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Riemannian Geometry

sheet 04

Exercise 1. Let (M, g) be a connected Riemannian metric of dimension at least 3. Assume that for any point $p \in M$, the sectional curvature K is independent of the plane. I.e., for all 2-planes $\sigma \subset T_pM$, one has $K_p(\sigma) = K_0(p)$ for some $K_0(p) \in \mathbb{R}$. Show that then the sectional curvature is already globally constant, i.e. $K_0(p) = K_0(p')$ for all $p \in \mathbb{R}$.

Exercise 2. Let M be a complete Riemannian manifold, and let $n \subset M$ be a closed submanifold of M. Let $p \in M \setminus N$ and let $d(p, N) = \inf\{d(p, q) \mid q \in N\}$ be the distance from p to N. Show that there exists a point $q_{min} \in N$ such that $d(p, N) = d(p, q_{min})$ and that a minimizing geodesic joining p and q_{min} is orthogonal to N at q_{min}.

Exercise 3. Given a complete Riemannian metric on $\mathbb{R}^2$, show that

$$\lim_{r\to\infty} \big(\inf_{x^2+y^2 \geq r^2} K(x, y)\big) \leq 0,$$

where $(x, y) \in \mathbb{R}^2$ and $K(x, y)$ denotes the sectional curvature of the given metric at (x, y).

Exercise 4. Let M be a Riemannian manifold of constant sectional curvature K, and let $\gamma : [0, l] \to M$ be a geodesic parametrized by arc length. Further, let J be a Jacobi field along γ, normal to γ'. Show that in this case the Jacobi equation simplifies to

$$\frac{D^2 J}{dt^2} + KJ = 0.$$

Given a parallel field $w(t)$ along γ with $\langle \gamma'(t), w(t)\rangle = 0$ and $|w(t)| = 1$, find a solution for the Jacobi-equation with initial conditions $J(0) = 0$ and $J'(0) = w(0)$. Distinguish three cases according to the sign of K.