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Differentiable manifolds

Retake Exam

What to do first:

- Please stow away your mobile phone in a place where it is not distracting you and turn it off. No cheat sheets, pocket calculators etc. are permitted.
- Check that your copy of this exam contains **6 exercises**.
- Do not use the colors red or green. Write with a pen and not with a pencil.
- Put your name on top of **every** sheet of paper you hand in. Make an extra effort so that we can decipher it.
- Different exercises are to be solved on different sheets of paper. Make sure we know what you are solving.
- Hand in only one solution per exercise. If you hand in several solutions for one question we will invariably grade the worst version and ignore all others. Cross out clearly whatever you want us to ignore.
- Fill out the following:

Name: _____ Surname: _____

Matrikelnr.: _____ Semester: _____

Subject: Bachelor Mathematics ❑ Master Mathematics ❑

Bachelor Physics ❑ Master Physics ❑

TMP ❑

Other : _____

You have **120 Minutes**.

Good luck!

	1	2	3	4	5	6	$\sum$
	/6	/8	/10	/12	/8	/6	/50

Name: _____

Exercise 1. [6 Points]

1. Let $f : M \longrightarrow N$ be a smooth map. When is $p \in N$ a regular value?
2. Let A be a non-degenerate symmetric $n \times n$-matrix and $0 \neq b \in \mathbb{R}$. Show that b is a regular value of

$$\begin{aligned} B : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto x^T A x. \end{aligned}$$

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1. By definition, $p \in N$ is a regular value of f if and only if all its preimages $x \in f^{-1}(p)$ are regular points in M. A regular point $x \in M$ of map $f : M \to N$ is a point where the differential Df_x is surjective.

   (2 points)

2. To check that $b \neq 0$ is a regular value, we should check that the differential of B at all preimages of b is surjective. Since $\mathbb{R}$ is a one-dimensional manifold, the differential is surjective iff it is non-zero. We consider a map between euclidean spaces, so the coordinate charts can be chosen to be just the identity maps.

   The differential of B at point $x \in \mathbb{R}$ evaluated on tangent vector $h \in \mathbb{R}^n$ is

$$Df_x(h) = 2h^T A x$$

   so Df_x is surjective iff $x \neq 0$. We assumed that A is a non-degenerate quadratic form so the only singular value in $\mathbb{R}$ is $b = 0$.

   (2 points for differential, 2 points for surjectivity)

Name: _____

Exercise 2. [8 Points]

Let (M, g) be a Riemannian manifold.

1. What is the Levi-Civita connection of (M, g)?
2. Let $f : M \longrightarrow M$ be a diffeomorphism and $h = f^*g$ the Riemannian metric $h(X, Y) = g(Df(X), Df(Y))$. Show that if ∇ is the Levi-Civita connection of (M, g) then the Levi-Civita connection $\widehat{\nabla}$ of (M, h) satisfies

$$Df(\widehat{\nabla}_X Y) = \nabla_{Df(X)}(Df(Y)).$$

Levi-Civita connection of (M, g) is a connection on M that is torsion-free,

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

and metric,

$$Z\,[g(X, Y)] = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

(2 points)
Since f is assumed to be a diffeomorphism, its differential is an isomorphism at every point of M. We can thus define a connection $\widehat{\nabla}$ using

$$Df(\widehat{\nabla}_X Y) = \nabla_{Df(X)}(Df(Y)).$$

and check that it is the (neccessarily unique) Levi-Civita connection with respect to metric f^*g. First of all, we should check that it is a connection, i.e. $\mathcal{C}^\infty$ linear in first argument and $\mathbb{R}$-linear in the second one. For example,

$$\begin{aligned} Df(\widehat{\nabla}_X(Y + \alpha Z)) &= \nabla_{Df(X)}(Df(Y + \alpha Z)) \\ &= \nabla_{Df(X)}(Df(Y)) + \alpha \nabla_{Df(X)}(Df(Z)) \\ &= Df(\widehat{\nabla}_X Y) + \alpha Df(\widehat{\nabla}_X Z) \end{aligned}$$

for $\alpha \in \mathbb{R}$. Next we should verify that this connection is torsion-free. Since Df is isomorphism, we can compute

$$Df\Big(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X - [X, Y]\Big) = \nabla_{Df(X)} Df(Y) - \nabla_{Df(Y)} Df(X) - [Df(X), Df(Y)] = 0.$$

Similarly, for metricity we have

$$\begin{aligned} \mathcal{L}_Z(h(X, Y)) &= f^*(\mathcal{L}_{Df(Z)} g(Df(X), Df(Y))) \\ &= f^*g(\nabla_{Df(Z)} Df(X), Df(Y)) + f^*g(Df(X), \nabla_{Df(Z)} Df(Y)) \\ &= f^*g(Df(\widehat{\nabla}_Z X), Df(Y)) + f^*g(Df(X), Df(\widehat{\nabla}_Z Y)) \\ &= h(\widehat{\nabla}_Z X, Y) + h(X, \widehat{\nabla}_Z Y) \end{aligned}$$

which was to be proved.
(2 points for tensoriality, 2 points for torsionfreeness, 2 points for metricity)

Name: _____

Exercise 3. [10 Points]

1. State Stokes theorem.
2. Consider the volume form $\omega = dx_1 \wedge dx_2 \ldots \wedge dx_n$ on $\mathbb{R}^n$. Prove that ω is exact.
3. What is the de Rham cohomology group $H^n(M)$ for compact, oriented, connected manifold M of dimension n?
4. Let M be a compact, connected, oriented manifold of dimension n and $\partial M = \emptyset$. Show that for every smooth map $F : M \longrightarrow \mathbb{R}^n$ there is a point $p \in M$ where DF is not injective (hint: both manifolds are of the same dimension).

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1. Let M be a smooth, compact, oriented manifold of dim n with boundary and $i : \partial M \to M$ be the inclusion map of the boundary into M. Then for $\omega \in \Omega^{(n-1)}$ we have

$$\int_{\partial M} i^*\omega = \int_M d\omega.$$

   To get the correct sign, we should orient the boundary by the outward normal first convention. (2 points)

2. The volume form in R^n can be written as an exact form, for example in the following way,

$$\omega = d\alpha = d(x_1 dx_2 \wedge ... \wedge dx_n).$$

   (2 points)

3. $H^n(M) = \mathbb{R}$.
   (2 points)

4. Since the dimension of M is the same as the dimension of R^n then if F is injective it is also bijective. Therefore, F maps a volume form to a volume form (possibly up to an overall sign). Thus, if ω is the volume form in R^n then $F^*\omega$ is the volume form in M. Since the integral of a volume form is non-vanishing (and finite since M is compact) we find,

$$0 \neq \int_M F^*\omega = \int_M F^* d\alpha = \int_M d(F^*\alpha) = \int_{\partial M} i^* F^* \alpha,$$

   where we used that $\omega = d\alpha$ in R^n and the Stokes theorem. However, M does not have a boundary, therefore the last integral is zero, this contradicts the assumption that F is injective.
   (4 points)

Name: _____

Exercise 4. [12 Points]

1. Let M, N be two compact, oriented, connected manifolds of the same dimension n. Define the mapping degree of a smooth map $f : M \longrightarrow N$.

2. Let $M = N = S^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ with the same orientation and consider the smooth map

$$\begin{aligned} f : S^2 &\longrightarrow S^2 \\ (x, y, z) &\longmapsto \frac{(x^2 - y^2, 2xy, z)}{\|(x^2 - y^2, 2xy, z)\|}. \end{aligned}$$

   Show that $(1, 0, 0)$ is a regular value of f and verify that the mapping degree of f is two.

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1. According to the lecture $H^n(M) \simeq \mathbb{R}$ for closed, connected and oriented manifolds of dimension n. Let $\omega \in \Omega^n(N)$ represent a non-trivial cohomology class (i.e. $\int_N \omega \neq 0$). Then the degree of f is

$$\deg(f) = \frac{\int_M f^*\omega}{\int_N \omega}.$$

   (2 points)

2. The preimage of $(1, 0, 0)$ under the map f is the set of the two points $p_\pm = (\pm 1, 0, 0)$. The differential of the map is

$$\begin{aligned} \partial_x f &= 2N \left\{x(2x^2y^2 + 2y^4 + z^2), y(-x^4 + y^4 + z^2), -x(x^2 + y^2)z\right\} \\ \partial_y f &= 2N \left\{-y(2x^4 + 2x^2y^2 + z^2), x(x^4 - y^4 + z^2), -y(x^2 + y^2)z\right\} \\ \partial_z f &= N \left\{-(x^2 - y^2)z, -2xyz, (x^2 + y^2)^2\right\}, \end{aligned}$$

   with $N = ((x^2 + y^2)^2 + z^2)^{-3/2}$. At the points $p_\pm$ the derivatives are

$$\begin{aligned} \partial_x f(p_\pm) &= \{0, 0, 0\} \\ \partial_y f(p_\pm) &= \{0, \pm 2, 0\} \\ \partial_z f(p_\pm) &= \{0, 0, 1\}. \end{aligned}$$

   A basis of tangent space at the points $p_\pm$ can be chosen to be $B = \{\partial_y, \partial_z\}$. The differential map Df transforms it to

$$\begin{aligned} Df_{p_\pm}(\partial_y) &= \pm 2\partial_y \\ Df_{p_\pm}(\partial_z) &= \partial_z, \end{aligned}$$

   so we see that $Df_{p_\pm}$ is surjective. Thus, $(1, 0, 0)$ is a regular value.
   The basis B at p_+ has a positive orientation which under the map f transforms to a positively orientated basis. The basis B at p_- has a negative orientation and under the map f transforms to a negatively orientated basis. Therefore, the map f is orientation preserving at both points $p_\pm$ and the degree of the map f is 2.
   (2 points for preimages, 5 points for surjectivity, 3 points for orientation)

Name: _____

Exercise 5. [8 Points]

Let N be the group

$$\left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| \, x, y, z \in \mathbb{R} \right\}$$

We define a smooth structure $\mathcal{A}$ on N using the single chart

$$\begin{aligned} \varphi : N &\longrightarrow \mathbb{R}^3 \\ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) &\longmapsto (x, y, z). \end{aligned}$$

1. Prove that the inversion

$$\begin{aligned} I : N &\longrightarrow N \\ A &\longmapsto A^{-1} \end{aligned}$$

and the group multiplication

$$\begin{aligned} \mu : N \times N &\longrightarrow N \\ (A, B) &\longmapsto AB \end{aligned}$$

are smooth.

2. Compute the image of $\partial_x, \partial_y, \partial_z$ under the differential of

$$\begin{aligned} Ad_A : N &\longrightarrow N \\ X &\longmapsto AXA^{-1} \end{aligned}$$

at the identity element of N. Parametrize $A \in N$ such that $\varphi(A) = (a, b, c)$.

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1. We can verify that the matrix inverse of the matrix

$$X = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

is

$$X^{-1} = \begin{pmatrix} 1 & -x & -z + xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}$$

so in our local chart the matrix multiplication is given simply by the map

$$(x, y, z) \mapsto (-x, -y, -z + xy).$$

This is obviously smooth (polynomials are continuous). As for the multiplication, it is enough to check that

$$AX = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a + x & c + z + ay \\ 0 & 1 & b + y \\ 0 & 0 & 1 \end{pmatrix}$$

is smooth which is again clear.

(2+2 points)

2. We can first compute the expression for Ad_A in terms of local charts:

$$AXA^{-1} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a & -c+ab \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & z+ay-bx \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

so in local coordinates, the map Ad_A is represented by a smooth map

$$(x, y, z) \mapsto (x, y, z + ay - bx).$$

The differential of this map maps basis vectors at identity as follows:

$$\begin{aligned} \partial_x &\mapsto \partial_x - b\partial_z \\ \partial_y &\mapsto \partial_y + a\partial_y \\ \partial_z &\mapsto \partial_z. \end{aligned}$$

Alternatively, since we are working with a matrix group, we can represent the vector ∂_x at identity as a matrix

$$M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The adjoint action maps this to

$$AM_1A^{-1} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -a & -c+ab \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is a representative of $\partial_x - b\partial_y$.

The vector ∂_y at identity element is represented by matrix

$$M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Its left-translate to point $X \in M$ is represented by

$$XM_1 = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so the left-invariant vector field we are looking for is

$$\partial_y + x\partial_z$$

(4 points)

Name: _____

Exercise 6. [6 Points]

Define the torsion and the curvature tensor of connection ∇ and show that if the torsion vanishes, the first Bianchi identitfy

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

is satisfied.

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1. The torsion tensor is

$$\begin{aligned} T : \chi(M) \times \chi(M) &\longrightarrow \chi(M) \\ (X,Y) &\longmapsto \nabla_X Y - \nabla_Y X - [X,Y]. \end{aligned}$$

   The curvature tensor is $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ where X, Y, Z are smooth vector fields on M.
   (2 points)

2. The left hand side of the Bianchi identity is,

$$\begin{aligned} B(X,Y,Z) &= R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &+ \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X \\ &+ \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y \\ &= \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X) \\ &- \nabla_{[Y,Z]} X - \nabla_{[X,Y]} Z - \nabla_{[Z,X]} Y. \end{aligned}$$

   Using the fact that the torsion vanishes we find,

$$\begin{aligned} B(X,Y,Z) = \nabla_X([Y,Z]) + \nabla_Y([Z,X]) + \nabla_Z([X,Y]) \\ - \nabla_{[Y,Z]} X - \nabla_{[X,Y]} Z - \nabla_{[Z,X]} Y. \end{aligned}$$

   Using the vanishing of the torsion again,

$$\begin{aligned} B(X,Y,Z) &= \nabla_X([Y,Z]) + \nabla_Y([Z,X]) + \nabla_Z([X,Y]) \\ &- (\nabla_X([Y,Z]) + [[Y,Z],X]) - (\nabla_Z([X,Y]) + [[X,Y],Z]) \\ &- (\nabla_Y Z([Z,X]) + [[Z,X],Y]) \\ &= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \end{aligned}$$

   The last line is zero because of the Jacobi identity for Lie algebra.
   (4 points)