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Prof. D. Kotschick
Dr. Jonas Stelzig

Riemannian Geometry

sheet 07

Exercise 1 (disk model for hyperbolic 2-space)**.** Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ the unit disc with metric given by $(g^D)_{(x,y)} = \frac{4}{(1-(x^2+y^2))^2} \cdot g^{Euc}$ where g^{Euc} is the usual Euclidean metric. Show that there is an isometry between (D, g^D) and $(\mathbb{H}, g^{hyp})$ from exercise 3 of sheet 2.

Exercise 2.

1. Let $\varphi : M \to M$ be an isometry of a Riemannian manifold and assume the set of fixed points $F = \{x \in M \mid \varphi(x) = x\}$ is a one-dimensional submanifold of M. Show that every curve parametrized by arc length with image in F is a geodesic.
2. (Re-)deduce that meridians on a surface of revolution are geodesics (c.f. exercise 4, sheet 2).

Exercise 3. Let $f \in O(n)$ be an orthogonal linear transformation of $\mathbb{R}^n$ s.t. $\det f = (-1)^{n+1}$. Show that f has a fixed point, i.e. that there is an $x \in \mathbb{R}^n$ with $f(x) = x$.

Exercise 4 (lens spaces)**.** Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ via $(x_1, ..., x_4) \mapsto (x_1 + ix_2, x_3 + ix_4)$. Then

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\right\}.$$

Let p, q be two coprime integers with $p > 2$ and $\zeta_p = e^{\frac{2\pi i}{p}}, \zeta_q = e^{\frac{2\pi i q}{p}} \in \mathbb{C}$. Define a diffeomorphism $h : S^3 \to S^3$ via $h(z_1, z_2) = (\zeta_p z_1, \zeta_q z_2)$.

1. Show that $G = \{\mathrm{id}, h, h^2, ..., h^{p-1}\}$ is a group of isometries of S^3 with its usual metric and that the action on S^3 is free and proper.
2. Equip S^3/G be the Riemannian manifold with the metric induced by the projection $\pi : S^3 \to S^3/G$. Show that all geodesics of S^3/G are closed but that they can have different lengths.