LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Riemannian Geometry

sheet 08

Exercise 1. Let (M, g) be a Riemannian manifold. A conformal deformation of g is a metric of the form $g_\varphi := e^{2\varphi} \cdot g$ for some $\varphi \in C^\infty(M)$. Define a vector field and a quadratic form on $\mathfrak{X}(M)$ via $\nabla\varphi$ and $Hess(\varphi)(X, Y)$ by $g(\nabla\varphi, X) = X(\varphi)$ and $Hess_\varphi(X, Y) := g(\nabla_X \nabla\varphi, Y)$.

1. Show that the Levi-Civita connections and sectional curvatures of g and g_φ are related as follows, where we assume X, Y orthonormal in the second statement.

$$\begin{aligned}\nabla^\varphi_X Y =& \nabla_X Y + X(\varphi)Y + Y(\varphi)X - g(X, Y)\nabla\varphi \ ,\\ K_\varphi(X, Y) =& \frac{1}{e^{2\varphi}}(K(X, Y) - Hess_\varphi(X, X) - Hess_\varphi(Y, Y) \\ & - g(\nabla\varphi, \nabla\varphi) + g(X(\varphi), X(\varphi)) + g(Y(\varphi), Y(\varphi)))\ .\end{aligned}$$

2. Apply this formula to calculate the curvature of g^{hyp} on $\mathbb{H}$ from the previous sheets.

Exercise 2.

1. Let G be a finite group of odd order acting on an orientable manifold M. Show that the action is orientation preserving.
2. For which integers n, k with k odd does there exist a free action of $\mathbb{Z}/k$ on S^n?

Exercise 3. Let G be a Lie group with a bi-invariant metric g, i.e. one which is invariant under both left and right translations. Show the following formulae for Lie-bracket, Levi-Civita connection and curvature hold for all left-invariant vector fields X, Y, Z:

1. $g([X, Y], Z) = g(X, [Y, Z])$
2. $\nabla_X Y = \frac{1}{2}[X, Y]$
3. $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$

Deduce that for such a metric $K \geq 0$ everywhere.

Exercise 4. Let G be a Lie group and X a left-invariant vector field. The trajectories of X determine a mapping $\varphi : (-\varepsilon, \varepsilon) \to G$ with $G(0) = e$ and $\varphi'(t) = X(\varphi(t))$.

1. Prove that $\varphi(t)$ is defined for all $t \in \mathbb{R}$ and that $\varphi(t+s) = \varphi(t)\cdot\varphi(s)$. We say φ is a one-parameter subgroup.
2. Using the previous exercise, show that if G admits a bi-invariant metric, the one-parameter subgroups are the geodesics through $e \in G$.