ADVANCED ANALYSIS
Exercise sheet 9 – 12.01.2023

Reminder: $f_k \longrightarrow f$ means that f_k converges strongly to f and $f_k \rightharpoonup f$ means that f_k converges weakly to f. In both cases, one needs to specify in which space (i.e. in which manner or "topology").

Ex.1.1 Let $1 < p < \infty$. Assume that $\{u_j\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Prove that there exists a subsequence $\{u_{i_k}\}$ and $u \in W^{1,p}(\Omega)$ such that $u_{i_k} \rightharpoonup u$ in L^p and $Du_{i_k} \rightharpoonup Du$ in $L^p(\Omega)$.

Ex.1.2

1. Let $\{f_k\} \subset L^p \cap L^q(\mathbb{R}^n)$ such that $f_k \rightharpoonup f$ in $L^p(\mathbb{R}^n)$ and $\sup_k \|f_k\|_{L^q} < \infty$. Show that $f_k \rightharpoonup f$ in $L^q(\mathbb{R}^n)$ as well.
   Hint: First show that $\{f_k\}$ converges weakly in L^q up to a subsequence and then show that the limit does not depend on the subsequence (and conclude).
2. Let $n \geq 3$, $\{f_k\} \subset D^1(\mathbb{R}^n)$ such that $\nabla f_k \rightharpoonup g$ in L^2. Show that $f_k \rightharpoonup f$ in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ for some $f \in D^1(\mathbb{R}^n)$ such that $\nabla f = g$.
3. Let $\{f_j\} \subset H^1(\mathbb{R}^n)$ such that $f_j \rightharpoonup f$ in $L^2(\mathbb{R}^n)$ and $\nabla f_j \rightharpoonup g_i$ in $L^2(\mathbb{R}^n)$ ($\rightharpoonup$ means weakly).
   (a) Recall what $f_j \rightharpoonup f$ in L^2 and $\nabla f_j \rightharpoonup g_i$ in $L^2(\mathbb{R}^n)$ means.
   (b) Show that $f \in H^1(\mathbb{R}^3)$ and that $\nabla f = g$ in the distributional sense.

Ex.1.3
Show that the characteristic function of a (measurable) set in $\mathbb{R}^n$ having finite and positive measure is never in $H^1(\mathbb{R}^n)$.

Ex.1.4(Heat equation)
For $t > 0$, and $\varphi \in C_c^\infty(\mathbb{R}^n)$, we define $e^{t\Delta}[\varphi]$ to be the function

$$e^{t\Delta}[\varphi] = \left(e^{-t|2\pi k|^2}\widehat{\varphi}(k)\right)^\vee .$$

1. Show that for $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have
   $$e^{t\Delta}[\varphi] = K_t * \varphi$$
   for a certain function K_t.
2. Using this, show that $e^{t\Delta}$ can be continuously extended to all $L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2$.
   Hint: what does "continuously extended" mean ?
3. For $1 \leq p \leq 2$, $t > 0$ and $f \in L^p(\mathbb{R}^n)$, show that

(a) $g_t := e^{t\Delta}[f]$ is C^∞

(b) Show that the following limit exists in L^p (strongly)

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t := \lim_{\varepsilon \to 0} \frac{g_{t+\varepsilon} - g_t}{\varepsilon}.$$

(c) Show that

$$\Delta g_t = \frac{\mathrm{d}}{\mathrm{d}t} g_t$$
$$g_t \underset{t \to 0}{\longrightarrow} f \text{ strongly in } L^p.$$