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Riemannian Geometry

sheet 06

Exercise 1. Prove that every isometry of $\mathbb{R}^n$ (with the Euclidean metric) is an affine transformation.

Exercise 2. Let M be a complete connected Riemannian manifold. A point $p \in M$ is called a pole, if for all nonzero Jacobi fields J along geodesics starting at p with $J(0) = 0$ one has $J(t) \neq 0$ for all $t > 0$.

1. Show that any simply connected M which has a pole $p \in M$ is diffeomorphic to $\mathbb{R}^n$.
2. Show that the paraboloid $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z\}$ satisfies the assumptions of the previous point, but has a point with positive sectional curvature.

Exercise 3. Let $(M, < \ , \ >_M)$ and $(N, < \ , \ >_N)$ be two Riemannian manifolds with isometry groups I_M and I_N. Prove that there is an inclusion from $I_M \times I_N$ into the isometry group $I_{M\times N}$ of the product $(M \times N, < \ , \ >_M + < \ , \ >_N)$. Show by example that this inclusion is in general not an equality.

Exercise 4. Let $(M, < \ , \ >)$ be a Riemannian manifold and let $p : N \to M$ be a local diffeomorphism onto M. Equip N with the induced metric making p into a local isometry.

1. Assume p is a covering space map. Show that N is complete if and only if M is complete
2. Show by example that the previous statement fails in general if p is only a local diffeomorphism.