Stochastic Self-Duality for Consistent Particle Systems on General State Spaces

16. DoktorandInnentreffen der Stochastik

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Overview

1. **Known facts - discrete spaces.**
   Self-dualities of Markov processes describing the evolution of particles on a discrete set.
2. **Research - general spaces.**
   What happens if we replace the discrete space by a much more general space?

Joint work with

- Sabine Jansen (LMU)
- Frank Redig (TU Delft)
- Simone Floreani (TU Delft)

Stochastic Duality

Definition (Stochastic duality of Markov processes)

Let $X = (\Omega_1, \mathcal{F}_1, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{X}})$ and $Y = (\Omega_2, \mathcal{F}_2, (Y_t)_{t\geq 0}, (\mathbb{P}^y)_{y\in\mathbb{Y}})$ be two (time-continuous) Markov processes with state spaces $\mathbb{X}, \mathbb{Y}$. X and Y are dual with respect to $H : \mathbb{X}\times\mathbb{Y} \to \mathbb{R}$ if and only if for all $x \in \mathbb{X}, y \in \mathbb{X}$ and $t \geq 0$

$$\mathbb{E}_x H(X_t, y) = \mathbb{E}^y H(x, Y_t).$$

Semigroup[1] notation: $P_t H(\cdot, y)(x) = S_t H(x, \cdot)(y)$

Self-duality: $X = Y$.

[1] Markov semigroup: $P_t f(x) := \mathbb{E}_x f(X_t)$

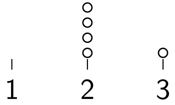
Particle configurations

We are interested in Markov processes describing the time-evolution of particles.
Let E be a countable set (e.g. $E = \{1, \ldots, N\}$, $E = \mathbb{Z}^d$, graph). Consider

$$\mathbb{X} := \mathbb{N}_0^E := \{(x_k)_{k \in E} : x_k \in \mathbb{N}_0\} .$$

x_k = "number of particles at position k".

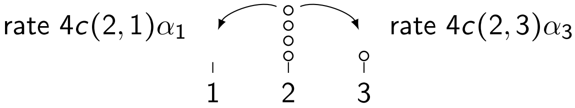
Example: $E = \{1, 2, 3\}$: $x = (x_1, x_2, x_3) = (0, 4, 1) =$



The Independent Random Walker, IRW

We introduce three models by their (formal) generators[2] ($(\delta_k)_\ell := \delta_{k,\ell}$). Fix $\alpha_k \in \mathbb{N}$, $c : E \times E$ is an arbitrary symmetric function (spatial component). For $f : \mathbb{N}_0^E \to \mathbb{R}$:

$$Lf(x) = \sum_{k \in E} \sum_{\ell \in E} (f(x - \delta_k + \delta_\ell) - f(x)) c(k, \ell) \alpha_\ell x_k$$

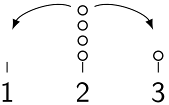


[2] Generator of a Markov process: $Lf(x) := \partial_{t|t=0} P_t f(x)$

The Symmetric Inclusion Process, SIP

$$Lf(x) = \sum_{k \in E} \sum_{l \in E} (f(x - \delta_k + \delta_l) - f(x))c(k,l)(\alpha_l + x_l)x_k$$

rate $4c(2,1)(\alpha_1 + 0)$

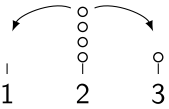


rate $4c(2,3)(\alpha_3 + 1)$

The Symmetric Exclusion Process, SEP

$$Lf(x) = \sum_{k \in E} \sum_{l \in E} (f(x - \delta_k + \delta_l) - f(x)) c(k,l)(\alpha_l - x_l) x_k$$

rate $4c(2,1)(\alpha_1 - 0)$



rate $4c(2,3)(\alpha_3 - 1)$

Consistency

"the action of removing a particle uniformly at random commutes with the dynamic"
These three models share the following[3] relation: $\mathcal{A}L = L\mathcal{A}$ for the so-called lowering operator $\mathcal{A}f(x) := \sum_{k\in E} x_k f(x-\delta_k)$.
In terms of expectations, for each $f : \mathbb{N}_0^E \to \mathbb{R}$, $t \geq 0$, $x \in \mathbb{N}_0^E$

$$\mathbb{E}_x\left[\sum_{k\in E} f(X_t-\delta_k)X_t\right] = \sum_{k\in E} x_k \mathbb{E}_{x-\delta_k}\left[f(X_t)\right].$$

[3] There is a link to Lie-algebras

Conservation of the number of particles

For all $t \geq 0$ and $X_0 \in \mathbb{N}_0^E$

$$\sum_{k \in E} (X_0)_k = \sum_{k \in E} (X_t)_k.$$

Thus we can define also the dynamics of exactly n-particles. We denote the Markov-semigroup by $(p_t^{[n]})_{t \geq 0}$

Duality with falling factorials

Theorem (Carinci, Giardinà, Redig, '19)

Let ϱ be a reversible measure (i.e. detailed balance), $(n)_k := n(n-1)\cdots(n-k+1)$. Then, a self-duality function for IRW, SIP, SEP is $H(x,y) := \frac{1}{\varrho(\{x\})}\prod_{k\in E}(y_k)_{x_k}\frac{1}{(x_k)!}$.

Reversible measures

Define $\varrho = \bigotimes_{k \in E} \varrho_{\alpha_k}$ with

$$\varrho_a = \begin{cases} \mathrm{Poi}(a) & \text{IRW} \\ \mathrm{NegativeBinomial}(p, a) & \text{SIP} \\ \mathrm{Binomial}(p, a) & \text{SEP} \end{cases}$$

for a fixed $p \in (0, 1)$.

Orthogonal Polynomials

Let $(P_n(\cdot, a))_{n\in\mathbb{N}_0}$ be the orthogonal Polynomials

$$\begin{cases} \text{Charlier} & \mathrm{Poi}(\alpha) \text{ (IRW)} \\ \text{Meixner} & \mathrm{NegativeBinomial}(p, a) \text{ (SIP)} \\ \text{Krawtchouk} & \mathrm{Binomial}(p, a) \text{ (SEP)} \end{cases}$$

Consider the multivariate polynomials (orthogonal for ϱ)

$$P_y(x, \alpha) := \prod_{k\in E} P_{y_k}(x_k, \alpha_k)$$

Let $\mathcal{P}_n := \left\{x \mapsto \sum_{|\ell|\leq n} a_\ell x^\ell : a_\ell \in \mathbb{R}\right\}$, $x^\ell := \prod_{k\in E} x_k^{\ell_k}$, $|\ell| = \sum_{k\in E} \ell_k$. Then $P_y(\cdot, \alpha)$ is the orthogonal projection of $x \mapsto \prod_{k\in E} x_k^{y_k}$ onto $\mathcal{P}_{|y|} \ominus \mathcal{P}_{|y|-1} = \mathcal{P}_{|y|-1}^{\perp} \cap \mathcal{P}_{|y|}$ in $L^2(\varrho)$

Orthogonal Duality

Theorem (Franceschini, Giardinà, 19')

$H(x, y) := P_x(y, \alpha)$ *forms a duality function for the three models.*

Generalization?

Replace the discrete E by a much more general Polish space E (e.g. $\mathbb{R}^d$, Banach spaces)
My Question: Generalization of all the objects and the resulting theorems?

We are now looking at all the slides again and see what happens at each step.

Stochastic Duality

Definition (Stochastic duality of Markov processes)

Let $X = (\Omega_1, \mathcal{F}_1, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{X}})$ and $Y = (\Omega_2, \mathcal{F}_2, (Y_t)_{t\geq 0}, (\mathbb{P}^y)_{y\in\mathbb{Y}})$ be two (time-continuous) Markov processes with state spaces $\mathbb{X}, \mathbb{Y}$. X and Y are dual with respect to $H : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ if and only if for all $x \in \mathbb{X}, y \in \mathbb{X}$ and $t \geq 0$

$$\mathbb{E}_x H(X_t, y) = \mathbb{E}^y H(x, Y_t).$$

Semigroup[4] notation: $P_t H(\cdot, y)(x) = S_t H(x, \cdot)(y)$

Self-duality: $X = Y$.

Definition (Intertwiner)

An operator U is called intertwiner, if $UP_t = S_t U$ for all $t \geq 0$.

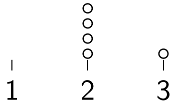
[4]Markov semigroup: $P_t f(x) := \mathbb{E}_x f(X_t)$

Particle configurations

We are interested in Markov processes describing the time-evolution of particles.
Let E be a countable set (e.g. $E = \{1, \dots, N\}$, $E = \mathbb{Z}^d$, graph). Consider

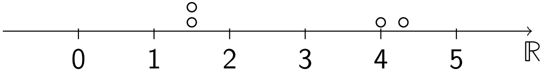
$$\mathbb{X} := \mathbb{N}_0^E := \{(x_k)_{k\in E} : x_k \in \mathbb{N}_0\}.$$

x_k = "number of particles at position k".

Example: $E = \{1, 2, 3\}$: $x = (x_1, x_2, x_3) = (0, 4, 1) =$ 

Let E be a Polish space. Consider the set of measures

$$\mathbb{X} := \mathbf{N}(E) := \left\{ \sum_{k=1}^{n} \delta_{x_k} : x_k \in E, n \in \mathbb{N}_0 \cup \{\infty\} \right\}$$

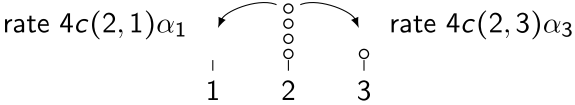
Example: $E = \mathbb{R}$: $2\delta_{1.5} + \delta_4 + \delta_{4.3} =$ 

Therefore, we look at measure-valued Markov processes.

The Independent Random Walker, IRW

We introduce three models by their (formal) generators[5] ($(\delta_k)_\ell := \delta_{k,\ell}$). Fix $\alpha_k \in \mathbb{N}$, $c : E \times E$ is an arbitrary symmetric function (spatial component). For $f : \mathbb{N}_0^E \to \mathbb{R}$:

$$Lf(x) = \sum_{k \in E} \sum_{\ell \in E} (f(x - \delta_k + \delta_\ell) - f(x)) c(k, \ell) \alpha_\ell x_k$$



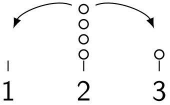
Let Z be an arbitrary Markov process with reversible measure α on E (e.g. Brownian motion). For an initial condition $\eta_0 = \sum_{k=1}^n \delta_{z_i} \in \mathbf{N}(E)$ define the Markov process $\eta_t := \sum_{k=1}^n \delta_{Z_{k,t}}$ with $(Z_{k,t})_{t \geq 0}$ independent copies of Z with initial condition $Z_{k,0} = z_k$.

[5] Generator of a Markov process: $Lf(x) := \partial_{t|t=0} P_t f(x)$

The Symmetric Inclusion Process, SIP

$$Lf(x) = \sum_{k \in E} \sum_{l \in E} (f(x - \delta_k + \delta_l) - f(x)) c(k,l)(\alpha_l + x_l) x_k$$

rate $4c(2,1)(\alpha_1 + 0)$



rate $4c(2,3)(\alpha_3 + 1)$

Let α be a finite measure on E. Consider

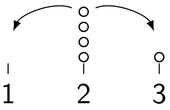
$$Lf(\mu) = \iint c(x,y)(f(\mu - \delta_x + \delta_y) - f(\mu))(\mu + \alpha)(\mathrm{d}y)\mu(\mathrm{d}x)$$

for $f : \mathbf{N}(E) \to \mathbb{R}$, $\mu \in \mathbf{N}(E)$.

The Symmetric Exclusion Process, SEP

$$Lf(x) = \sum_{k \in E} \sum_{l \in E} (f(x - \delta_k + \delta_l) - f(x)) c(k,l) (\alpha_l - x_l) x_k$$

rate $4c(2,1)(\alpha_1 - 0)$



rate $4c(2,3)(\alpha_3 - 1)$

No (direct) generalization.

Consistency

"the action of removing a particle uniformly at random commutes with the dynamic"
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In terms of expectations, for each $f : \mathbb{N}_0^E \to \mathbb{R}$, $t \geq 0$, $x \in \mathbb{N}_0^E$

$$\mathbb{E}_x\left[\sum_{k\in E} f(X_t - \delta_k) X_t\right] = \sum_{k\in E} x_k \mathbb{E}_{x-\delta_k}\left[f(X_t)\right].$$

Generalized SIP and independent Markov processes: the operator $\mathcal{A}f(\mu) := \int f(\mu - \delta_x)\mu(\mathrm{d}x)$ satisfies $\mathcal{A}L = \mathcal{A}L$, i.e. for each $f : \mathbf{N}(E) \to \mathbb{R}$, $t \geq 0$, $\eta \in \mathbf{N}(E)$

$$\mathbb{E}_\eta\left[\int f(\eta_t - \delta_x)\eta_t(\mathrm{d}x)\right] = \int \mathbb{E}_{\eta-\delta_x}\left[f(\eta_t)\right]\eta(\mathrm{d}x)$$

[6] There is a link to Lie-algebras

Conservation of the number of particles

For all $t \geq 0$ and $X_0 \in \mathbb{N}_0^E$

$$\sum_{k \in E} (X_0)_k = \sum_{k \in E} (X_t)_k.$$

Thus we can define also the dynamics of exactly n-particles. We denote the Markov-semigroup by $(p_t^{[n]})_{t \geq 0}$

Both the generalized IRW and the generalized SIP conserve the number of particles, i.e. for each $t \geq 0$, $(\eta_0)(E) = \eta_t(E)$.

Duality with falling factorials

Theorem (Carinci, Giardinà, Redig, '19)

Let ϱ be a reversible measure (i.e. detailed balance), $(n)_k := n(n-1)\cdots(n-k+1)$. Then, a self-duality function for IRW, SIP, SEP is $H(x,y) := \frac{1}{\varrho(\{x\})}\prod_{k\in E}(y_k)_{x_k}\frac{1}{(x_k)!}$.

Generalize the falling factorial with factorial measures, i.e.

$$J_k(f_k,\mu) := \int f_k \,\mathrm{d}\mu^{(k)}$$

$$:= \int f_k(x_1,\ldots,x_n)(\mu-\delta_{x_1}-\ldots-\delta_{x_{k-1}})(\mathrm{d}x_n)\cdots(\mu-\delta_{x_1})(\mathrm{d}x_2)\mu(\mathrm{d}x_1)$$

for $f_k : E^k \to \mathbb{R}$, $\mu \in \mathbf{N}(E)$

Theorem (Floreani, Jansen, Redig, W.)

Let η be a consistent and conservative Markov process (there are also other examples). Then $P_t J_k(f_k,\cdot)(\mu) = J_k(p_t^{[k]} f_k,\mu)$, i.e.

$$\mathbb{E}_\eta\left[\int f(\delta_{x_1}+\ldots+\delta_{x_k})\eta_t^{(k)}(\mathrm{d}(x_1,\ldots,x_k))\right] = \int \mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_k}}[f(\eta_t)]\,\eta^{(k)}(\mathrm{d}(x_1,\ldots,x_k)).$$

Reversible measures

Define $\varrho = \bigotimes_{k \in E} \varrho_{\alpha_k}$ with

$$\varrho_a = \begin{cases} \mathrm{Poi}(a) & \text{IRW} \\ \mathrm{NegativeBinomial}(p, a) & \text{SIP} \\ \mathrm{Binomial}(p, a) & \text{SEP} \end{cases}$$

for a fixed $p \in (0, 1)$.

A random variable $\xi \sim \varrho$ (Poi, NegativeBinomial) satisfies for distinct $k_1, \ldots, k_l \in E$

1. $(\xi_{k_1}, \ldots, \xi_{k_l})$ are independent,
2. $\xi_{k_1} + \ldots + \xi_{k_l} \sim \varrho_{\alpha_{k_1} + \ldots + \alpha_{k_l}}$

This leads to Lévy processes on general spaces (in our examples: Pascal and Poisson process), namely a measure ζ on $\mathbf{N}(E)$ s.t. $\zeta \sim \varrho$ implies

1. $\xi(A_1), \ldots, \xi(A_N)$ are independent for pairwise disjoint measurable sets $A_1, \ldots, A_n \subset E$,
2. $\xi(A) \sim \varrho_{\alpha(A)}$ for each measurable $A \subset E$

Orthogonal Polynomials

Let $(P_n(\cdot, a))_{n\in\mathbb{N}_0}$ be the orthogonal Polynomials

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Consider the multivariate polynomials (orthogonal for ϱ)

$$P_y(x,\alpha) := \prod_{k\in E} P_{y_k}(x_k,\alpha_k)$$

Let $\mathcal{P}_n := \left\{x \mapsto \sum_{|\ell|\leq n} a_\ell x^\ell : a_\ell \in \mathbb{R}\right\}$, $x^\ell := \prod_{k\in E} x_k^{\ell_k}$, $|\ell| = \sum_{k\in E} \ell_k$. Then $P_y(\cdot,\alpha)$ is the orthogonal projection of $x \mapsto \prod_{k\in E} x_k^{y_k}$ onto $\mathcal{P}_{|y|} \ominus \mathcal{P}_{|y|-1} = \mathcal{P}_{|y|-1}^\perp \cap \mathcal{P}_{|y|}$ in $L^2(\varrho)$

Let $\mathcal{P}_n := \left\{\mu \mapsto \sum_{k=0}^n \int f_k \,\mathrm{d}\mu^{\otimes k}\right\}$. Define $I_n(f_n,\cdot)$ as the projection[7] of $J_n(f_n,\cdot)$ onto $\mathcal{P}_{n-1}^\perp \cap \mathcal{P}_n$.

[7] Known as chaos decomposition. Also related to multiple stochastic integrals and Malliavin calculus

Orthogonal Duality

Theorem (Franceschini, Giardinà, 19')

$H(x,y) := P_x(y,\alpha)$ forms a duality function for the three models.

Theorem (Floreani, Jansen, Redig, W.)

Let $(\eta_t)_{t\geq 0}$ be a consistent and conservative Markov process. Assume that a reversible measure is given by the distribution of a Lévy process. Then,

$$P_t I_n(f_n, \cdot)(\mu) = I_n(p_t^{[n]} f_n, \mu)$$

for all $f_n : E^n \to \mathbb{R}$, $\mu \in \mathbf{N}(E)$, $t \geq 0$.

"I_n intertwines the process with arbitrary (also infinite) many particles with the same process on n particles"