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Riemannian Geometry

sheet 11

Exercise 1. Given two natural numbers n, m, show that the growth functions $x \mapsto x^n$ and $x \mapsto x^m$ are equivalent if and only if $n = m$.

Exercise 2. Consider the free abelian group $\mathbb{Z}^2$. It is the fundamental group of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$.

1. Compute the growth function with respect to the generating set given by the standard basis vectors of $\mathbb{R}^2$ and their inverses.
2. Compute the growth function with respect to the generating set that figures in the comparison with the growth of Riemannian volume on the universal covering, discussed in the lecture on 9 July.

Exercise 3. Let F_2 be the free group on two generators a, b. Its elements are all words (including the empty one) that one can write using the letters a, b, a^{-1}, b^{-1}, subject only to the relation of cancellation between a and a^{-1}, resp. b and b^{-1}. Compute the growth function with respect to the generating set $\{a, b, a^{-1}, b^{-1}\}$.

Exercise 4. Let

$$H := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\} .$$

1. Show that one has a presentation in terms of generators and relations as $H = \langle x, y, z \mid [x, z] = [y, z] = 0, [x, y] = z \rangle$.

For $w \in H$ denote by $l(w)$ the minimal length of a word in $S := \{x, y, z, x^{-1}, y^{-1}, z^{-1}\}$ representing w. Let $m, n, k \in \mathbb{Z}$.

2. Show that $l(x^m \cdot y^n \cdot z^k) \leq |m| + |n| + 6\sqrt{|k|}$.
3. Show that $|m| + |n| \leq l(x^m \cdot y^n \cdot z^k)$ and $|k| \leq l(x^m \cdot y^n \cdot z^k)^2$.
4. Show that $\frac{1}{2} \cdot (|m| + |n| + \sqrt{|k|}) \leq l(x^m \cdot y^n \cdot z^k)$.
5. Conclude that the growth function of H with respect to S is equivalent to a polynomial function of degree 4.