\) satisfying the following property: For each point \(x \in X\) exists an open neighbourhood \(U \subset X\) and a local model \((A, \mathcal{O}_A)\) with an isomorphism of \(\mathbb{C}\)-ringed spaces
\[
(U, \mathcal{O}_X|U) \simeq (A, \mathcal{O}_A),
\]
see [5, Chap. 0.14].

For the local model \((A, \mathcal{O}_A)\) each stalk \(\mathcal{O}_{A,x}, x \in A\), is a local \(\mathbb{C}\)-algebra. A morphism
\[
f : A \to B
\]
between two \(\mathbb{C}\)-algebras has to map the maximal ideals, i.e.
\[
f(\mathfrak{m}_A) \subset \mathfrak{m}_B.
\]

**Definition 7.10 (Nilradical and reduction of a complex space).**

1. For a complex space \((X, \mathcal{O})\) the nilradical
\[
\mathcal{N} \subset \mathcal{O}_X
\]
is the sheafification of the presheaf on \(X\)
\[
\{f \in \mathcal{O}_X(U) : f^k = 0 \text{ for a suitable } k \in \mathbb{N}\}, \ U \subset X \text{ open}
\]
2. The complex space \((X, \mathcal{O}_X)\) is reduced if \(\mathcal{N} = 0\).
3. The reduction of a complex space \((X, \mathcal{O}_X)\) is the reduced complex space \((X, \mathcal{O}_X/\mathcal{N})\).

The nilradical \(\mathcal{N} \subset \mathcal{O}_X\) of a complex space is a coherent \(\mathcal{O}_X\)-module.

The definition of a Stein space is literally the same as the classical Definition 6.10 for Stein manifolds.

**Definition 7.11 (Stein space).** A complex space \((X, \mathcal{O}_X)\) with second countable topology is a Stein space if it satisfies all of the following properties:

1. Holomorphically convex: For each compact \(K \subset X\) the holomorphically convex hull
\[
\hat{K} := \bigcap_{f \in \mathcal{O}(X)} \{x \in X : |f(x)| \leq \|f\|_K\}
\]
is compact.
2. **Holomorphically separable**: For each pair of points \( x, y \in X \) exists a global holomorphic function \( f \in \mathcal{O}(X) \) with \( f(x) \neq f(y) \).

3. **Locally holomorphically uniformizable**: Each point \( x \in X \) has an open neighbourhood \( U \subset X \) and finally many global holomorphic functions 
   \[ f_1, \ldots, f_k \in \mathcal{O}(X) \]
   such that the restriction 
   \[ (f_1, \ldots, f_k)|_U : U \to A \subset \Delta \]
   induces an isomorphism of \( (U, \mathcal{O}_X|U) \) onto a local model \( (A, \mathcal{O}_A) \) with an analytic subset \( A \subset \Delta \) in a polydisc \( \Delta \subset \mathbb{C}^k \).

Due to Grauert’s simplification of the definition: A complex space \( X \) is already a Stein space if \( X \) is holomorphically convex and holomorphically spreadable, compare Remark 6.11.

**Proposition 7.12 (Reduction and normalization of a Stein space).**

- A complex space \( X \) is a Stein space iff its reduction \( X^{\text{red}} \) is a Stein space.
- A reduced complex space \( X \) is a Stein space iff its normalization \( X^{\text{norm}} \) is a Stein space.

For the proof see [20, Prop. 52.19] and [5, Chap. 2.32, Cor.]

### 7.3 Affine schemes

Grothendieck showed how to geometrize algebra: Each commutative ring with unity can be represented as a topological space equipped with a canonical structure sheaf. The resulting pair is named the **affine scheme** of the ring. All commutative rings in the following will be assumed to have a unit.

**Definition 7.13 (Affine scheme).** Consider a commutative ring \( A \).

- **Spectrum \( \text{Spec} \ A \):** Denote by
  \[ \text{Spec} \ A := \{ p \subset A : \text{p prime ideal} \} \]
  the set of prime ideals of \( A \).
• Zariski topology on $\text{Spec } A$: For each ideal $a \subset A$ denote by

$$V(a) := \{ p \in \text{Spec } A : a \subset p \}$$

the set of prime ideals lying over $a$, i.e. containing $a$. For an element $f \in A$ denote by

$$D_f := \{ p : f \notin p \}$$

the set of prime ideals which do not contain $f$. The Zariski topology of set $\text{Spec } A$ has the closed sets $V(a)$, $a \subset A$, respectively the base of open sets

$$D_f, f \in A.$$

• Sheaves on $\text{Spec } A$: For each basic open set $D(f) \subset \text{Spec } A$ denote by $A_f$ the localization of $A$ by the multiplicatively closed set \{ $f^n : n \in \mathbb{N}$ \}. The attachment

$$D(f) \mapsto A_f, f \in A,$$

defines the structure sheaf $\mathcal{O}_{\text{Spec } A}$ on $\text{Spec } A$. More general, for each $A$-module $M$ the attachment

$$D(f) \mapsto M_f := M_{\{ f^n : n \in \mathbb{N}\}} = A_f \otimes_A M, f \in A,$$

defines the $\mathcal{O}$-module sheaf $\tilde{M}$ on $\text{Spec } A$.

• Stalks: The structure sheaf $\mathcal{O} = \mathcal{O}_{\text{Spec } A}$ of $\text{Spec } A$ has at a point $p \in \text{Spec } A$ the stalk

$$\mathcal{O}_p = A_p,$$

the localization of $A$ by the multiplicatively closed set $A \setminus p$. The stalk $\mathcal{O}_p$ is a local ring. The quotient field

$$k(p) := \mathcal{O}_p / m_p,$$

with $m_p \subset \mathcal{O}_p$ the maximal ideal, is named the residue field at the point $p \in \text{Spec } A$. Moreover for each $A$-module $M$ and each point $p \in X$ the sheaf $\tilde{M}$ on $\text{Spec } A$ has the stalk

$$\tilde{M}_p = M_p,$$

the localisation with respect to $A \setminus p$.

• Coherent $\mathcal{O}$-modules: Each $A$-module $M$ can be recovered from the $\mathcal{O}$-module $\tilde{M}$ according to

$$\Gamma(\text{Spec } A, \tilde{M}) = M,$$

in particular
The $\mathcal{O}$-modules of the form $\tilde{M}$ are named \textit{quasi-coherent}. If $M$ is even finitely generated then $\tilde{M}$ is named \textit{coherent}.

• \textit{Affine scheme:} An \textit{affine scheme} is a pair of the form $X = (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$.

\begin{itemize}
  \item \textbf{Remark 7.14 (Affine scheme).}
  \begin{enumerate}
    \item \textit{Modules and quasicoherent sheaves:} Consider a fixed affine scheme $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$.

    The canonical covariant functor

    \begin{align*}
      \mathcal{A}\text{-mod} & \to \mathcal{A}\text{-mod} \\
      M & \mapsto \tilde{M} \text{ and } (g : M_1 \to M_2) \mapsto (\tilde{g} : \tilde{M}_1 \to \tilde{M}_2)
    \end{align*}

    from the category of $A$-modules to the category of $\mathcal{O}_{\text{Spec } A}$-modules is fully faithful, i.e. bijective on morphisms. The functor maps exact sequences to exact sequences. See [18, Chap. II, Prop. 5.2].

    \item \textit{Change of rings and change of affine schemes:} The pullback of a prime ideal along a morphism of commutative rings is again a prime ideal: Consider a morphism

    \[
    \phi : A \to B
    \]

    of commutative rings. Then for each prime ideal $p \subset B$ the inverse image

    \[
    \phi^{-1}(p) \subset A
    \]

    is also a prime ideal. As a consequence, each morphism between two commutative rings induces a morphism in the opposite direction between their affine schemes.

    There exists a contravariant functor from the category of commutative rings to the category of locally ringed spaces:

    \begin{itemize}
      \item \textit{Mapping objects:}

      \[
      A \mapsto (\text{Spec } A, \mathcal{O}_{\text{Spec } A})
      \]

      \item \textit{Mapping morphisms:}

      \[
      (\phi : A \to B) \mapsto ((f, f^\sharp) : \text{Spec } B, \mathcal{O}_{\text{Spec } B} \to \text{Spec } A, \mathcal{O}_{\text{Spec } A})
      \]

      Here $f : \text{Spec } B \to \text{Spec } A$ is the pullback of prime ideals. And

      \[
      f^\sharp : \mathcal{O}_{\text{Spec } A} \to f_* \mathcal{O}_{\text{Spec } B}
      \]

      is defined on sections over basis open sets as

  \end{enumerate}
\end{itemize}
\[ \mathcal{O}_{\text{Spec } A}(D_g) = A_g \rightarrow f, \mathcal{O}_{\text{Spec } B}(D_g) := \mathcal{O}_{\text{Spec } B}(f^{-1}(D_g)) = \mathcal{O}_{\text{Spec } B}(D_{\Phi}) = B_{\Phi} \]

The functor is a full functor, i.e. it is surjective on morphisms, see [18, Chap II, Prop. 2.3].

**Remark 7.15 (Cohomology of affine schemes).** Consider an affine scheme
\[ X = (\text{Spec } A, \mathcal{O}_{\text{Spec } A}) \]
with a Noetherian ring \( A \) and an \( A \)-module \( M \).

1. **Injective resolution:** There exists an injective resolution of \( A \)-modules
\[ 0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \ldots \]
see [18, Chap. III, Prop. 2.1].

2. **Flasque resolution:** The exactness of the functor from Remark 7.14 induces the resolution of \( \tilde{M} \)
\[ 0 \rightarrow \tilde{M} \rightarrow \tilde{I}^0 \rightarrow \tilde{I}^1 \rightarrow \ldots \]
All \( \mathcal{O}_X \)-modules \( \tilde{I}^j, j \in \mathbb{N}, \) are flasque, see [18, Chap. III, Prop. 3.4], and therefore acyclic. Hence
\[ H^j(X, \tilde{M}) = \frac{\ker \left[ \Gamma(X, \tilde{I}^j) \rightarrow \Gamma(X, \tilde{I}^{j+1}) \right]}{\text{im} \left[ \Gamma(X, \tilde{I}^{j-1}) \rightarrow \Gamma(X, \tilde{I}^{j}) \right]} \]

**Proposition 7.16 (Theorem B for affine schemes).** On an affine scheme
\[ X = (\text{Spec } A, \mathcal{O}_{\text{Spec } A}) \]
each quasi-coherent \( \mathcal{O}_X \)-module \( \tilde{M} \) is acyclic.

For a proof under the additional assumption that \( A \) is Noetherian see [18, Chap. III, Theor. 3.5].

**Theorem 7.17 (Characterization of affine schemes).** For a Noetherian scheme \( X \) the following properties are equivalent:

1. The scheme \( X \) is affine, i.e. there exists a commutative ring \( A \) with unit such that \( X = (\text{Spec } A, \mathcal{O}_{\text{Spec } A}). \)

2. All quasi-coherent \( \mathcal{O}_X \)-modules are acyclic.

3. Each quasi-coherent ideal sheaf \( \mathcal{I} \subset \mathcal{O}_X \) satisfies
\[ H^1(X, \mathcal{I}) = 0. \]
7.3 Affine schemes

The proof is due to Serre, see [18, Chap. III, Theor. 3.7].

Due to Theorem 6.24 the Stein manifolds from complex analysis correspond to complex affine schemes without singularities from algebraic geometry.
List of results

Chapter 1. Holomorphic functions of several variables

Convergence of power series in several variables (Prop. 1.2)
Holomorphy and partial holomorphy (Theor. 1.6)
Integral formula with Cauchy kernel (Cor. 1.7)
Identity theorem (Theor. 1.10)
Hartogs’ continuity theorem (Prop. 1.14)
Weierstrass’ convergence theorem (Theor. 1.16)
Fréchet space (Def. 1.17)
Solution of the \( \partial \)-problem (Theor. 1.24)
Dolbeault’s Lemma (Theor. 1.27)

Chapter 2. Complex manifolds and sheaves

Complex manifold (Def. 2.2)
Sheaf and presheaf (Def. 2.12)
Stalk of a presheaf (Def. 2.15)
Sheaf of meromorphic functions (Def. 2.19)
Exact presheaf sequence resp. sheaf sequence (Def. 2.22)
List of results

Chapter 3. Sheaf cohomology

Čech cohomology group of a sheaf (Def. 3.7)
Long exact cohomology sequence (Theor. 3.12)
Canonical flabby resolution of a sheaf (Theor. 3.15)
Leray’s theorem (Theor. 3.19)
Acyclicity of the sheaf of smooth differential forms (Prop. 3.21)
Dolbeault’s theorem for holomorphic differential forms (Theor. 3.22)
Acyclicity of the sheaf of holomorphic differential forms on polydiscs (Cor. 3.23)
Solving the Cousin problems for polydiscs (Cor. 3.24)

Chapter 4. Local theory and coherence of sheaves

Weierstrass polynomial (Def. 4.1)
Laurent splitting (Theor. 4.3)
Weierstrass preparation theorem (Theor. 4.7)
Weierstrass division theorem (Theor. 4.9)
Hilbert-Rückert ideal basis theorem (Theor. 4.13)
Hilbert’s syzygy theorem from local analytic geometry (Theor. 4.24)
Coherent $\mathcal{O}$-module (Def. 4.26)
Coherence of the structure sheaf (Theor. 4.28)
List of results

Coherence of the ideal sheaf of an analytic submanifold (Prop. 4.29)
Coherence in short exact sequences (Prop. 4.33)

Chapter 5. Cartan’s lemma for holomorphic matrices
Local openness criterion for strictly differentiable maps (Prop. 5.4)
Approximation of holomorphic matrix functions (Cor. 5.7)
Cartan’s lemma for holomorphic matrix functions (Theor. 5.8)
Hilbert’s syzygy theorem for coherent $\mathcal{O}$-modules (Theor. 5.14)
Acyclicity of coherent $\mathcal{O}$-modules after shrinking to a polydisc (Cor. 5.15)
Coherent sheaves are Fréchet sheaves (Prop. 5.17)

Chapter 6. Theorem B and Theorem A on Stein manifolds
Holomorphic convexity (Def. 6.1)
Analytic polyhedron (Def. 6.6)
Stein manifold (Def. 6.10)
Embedding of analytic polyhedra of Stein manifolds (Theor. 6.13)
Runge pair (Def. 6.18)
Runge approximation for coherent sheaves (Theor. 6.21)
Theorem B for coherent $\mathcal{O}$-modules (Theor. 6.22)
Theorem A for coherent $\mathcal{O}$-modules (Theor. 6.23)
Cohomological characterization of Stein manifolds (Theor. 6.24)
Kugelsatz: General form (Prop. 6.26)
Hilbert’s Nullstellensatz on Stein manifolds (Theor. 6.29)
Holomorphic de Rahm theorem on Stein manifolds (Theor. 6.31)

FINIS
References

The main reference for the present text are the notes [6].

An excellent textbook is [20] by L. Kaup and B. Kaup. Their book covers also the theory of Stein spaces, i.e. it considers also singularities. In addition, it proves the fundamental theorems from local analytic geometry.

17. Hörmander, Lars: An Introduction to Complex Analysis in Several Variables. North Holland, Amsterdam (1973)
Index

\( \partial \)-problem, 24
\( d'' \)-equation, 26
Čech class
definition, 76
Čech cohomology
as inductive limit, 85
with respect to a covering, 77
acyclicity criterion, 163
affine scheme
cohomology, 228
definition, 225
equivalent characterization, 228
theorem \( B \), 228
analytic polyhedron
definition, 183
embedding theorem, 189
is relatively-holomorphically convex, 184
provides a neighbourhood basis, 185
analytic submanifold, 45, 46

Banach space of bounded holomorphic functions, 147

Cartan lemma
for holomorphic functions, 66
Cartan’s lemma
bounded holomorphic functions, 148
invertible matrix functions, 160
Cauchy integral formula, 11
chain homotopy, 80
chart, 37
coboundary
definition, 76
coboundary operator, 77
cochain
definition, 76
cocycle
cocycle relation, 77
definition, 76
coherence in short exact sequences, 143
cohomology
axiomatic theory, 106
computation by flabby resolution, 98
Leray’s theorem, 99
of affine scheme, 228
with compact support, 222
complex atlas, 37
complex manifold
as ringed space, 61
definition, 39
complex space, 223
Cousin induction, 64
Cousin problem
additive, 62
cohomological formulation, 78
definition, 62
multiplicative, 62
solution, 63
solution for product domains, 69
solution on polydisc, 104
solution on polydiscs, 71
solution on Stein manifold, 212
de Rahm theorem
Abstract form, 96
holomorphic form, 216
density of restriction
holomorphic functions, 154
invertible matrix functions, 156
density on restriction
Runge pairs, 196
differential forms, 20
distinguished polynomial, 110
Dolbeault’s lemma, 31
Dolbeault’s theorem, 104
domain of holomorphy
   definition, 178
   equivalence for domain in \(\mathbb{C}^n\), 182
double complex, 98
duality
   on compact Riemann surfaces, 221
   on Stein manifolds, 222
embedding, 45
exactness
   long exact cohomology sequence, 86
   presheaf sequence, 57
   sheaf sequence, 57
   exhaustion, 40
   exponential sequence, 58
   exterior derivation, 22
field of meromorphic functions, 55
Fréchet space
   definition, 18
   group of cocycles, 176
   of differential forms, 23
Hartogs theorem, 11
higher direct image sheaf, 145
holomorphic function, 39
holomorphic differential form, 22
holomorphic function, 7, 39
holomorphic map, 39
holomorphic structure sheaf, 50
holomorphically convex
   criterion, 179
   definition, 178
holomorphically convex hull, 178
holomorphically spreadable, 188
homological dimension, 128, 166
homological dimension
   definition, 128
   in exact sequences, 128
identity theorem, 11
immersion, 42, 44
inductive limit, 55
integral domain, 13
Krull lemma, 128
local isomorphism, 41, 42, 44
local model, 60
local openness criterion, 150
locally finite covering, 39
logarithm, 106
long exact cohomology sequence
   existence, 86
   for presheaves, 90
   for sheaves, 91
manifold, 37
meromorphic function, 54
Mittag-Leffler induction, 29
Nakayama lemma, 127
nilradical, 224
Osgood’s lemma, 9
paracompact, 40
parameter representation, 45
power series, 6
presheaf
   definition, 46
   morphism, 47
   section, 46
   sheafification, 52
product domain, 63
refinement, 80, 82
relatively compact, 40
relatively-holomorphically convex, 178
Ring of convergent powers series
   factorial, 122
   Noetherian, 121
   normal, 122
ringed space
   definition, 59
   local model, 60
   morphism, 59
Runge approximation
   coherent sheaf, 200
   structure sheaf, 197
Runge pair
   of analytic polyhedra, 199
sheaf
   acyclic, 92
   acyclicity of coherent sheaf on polydisc, 171
   acyclicity of smooth structure sheaf, 102
   acyclicity of structure sheaf of polydisc, 104
   canonical flabby resolution, 95
   coherence of ideal sheaf, 138
   coherent, 130
   constant sheaf, 50
   definition, 48
   direct image, 59
   example, 49
flabby, 92
flabby sheaves are acyclic, 92
Fréchet sheaf, 172
holomorphic structure sheaf, 49
invertible, 164
locally free sheaf, 164
locally free sheaf on a polydisc, 164
morphism, 48
of fractions, 54
of meromorphic functions, 54
of modules, 129
of smooth differential forms, 104
quotient sheaf, 53
relation-finite, 130
smooth structure sheaf, 49
stalk
of a presheaf, 51
of coherent sheaf, 139
of structure sheaf, 55
Stein manifold
affine closed embedding, 219
character theorem, 220
definition, 187
equivalent characterization, 209
exhaustion by analytic polyhedra, 189
Theorem A, 208
Theorem B, 204
Theorem B for analytic polyhedra, 194
Stein space
definition, 224
reduction and normalization, 225
strict differentiability, 149
submersion, 42–44
theorem
Nakayama lemma, 127
Grauert’s coherence theorem, 146
Hartogs’ continuity result, 15
Hensel’s lemma, 123
Hilbert syzygy theorem for modules, 129
Hilbert’s syzygy theorem for coherent sheaf, 167
holomorphic de Rahm theorem, 216
identity theorem, 11
Krull lemma, 128
Kugelsatz, 16, 212
Laurent splitting, 112
Leray’s theorem, 99
maximum modulus theorem, 14
Oka’s coherence theorem, 131
openness of holomorphic maps, 13
Rückert’s theorem, 121
Weierstrass division theorem, 118
Weierstrass preparation theorem, algebraic, 116
Weierstrass preparation theorem, analytic, 114
Weierstrass’ convergence theorem, 17
topology of compact convergence, 19
transition map, 37
Weierstrass polynomial, 110