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Riemannian Geometry

sheet 01

Exercise 1. Let (M, g) be a Riemannian manifold and $\gamma : [a, b] \to M$ a smooth curve. Denote by $\frac{D}{dt} : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ the operator induced by the Levi-Civita connections on the vector fields along γ, as introduced in the lecture. Show that for all $V, W \in \mathfrak{X}(\gamma)$ the following equation holds:

$$\frac{d}{dt} g(V, W) = g\left(\frac{D}{dt} V, W\right) + g\left(V, \frac{D}{dt} W\right) \ .$$

Exercise 2. Let (M, g) be a Riemannian manifold and let $Z \subseteq M$ be a submanifold. Restriction of g to TZ defines a Riemannian metric on Z. Compute the Levi-Civita connection for this restricted metric in terms of the Levi Civita connection for g on M.

Exercise 3. Let $Z \subseteq \mathbb{R}^{n+1}$ be an n-dimensional submanifold. Consider $\mathbb{R}^{n+1}$ with the standard Euclidean metric and Z with the induced metric as in the previous exercise.

1. Let $\gamma : [a, b] \to Z \subseteq \mathbb{R}^{n+1}$ be a smooth curve. Show that γ defines a geodesic on Z if at every point $t \in [a, b]$, with respect to the standard metric on $\mathbb{R}^{n+1}$, the second derivative $\ddot{\gamma}(t)$ is orthogonal to $T_{\gamma(t)} Z \subseteq T_{\gamma(t)} \mathbb{R}^{n+1} = \mathbb{R}^{n+1}$.
2. Apply this to conclude that the geodescics on a sphere are exactly the great circles.

Exercise 4. Denote by (x, y) the standard coordinates on $\mathbb{R}^2$. Let $c : (a, b) \to \mathbb{R}^2, c(t) = (f(t), g(t))$ be a smooth curve with $f'(t)^2 + g'(t)^2 \neq 0$ and $f(t) \neq 0$ for all $t \in (a, b)$.

1. Let $U = (r, s) \times (a, b) \subseteq \mathbb{R}^2$. Show that the function

$$\begin{aligned} \varphi : U &\longrightarrow \mathbb{R}^3 \\ (u, v) &\longmapsto (f(v)\cos(u), f(v)\sin(u), g(v)) \end{aligned}$$

   is an immersion. The image $\varphi(U)$ is called the surface of revolution generated by c.

2. Show that the metric induced by the standard metric on $\mathbb{R}^3$ on $\varphi(U)$ is given in the coordinates (u, v) by the following 2×2 matrix:

$$\begin{pmatrix} f^2 & 0 \\ 0 & (f')^2 + (g')^2 \end{pmatrix} \ .$$

3. Show that for a curve $\gamma : (x, y) \to U, \gamma(t) = (u(t), v(t))$ the condition to define a geodesic can be expressed as

$$\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0 \tag{1}$$

and

$$\frac{d^2v}{dt^2} - \frac{ff'}{(f')^2 + (g')^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}\left(\frac{dv}{dt}\right)^2 = 0\ . \tag{2}$$