Intertwining for continuum interacting particle systems

Workshop: Population Genetics, Interacting Particle Systems and Stochastic Flows: a duality perspective

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Overview

Joint work with

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1. **Known facts - discrete spaces.**
   Self-dualities of Markov processes describing the evolution of particles on a discrete set.
2. **My Research - general spaces.**
   What happens if we replace the discrete space by a much more general space?

Duality functions

Definition (Stochastic duality of Markov processes)

Let $X = (\Omega_1, \mathcal{F}_1, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{X}})$ and $Y = (\Omega_2, \mathcal{F}_2, (Y_t)_{t\geq 0}, (\mathbb{P}^y)_{y\in\mathbb{Y}})$ be two (time-continuous) Markov processes with state spaces $\mathbb{X}, \mathbb{Y}$. X and Y are dual with respect to $H : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ if and only if for all $x \in \mathbb{X}, y \in \mathbb{X}$ and $t \geq 0$

$$\mathbb{E}_x H(X_t, y) = \mathbb{E}^y H(x, Y_t).$$

Semigroup notation: $P_t H(\,\cdot\,, y)(x) = S_t H(x,\,\cdot\,)(y)$.

We consider self-duality, i.e, $X = Y$.

Self-Duality for SIP (falling factorials)

Consider the symmetric Inclusion Process (SIP) generated by

$$Lf(x) = \sum_{k \in E} \sum_{\ell \in E} (f(x - \delta_k + \delta_\ell) - f(x)) c(k, \ell)(\alpha_\ell + x_\ell) x_k$$

on the configuration space $\mathbb{N}_0^E$ with finite $E = \{1, \ldots, N\}$ and symmetric conductances $c(k, \ell) = c(\ell, k) \geq 0$, $k, \ell \in E$ and $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_k \geq 0$.
A reversible measure (i.e. detailed balance) is given by

$$\rho_{\alpha,p} = \bigotimes_{k \in E} \mathrm{NegativeBinomial}(p, \alpha_k)$$

for each $p \in (0, 1)$.

Similarly:

- independent random walkers (IRW): Product of Poisson distributions
- symmetric exclusion process (SEP): Product of Binomial distributions

Theorem (Carinci, Giardinà, Redig, '19)

Let ρ *be the reversible measure,* $(n)_k := n(n-1)\cdots(n-k+1)$*. Then, a self-duality function for IRW, SIP, SEP is*

$$H(x,y) := \frac{1}{\rho(\{x\})} \prod_{k\in E} \frac{1}{x_k!} (y_k)_{x_k}.$$

Orthogonal dualities

Let $(M_n(\,\cdot\,, a, p))_{n\in\mathbb{N}_0}$ be the (monic) Meixner polynomials (orthogonal with respect to $\mathrm{NegativeBinomial}(p, a)$. Consider the multivariate orthogonal polynomials

$$P_x(y, \alpha) := \frac{1}{\rho(\{x\})} \prod_{k\in E} \frac{1}{x_k!} P_{x_k}(y_k, \alpha_k).$$

Theorem (Franceschini, Giardinà, 19')

$H(x, y) := P_x(y, \alpha)$ is a self-duality function for the SIP .

IRW, SEP similarly.

Consistency

"the action of removing a particle uniformly at random commutes with the dynamic"

SIP, SEP, IRW share the following relation: $\mathcal{A}P_t = P_t\mathcal{A}$ for the so-called lowering operator $\mathcal{A}f(x) := \sum_{k\in E} x_k f(x-\delta_k)$.
In terms of expectations, for each $f : \mathbb{N}_0^E \to \mathbb{R}$, $t \geq 0$, $x \in \mathbb{N}_0^E$

$$\mathbb{E}_x\left[\sum_{k\in E} f(X_t-\delta_k)X_t\right] = \sum_{k\in E} x_k \mathbb{E}_{x-\delta_k}\left[f(X_t)\right].$$

For reversible particle systems on discrete sets, the property of consistency is equivalent to self-duality (Carinci, Giardinà, Redig, '19).

Non-discrete spaces?

Question: How to generalize these dualities to the continuum? More precisely: Replace discrete E by $\mathbb{R}$.

Challenges: How to generalize ...

1. the configuration spaces?
2. the models?
3. consistency?
4. the concept of duality functions?
5. falling factorials?
6. reversible measures?
7. orthogonal polynomials?

Further challenges:

8. Algebraic properties?
9. Infinitely many particles?

Main Idea

SIP, SEP, IRW are consistent. The concept of consistency can be generalized naturally. It turns out that this is the right notation and starting point to obtain dualities.

Configurations

Let $E = \mathbb{R}$. Model a configuration as counting measures, i.e,

$$\mathbf{N} := \left\{ \sum_{k=1}^{n} \delta_{x_k} : x_k \in E, n \in \mathbb{N}_0 \cup \{\infty\} \right\}$$

Modern notation for Point processes (Last / Penrose).

Models in the continuum

- generalized Version of the SIP on the continuum (gSIP): Let α be a finite measure on $\mathbb{R}$ and $c : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ symmetric. Consider the process generated by

  $$Lf(\eta) = \iint c(x,y)(f(\eta - \delta_x + \delta_y) - f(\eta))(\eta + \alpha)(\mathrm{d}y)\eta(\mathrm{d}x)$$

  for $f : \mathbf{N} \to \mathbb{R}$, $\eta \in \mathbf{N}$.
- Independent Markov processes, e.g., free Kawasaki
- Strongly Consistent systems (Called compatibility by Le Jan, Raimond) - stochastic flows
  - Brownian motions
  - Correlated Brownian motions

Consistency

Define

$$\mathcal{A}f(\eta) := \int f(\eta - \delta_x)\eta(\mathrm{d}x).$$

We say that a Markov process is consistent if

$$\mathcal{A}P_t = \mathcal{A}P_t,$$

i.e. for each $f : \mathbf{N} \to \mathbb{R}$, $t \geq 0$, $\eta \in \mathbf{N}$ In other words,

$$\mathbb{E}_\eta \left[\int f(\eta_t - \delta_x)\eta_t(\mathrm{d}x) \right] = \int \mathbb{E}_{\eta-\delta_x} \left[f(\eta_t) \right] \eta(\mathrm{d}x)$$

Duality and Intertwiners

Let E be discrete. Let ρ be a reversible measure for X and H is a self-duality function. Put a linear operator

$$Tf(y) = \int H(x, y) f(x) \rho(\mathrm{d}x), \quad y \in E$$

for functions $f : E \to \mathbb{R}$. Then, T intertwines P_t with itself.

Example: intertwiners for discrete systems

- Duality $\frac{1}{\rho(\{x\})}\prod_{k\in E}\frac{1}{x_k!}(y_k)_{x_k}$ with falling factorials turns into

$$Tf(y) = \int H(x,y)f(x)\rho(\mathrm{d}x) = \sum_{x_1=0}^{y_1}\cdots\sum_{x_N=0}^{y_N}\binom{y_1}{x_1}\cdots\binom{y_N}{x_N}f(x)$$

- Duality $\frac{1}{\rho(\{x\})}\prod_{k\in E}\frac{1}{x_k!}P_{x_k}(y_k,\alpha_k)$ with orthogonal polynomials turns into

$$Tf(y) = \int H(x,y)f(x)\rho(\mathrm{d}x) = \sum_{x_1=0}^{\infty}\cdots\sum_{x_N=0}^{\infty}\frac{1}{x_1!}P_{x_1}(y_1,\alpha_1)\cdots\frac{1}{x_N!}P_{x_N}(y_N,\alpha_N)f(x)$$

Improve duality functions

Moreover, if S intertwines P_t with itself, and H is a self-duality function (for example the cheap-duality function $H(x,y) = \delta_{x,y}\rho(\{x\})$), then $\tilde{H}(x,y) = TH(\,\cdot\,,y)(x)$ is another self-duality function.

Why intertwiners?

- Intertwiners can be seen as a “lifting” of duality functions.
- The intertwiners can be generalized naturally
- These generalization lead to kernel operators without absolute continuity with respect to a reversible measure.

Generalized falling factorials

The generalization does already exist: Lennard's K-transform - special kind of Möbius transform. Put

$$Kf(\eta) = \sum_{\nu \hat{\leq} \eta, \nu(\mathbb{R})<\infty} f(\nu)$$

$\hat{\leq}$ means that e.g. $\nu = \delta_x$ occurs two times if $\eta = 2\delta_x$.
Other representation:

$$Kf(\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int f(\delta_{x_1} + \ldots + \delta_{x_n}) \eta^{(n)}(\mathrm{d}(x_1, \ldots, x_n))$$

$\eta^{(n)}$ falling factorial measure, can be seen as a generalization of falling factorials.

Link to the discrete world

Theorem (Redig, Jansen, Floreani, W., '21)

Let X be consistent. Then $KP_t = P_t K$.

Moreover, if X conserves the number of particles then

$$\mathbb{E}_\eta \left[\int f(\delta_{y_1} + \ldots + \delta_{y_n}) \eta_t^{(n)}(\mathrm{d}(y_1, \ldots, y_n)) \right] = \int \mathbb{E}_{\delta_{x_1} + \ldots + \delta_{x_n}} [f(\eta_t)] \, \eta^{(n)}(\mathrm{d}(x_1, \ldots, x_n))$$

Intertwines the dynamics of arbitrary many particles with the dynamic of only n-particles.

Intertwining for all consistent systems (not only SIP).

Recover self-duality of discrete systems

If $f(\eta) = \mathbb{1}_{\eta(D_1)=d_1} \cdots \mathbb{1}_{\eta(D_N)=d_N}$, $D_1, \ldots, D_N$ partition of $\mathbb{R}$, $d_1, \ldots, d_N \in \mathbb{N}_0$, then,

$$Kf(\eta) = \prod_{k=1}^{N} \frac{(\eta(D_k))_{d_k}}{d_k!}$$

In particular, if E is finite, then $K = T$.

How to generalize reversible measures?

Observation: Let E be finite, $X \sim \rho_{\alpha,p}$ and write

$$X_D := \sum_{k \in D} X_k, \quad D \subset E.$$

Then:

1. $X_D \sim \mathrm{NegativeBinomial}\left(\sum_{k \in D} \alpha_k, p\right)$ (Negative Binomial distribution forms a convolution semigroup);
2. X_D and $X_{D'}$ are independent for disjoint D, D'.

.

We look for a measure ρ on $\mathbf{N}$. If ρ is a probability measure, ρ is the distribution of a point process.

Question: Is for a measure α on $\mathbb{R}$ a point process ζ with

1. $\xi(A_1), \dots, \xi(A_N)$ are independent for pairwise disjoint measurable sets $A_1, \dots, A_n \subset \mathbb{R}$,
2. $\xi(A) \sim \mathrm{NegativeBinomial}(p, \alpha(A))$ for each measurable $A \subset \mathbb{R}$

Yes: theory of Lévy processes on general spaces (in our examples: Pascal process).

Indeed: The distribution of the Pascal process is reversible for the gSIP.

Infinite dimensional polynomials

Define polynomials $\mathcal{P}_n$ of degree $\leq n$ as the linear combinations of the functions

$$\eta \mapsto \int f_k \,\mathrm{d}\eta^{\otimes k}$$

with $k \leq n$, $f_k : \mathbb{R}^k \to \mathbb{R}$.

Motivation if is E is discrete. $\eta = x_1\delta_1 + \ldots + x_N\delta_N$, then

$$\int f_k \,\mathrm{d}\eta^{\otimes k} = \sum_{\ell_1,\ldots,\ell_n=1}^{N} f_k(\ell_1,\ldots,\ell_n) x_{\ell_1}\cdots x_{\ell_n}$$

is a multivariate polynomial of degree n. $x_1^n = \int f_k \,\mathrm{d}\eta^{\otimes n}$ can be recovered by

$$f_k(\ell_1,\ldots,\ell_n) = \begin{cases} 1 & \ell_1 = \cdots = \ell_n = 1 \\ 0 & \text{else.} \end{cases}$$

Orthogonal polynomials

For $f_n : \mathbb{R}^n \to \mathbb{R}$ put

$$I_n f_n = \text{orthogonal projection of } \left(\eta \mapsto \int f_n \, \mathrm{d}\eta^{\otimes n}\right) \text{ onto } \mathcal{P}_{n-1}^{\perp} \text{ in } L^2(\rho),$$

ρ reversible measure (e.g. of the Pascal process).

Studied by e.g. by Lytvynov, '02—Link to infinite dimensional analysis, chaos decompositions and multiple stochastic integrals. Poisson case: multiple Wiener Itô integrals.

Factorization property

Theorem (Redig, Jansen, Floreani, W., '21)

Suppose that ρ is the distribution of some finite completely independent point process. Let $N \geq 2$, $A_1, \ldots, A_N \subset \mathbb{R}$, measurable, pairwise disjoint, and $d_1, \ldots, d_N \in \mathbb{N}_0$. Further let $f_i : \mathbb{R}^{d_i} \to \mathbb{R}$, $i = 1, \ldots, N$ be bounded measurable functions that vanish on $\mathbb{R}^{d_i} \setminus A_i^{d_i}$. Set $n := d_1 + \cdots + d_N$. Then

$$I_n(f_1 \otimes \ldots \otimes f_n)(\eta) = I_{d_1} f_1(\eta) \cdots I_{d_n} f_n(\eta)$$

for ρ-almost all $\eta \in \mathbf{N}_{<\infty}$.

Properties of infinite dimensional Meixner Polynomials

Theorem (Redig, Jansen, Floreani, W., '21)

- $I_d \mathbb{1}_{A^d}(\eta) = M_d(\eta(A); \alpha(A); p)$
- *There are measures* λ_n *such that*

$$\int (I_n f_n)(I_m g_m) \, \mathrm{d}\rho = \mathbb{1}_{\{n=m\}} \int f_n g_m \, \mathrm{d}\lambda_n$$

for permutation invariant $f_n : \mathbb{R}^n \to \mathbb{R}$, $g_m : \mathbb{R}^m \to \mathbb{R}$, $n, m \in \mathbb{N}_0$.

Intertwining

Theorem (Redig, Jansen, Floreani, W., '21)

Assume that X is consistent and conserves the number of particles (then the n-particle semigroup $P_t^{[n]}$ acting on permutation invariant functions $f_n : E^n \to \mathbb{R}$ is well-defined). Assume that ρ is reversible. Then

$$P_t I_n = I_n P_t^{[n]}$$

For $f : \mathbf{N} \to \mathbb{R}$ put

$$Tf = \sum_{n=0}^{\infty} \frac{1}{n!} I_n f_n$$

by $f(\delta_{x_1} + \ldots + \delta_{x_n}) = f_n(x_1, \ldots, x_n)$. Then, $P_t T = T P_t$.

Recover discrete dualities

Theorem (Redig, Jansen, Floreani, W., '21)

T recovers duality functions for SIP (also IRW)!

Indeed, if $f(\eta) = \mathbb{1}_{\eta(D_1)=d_1} \cdots \mathbb{1}_{\eta(D_N)=d_N}$, $D_1, \ldots, D_N$ partition of $\mathbb{R}$, $d_1, \ldots, d_N \in \mathbb{N}_0$, then,

$$Tf(\eta) = \prod_{k=1}^{N} \frac{M_{d_k}(\eta(D_k); \alpha(D_k); p)}{d_k!}$$

Further challenges

- Duality and Lie-Algebra representations
- Infinitely many particles

Thank you!

S. Floreani, S. Jansen, F. Redig, S.W.: *Duality and intertwining for consistent Markov processes* arXiv:2112.11885 [math.PR], 32 pp.