Infinite Particle Systems in Non-Discrete Spaces: Orthogonal Intertwiners

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CS37-Wiener chaos, orthogonal polynomials, and intertwinings

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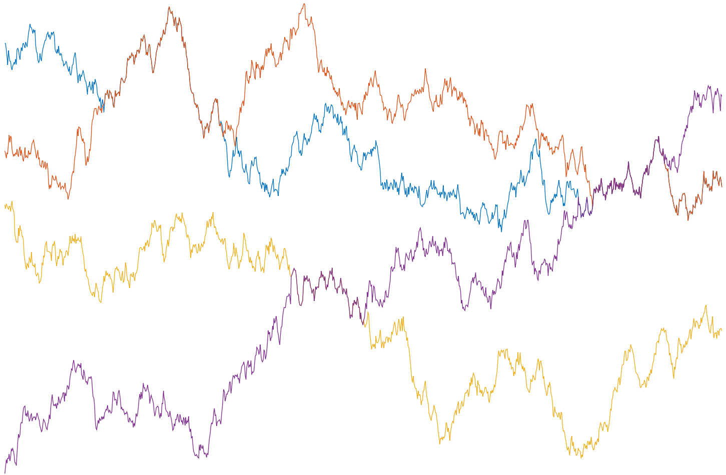
Particle system

$=$ a time-continuous Markov process describing the stochastic evolution of particles in a set E given a deterministic initial configuration

Uniform sticky Brownian motions with stickiness $\theta > 0$

Special case of a martingale problem introduced by Howitt, Warren '09.

Let $E = \mathbb{R}$.



Different notations

Unlabeled notation	Labeled notation
Markov process	family of Markov processes indexed via number of particles $N \in \mathbb{N}$
values in $\mathbf{N}$, the set of counting measures consisting of $\mu = \sum_{k=1}^{N} \delta_{x_k}$, $x_k \in \mathbb{R}$. $\mu(A) =$ number of particles in a subset A	values in $\mathbb{R}^N$
$(P_t)_{t \geq 0}$ Markov semigroup acting on functions $F : \mathbf{N} \to \mathbb{R}$	family of Markov semigroups $(P_t^{[N]})_{t \geq 0}, N \in \mathbb{N}$ acting on functions $f_N : \mathbb{R}^N \to \mathbb{R}$

n-th factorial measure intertwiner

For $f_n : \mathbb{R}^n \to \mathbb{R}$ and a configuration $\sum_{k=1}^N \delta_{x_k}$ put

$$J_n\left(f_n, \sum_{k=1}^N \delta_{x_k}\right) := \sum_{\substack{i_1,\dots,i_n=1 \\ \text{pairwise different}}}^N f_n(x_{i_1}, \dots, x_{i_n})$$

$$J_1(f_1, \delta_x + \delta_y + \delta_z) = f_1(x) + f_1(y) + f_1(z),$$
$$J_2(f_2, \delta_x + \delta_y + \delta_z) = f_2(x, y) + f_2(y, x) + f_2(x, z) + f_2(z, x) + f_2(y, z) + f_2(z, y)$$

Theorem 1 (W., '23)

J_n intertwines the dynamics of infinitely many sticky Brownian motions and their n-particle evolution, i.e.,

$$P_t J_n(f_n, \cdot)(\mu) = J_n\left(P_t^{[n]} f_n, \mu\right) \qquad f_n : \mathbb{R}^n \to \mathbb{R}, \quad \mu \in \mathbf{N}$$

Example

Given an initial configuration $\sum_{k=1}^{N} \delta_{x_k}$: What is the expected number of particles in a set A at a time $t > 0$?

$$
\begin{aligned}
\text{expected number of particles in a set } A \text{ at time } t \quad &= \quad P_t \left(\nu \mapsto \nu(A)\right) \left(\sum_{k=1}^{N} \delta_{x_k} \right) \\
&= \quad P_t J_1(\mathbf{1}_A, \,\cdot\,) \left(\sum_{k=1}^{N} \delta_{x_k} \right) \\
&= \quad J_1 \left(P_t^{[1]} \mathbf{1}_A, \mu \right) \\
&= \quad \sum_{k=1}^{N} P_t^{[1]} \mathbf{1}_A(x_k) \qquad =
\end{aligned}
$$

Let each particle evolve the one-particle dynamics separately. Sum of the probabilities that the respective particle is in the set A at time t.

The proof relies solely on compatibility (Le Jan, Raimond, '04)

$=$ *"the action of removing a particle commutes with the dynamics"*.

Le Jan, Raimond proved a one-to-one correspondence to stochastic flows.

Examples:

- independent particles:
  - independent random walkers (IRW)
  - free Kawasaki dynamics
- correlated Brownian motions
- coalescing Brownian motions

Theorem 2 (Redig, Jansen, Floreani, W., '21)

Consider a finite particle system that preserves the number of particles. Then,

consistency $\iff$ *J_n intertwines $P_t^{[n]}$ and P_t for all t and n, i.e.,*

$$P_t J_n(f_n, \cdot)(\mu) = J_n\left(P_t^{[n]} f_n, \mu\right), \qquad f_n : \mathbb{R}^n \to \mathbb{R}, \quad \mu \in \mathbf{N}$$

consistency = *"the action of removing a particle* **uniformly at random** *commutes with the dynamics"*.

Examples:

- symmetric inclusion process (SIP), symmetric exclusion process (SEP).
  - Theorem 2 recovers well-known self-dualities in terms of falling factorial polynomials.
- generalized SIP in the continuum = Moran process from population genetics

Infinite-dimensional orthogonal polynomials

Let ρ be a probability measure on $\mathbf{N}$, i.e., the distribution of a point process.

$$I_n(f_n, \cdot\,) \quad := \quad \begin{array}{l}\text{orthogonal projection of } J_n(f_n, \cdot\,) \\ \text{onto } \{J_k(u_k, \cdot\,), u_k : \mathbb{R}^k \to \mathbb{R}, 0 \le k \le n-1\}^\perp \\ \text{in } L^2(\rho).\end{array}$$

Keywords: infinite-dimensional orthogonal polynomials, (extended) Fock spaces, chaos decompositions, multiple stochastic integrals, non-Gaussian white noise analysis, Malliavin calculus

Let ρ be the distribution of a Pascal process with parameters $\theta \mathrm{Vol}_{\mathbb{R}}$ and arbitrary $p \in (0,1)$.

Theorem 3 (W., '23)

I_n intertwines the dynamics of infinitely many sticky Brownian motions and their n-particle evolution, i.e.,

$$P_t I_n(f_n, \cdot)(\mu) = I_n\left(P_t^{[n]} f_n, \mu\right) \qquad f_n : \mathbb{R}^n \to \mathbb{R}, \quad \mu \in \mathbf{N}$$

Corollary 4 (W., '23)

ρ is reversible for infinitely many sticky Brownian motions.

The proof of the corollary applies a result of Brockington, Warren '23: existence of a reversible measure for n ordered uniform sticky Brownian motions for each $n \in \mathbb{N}$.

Connection: reversible measures and orthogonal intertwiners

Theorem 5 (Redig, Jansen, Floreani, W., '21)

$$J_n \textit{ intertwines } P_t^{[n]} \textit{ and } P_t \textit{ for all } t \textit{ and } n \quad + \quad \rho \textit{ reversible}$$

$$\Downarrow$$

$$I_n \textit{ intertwines } P_t^{[n]} \textit{ and } P_t \textit{ for all } t \textit{ and } n$$

Together with Theorem 2:

finite number of particles $+\ \rho$ reversible $+$ consistency $\quad\Rightarrow\quad$ I_n intertwines $P_t^{[n]}$ and P_t

Applications:

- Theorem 5 recovers well-known self-dualities in terms of orthogonal polynomials for discrete particle systems (SIP, SEP, IRW)
- independent particles (ρ distribution of the Poisson process)
- generalized SIP (ρ distribution of the Pascal process)

Thank you!

- S. Floreani, S. Jansen, F. Redig, S.W.: *Duality and intertwining for consistent Markov processes*, arXiv:2112.11885 [math.PR], 32 pp.
- S.W.: *Orthogonal Intertwiners for Infinite Particle Systems In The Continuum*, arXiv:2305.03367 [math.PR], 24 pp.