LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Prof. D. Kotschick
Dr. Jonas Stelzig

Riemannian Geometry

sheet 05

Exercise 1. Let X be a topological space and $x \in X$ a point. Prove that, as sketched in the lecture, concatenation of paths induces a group structure on the set $\pi_1(X, x)$ of homotopy classes relative to the endpoints of loops starting and ending at x.

Exercise 2. Let $D := \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$ denote the closed ball in $\mathbb{R}^2$, with boundary $\partial D = S^1$. Show that there can be no continuos map $r : D \to S^1$ s.t. $r|_{S^1}$ is the identity.

Exercise 3. Recall that a Lie group is a manifold G with a point $e_G \in G$ and smooth maps $\circ : G \times G \to G$ and $(\ )^{-1} : G \to G$ satisfying the group axioms with e_G as neutral element, $\circ$ as composition and $(\ )^{-1}$ as inverse. Show that on every connected covering space $p : H \to G$ of a Lie group G and every point $e_H \in p^{-1}(e_G)$ there is again a unique structure of a Lie group with e_H as neutral element such that p becomes a smooth group homomorphism.

Exercise 4. Prove the following claim made in the lecture: The quotient of a manifold X by a free and proper action of a discrete group Γ is again a manifold and the projection $p : X \to X/\Gamma$ is a smooth covering space.

Recall that an action is called free if every nontrivial element of γ acts without fixed points and it is called proper if any two points $x, y \in X$ admit open neighborhoods $x \in U$, $y \in V$ such that the set $\{\gamma \in \Gamma \mid U \cap \gamma V \neq \emptyset\}$ is finite.