Mathematisches Institut LMU München
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ADVANCED ANALYSIS
Exercise sheet 12 – 31.01.2023

Ex.1.1 (Poincarés inequality)

Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and with the cone property. Let $1 \leq p < n$ and $g \in L^{p'}(\Omega)$ with $\int g = 1$, we want to prove that for $1 \leq q \leq np/(n-p)$, there exists some number $C_P > 0$ such that for all $f \in W^{1,p}(\Omega)$

$$\|f - \int fg\|_{L^q(\Omega)} \leq C_P \|\nabla f\|_{L^p(\Omega)} \tag{1}$$

I. Case $q < np/n - p$.

1. Recall the Sobolev inequality associated to $W^{1,p}(\Omega)$.

2. We prove (1) by contradiction. Assume that no such constant C_P exists. Justify that there exists $\{f_j\} \subset W^{1,p}(\Omega) \setminus \{0\}$ such that $\|f - \int fg\|_{L^q(\Omega)} \neq 0$ and
$$\frac{\|\nabla f\|_{L^p(\Omega)}}{\|f - \int fg\|_{L^q(\Omega)}} \underset{j\to\infty}{\longrightarrow} 0.$$

3. Denote $h_j = f_j - \int f_j g$. Show that $\{h_j\}$ is bounded in $W^{1,p}(\Omega)$.

4. Show that, up to a subsequence, $h_j \rightharpoonup h_\infty$ and compute for any $\varphi \in C_c^\infty(\Omega)$
   - $\lim_{j\to\infty} \int h_j \nabla\varphi$
   - $\lim_{j\to\infty} \int h_j g$.

5. Conclude that $h_\infty = 0$.

6. Show that $h_j \underset{j\to\infty}{\longrightarrow} h_\infty$ strongly in L^p.

7. Arrive at a contradiction.

II. Case $q = np/n - p$.

1. Justify that for some $C > 0$ and all $f \in W^{1,p}(\Omega)$, it holds
$$\|f - \int fg\|_{L^q(\Omega)} \leq C \left(\|f - \int fg\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)} \right)$$

2. Use the case **I** to conclude.

Ex.1.2 (Hardy's inequality, again)

Let $n \geq 3$, we want to show that for all $f \in H^1(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2} \mathrm{d}x \leq \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |\nabla f(x)|^2 \mathrm{d}x \tag{2}$$

1. Let $u \in C_c^\infty(\mathbb{R}^n)$ non-negative. Show by integration by part that
$$\int_{\mathbb{R}^n} u \nabla u \cdot \frac{x}{|x|^2} = -(n-2) \int_{\mathbb{R}^n} \frac{u^2}{|x|^2}.$$
2. Using the Cauchy-Schwarz inequality (Hölder), show that (2) holds for $u \in C_c^\infty(\mathbb{R}^n)$.
3. Conclude.

Ex.1.3 (Missing assumption in the Fatou lemma)

Find some sequence of measurable functions $\{f_j\}$, such that $\int |f_j| < \infty$ for all j and

$$\liminf_{j \to \infty} \int_{\mathbb{R}} f_j < \int_{\mathbb{R}} \liminf_{j \to \infty} f_j.$$