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Riemannian Geometry

sheet 10

Exercise 1. For any symmetric bilinear form ϕ on $\mathbb{R}^n$, one has

$$\int_{S^{n-1}} \phi(v,v)dv = \frac{1}{n}\,\mathrm{Vol}(S^{n-1})\,\mathrm{tr}(\phi)\ .$$

Exercise 2. Let (M,g) be a compact Riemannian manifold. Consider the scaled metric $h = \mu^2 g$ for some positive constant μ. Show that $\lambda(h) = \mu^{-1}\lambda(g)$, where λ denotes the volume entropy

$$\lambda(g) := \lim_{r\to\infty} \frac{\log \mathrm{Vol}_{\tilde{g}}(B_r^{\tilde{g}}(x))}{r}$$

where one considers balls in the universal covering $\tilde{M}$ with induced metric $\tilde{g}$ and some basepoint $x \in \tilde{M}$, and similarly for h.

Exercise 3. Consider $M = S^2 \times \mathbb{R}^2$ the product of sphere and the plane, each with standard metric. Let $(a,b) \in M$. For which (small) positive numbers r, s is $\mathrm{Vol}(B_r(a,b)) = \mathrm{Vol}(B_s(a) \times B_s(b))$?

Exercise 4. Let $g = dr^2 + f^2(r)d\theta^2$ with some smooth function f, nonzero outside the origin, define a metric on $\mathbb{R}^2$ in polar coordinates. Show that the metric is complete and that the volume is $2\pi \int_0^\infty f(r)dr$