Ex 1

(M,g) Riem. mfd. $\quad g_\varphi = e^{2\varphi} g$

$g(\nabla\varphi, X) = X(\varphi)$

$\mathrm{Hess}_\varphi(X,X) = g(\nabla_X \nabla\varphi, X)$

i) Claim: $\nabla^\varphi_X Y = \nabla_X Y + X(\varphi) Y + Y(\varphi) X - g(X,Y)\cdot\nabla\varphi$

Then by Koszul have

$$2g(\nabla_X Y, Z) = X g(Y,Z) + Y g(Z,X) - Z g(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]))$$

and

$$2g^\varphi(\nabla^\varphi_X Y, Z) = X(g^\varphi(Y,Z)) + Y g^\varphi(Z,X) - Z g^\varphi(X,Y) - g^\varphi(X,[Y,Z]) + g^\varphi(Y,[Z,X]) + g^\varphi(Z,[X,Y])$$

e.g. $X(e^{2\varphi} g(Y,Z)) = 2X(\varphi) e^{2\varphi} g(Y,Z) + e^{2\varphi} X(g(Y,Z))$

$$= \boxed{\begin{aligned} &2\cdot X(\varphi) g^\varphi(Y,Z) \\ &+ 2\cdot Y(\varphi) g^\varphi(Z,X) \\ &- 2 Z(\varphi) g^\varphi(X,Y) \end{aligned}} \begin{aligned} &+ e^{2\varphi} X g(Y,Z) \\ &+ e^{2\varphi} Y g(Z,X) \\ &+ e^{2\varphi} Z g(X,Y) \end{aligned}$$

$$- g^\varphi(X,[Y,Z]) + g^\varphi(Y,[Z,X]) + g^\varphi(Z,[X,Y])$$

$$= \boxed{} + 2\cdot g^\varphi(\nabla_X Y, Z)$$

$$\rightsquigarrow g^\varphi(\nabla^\varphi_X Y - \nabla_X Y, Z) = g^\varphi(X(\varphi) Y + Y(\varphi) X - \nabla\varphi \cdot g(X,Y), Z)$$

□

where we used $g^\varphi(X,Y)\cdot Z(\varphi) = e^{2\varphi} g(X,Y)\cdot g(\nabla\varphi, Z)$

$$= g^\varphi(\nabla\varphi \cdot g(X,Y), Z)$$

Part 2 Compute $R^\varphi(X,Y)Z$ in terms of $R(X,Y)Z$ etc.:

$$\nabla_X^\varphi \nabla_Y^\varphi Z = \nabla_X(\nabla_Y^\varphi Z + X(\varphi)\nabla_Y^\varphi Z + \nabla_Y^\varphi(Z)(\varphi)\cdot X - g(X, \nabla_Y^\varphi Z)\nabla\varphi$$

$$= \nabla_X \nabla_Y Z + X(Y(\varphi))\cdot Z + Y(\varphi)\cdot \nabla_X Z + X(Z(\varphi))\cdot Y + Z(\varphi)\cdot \nabla_X Y$$

$$- X(g(Y,Z))\nabla\varphi - g(Y,Z)\cdot \nabla_X \nabla\varphi$$

$$+ X(\varphi)\nabla_Y Z + X(\varphi)Y(\varphi)Z + X(\varphi)Z(\varphi)Y - X(\varphi)g(Y,Z)\nabla\varphi$$

$$+ (\nabla_Y Z)(\varphi)\cdot X + Y(\varphi)Z(\varphi)\cdot X + Z(\varphi)Y(\varphi)\cdot X - g(Y,Z)\nabla\varphi(\varphi)\cdot X$$

$$- g(X, \nabla_Y Z)\nabla\varphi - g(X, Y(\varphi)Z)\cdot\nabla\varphi - g(X, Z(\varphi)Y)\nabla\varphi$$

$$+ g(X, g(Y,Z)\nabla\varphi)\cdot\nabla\varphi$$

and

$$\nabla^\varphi_{[X,Y]} Z = \nabla_{[X,Y]} Z + [X,Y](\varphi)\cdot Z + Z(\varphi)\cdot[X,Y] - g([X,Y],Z)\cdot\nabla\varphi$$

Thus,

$$R^\varphi(X,Y)Z - R(X,Y)Z = \nabla_X^\varphi \nabla_\varphi^\varphi Z - \nabla_X\nabla_Y Z - \nabla_Y^\varphi \nabla_X^\varphi Z + \nabla_Y \nabla_X Z$$

$$- \nabla^\varphi_{[X,Y]} + \nabla_{[X,Y]} Z$$

= next p.

$= [X,Y](\varphi)\cdot Z$

$$= \underline{X(Y(\varphi))Z - Y(X(\varphi))\cdot Z} + Y(\varphi)\cdot\nabla_X Z - X(\varphi)\cdot\nabla_Y Z$$

$$+ X(Z(\varphi))\cdot Y - Y(Z(\varphi))\cdot X + Z(\varphi)\cdot(\underbrace{\nabla_X Y - \nabla_Y X}_{= [X,Y] \text{ since } \nabla \text{ t.f.}})$$

$$- \underbrace{X(g(Y,Z))}_{\boxed{g(\nabla_X Y,Z)} + g(Y,\nabla_X Z)}\nabla\varphi + \underbrace{Y(g(X,Z))}_{\boxed{g(\nabla_Y X,Z)} + g(X,\nabla_Y Z)}\nabla\varphi - g(Y,Z)\nabla_X\nabla\varphi + g(X,Z)\nabla_Y\nabla\varphi$$

$$+ X(\varphi)\nabla_Y Z - Y(\varphi)\nabla_X Z + \underbrace{X(\varphi)Y(\varphi)Z - Y(\varphi)X(\varphi)Z}_{=0}$$

$$+ X(\varphi)Z(\varphi)\cdot Y - Y(\varphi)Z(\varphi)\cdot X - X(\varphi)g(Y,Z)\cdot\nabla\varphi + Y(\varphi)g(X,Z)\cdot\nabla\varphi$$

$$+ (\nabla_Y Z)(\varphi)\cdot X - (\nabla_X Z)(\varphi)\cdot Y + Y(\varphi)\cdot Z(\varphi)\cdot X - X(\varphi)Z(\varphi)\cdot Y$$

$$+ Z(\varphi)Y(\varphi)X - Z(\varphi)X(\varphi)\cdot Y - g(Y,Z)\cdot\nabla\varphi(\varphi)\cdot X + g(X,Z)\cdot\nabla\varphi(\varphi)\cdot Y$$

$$- g(X,\nabla_Y Z)\nabla\varphi + g(Y,\nabla_X Z)\nabla\varphi - g(X, Y(\varphi)Z)\cdot\nabla\varphi + g(Y, X(\varphi)Z)\cdot\nabla\varphi$$

$$\underbrace{- g(X, Z(\varphi)\cdot Y)\cdot\nabla\varphi + g(Y, Z(\varphi)\cdot X)\cdot\nabla\varphi}_{=0}$$

$$+ g(Y,Z)\cdot g(X,\nabla\varphi)\cdot\nabla\varphi - g(X,Z)\cdot g(Y,\nabla\varphi)\cdot\nabla\varphi$$

$$- \underline{[X,Y](\varphi)\cdot Z} - Z(\varphi)\cdot[X,Y] + \underbrace{g([X,Y],Z)}_{\boxed{= g(\nabla_X Y,Z) - g(\nabla_Y X,Z)}}\cdot\nabla\varphi$$

$$= Y(\varphi)\nabla_X Z - X(\varphi)\nabla_Y Z + X(Z(\varphi))\cdot Y - Y(Z(\varphi))\cdot X$$

$$- g(Y,Z)\cdot\nabla_X\nabla\varphi + g(X,Z)\nabla_Y\nabla\varphi + X(\varphi)\nabla_Y Z - Y(\varphi)\nabla_X Z$$

$$+ (\nabla_Y Z)(\varphi)\cdot X - (\nabla_X Z)(\varphi)\cdot Y + Z(\varphi)Y(\varphi)\cdot X - Z(\varphi)X(\varphi)\cdot Y$$

$$- g(Y,Z)\cdot(\nabla\varphi)(\varphi)\cdot X + g(X,Z)\cdot(\nabla\varphi)(\varphi)\cdot Y - g(X,Z)\cdot Y(\varphi)\cdot\nabla\varphi + g(Y,Z)\cdot X(\varphi)\cdot\nabla\varphi$$

Thus, assuming X, Y to be ortho~~gonal~~normal (w.r.t. g), have

$$g(R^\varphi(X,Y)Y,X) - g(R(X,Y)Y,X)$$

$$= \boxed{\cancel{g(Y(\varphi)\nabla_X Y, X)} - \cancel{g(X(\varphi)\nabla_Y Y, X)}}$$

$$+ \underbrace{g(X(Y(\varphi))\cdot Y, X)}_{=0} - g\big(\underbrace{Y(Y(\varphi))\cdot X, X}_{=Y(Y(\varphi))}\big)$$

$$- \underbrace{g(Y,Y)}_{=1}\cdot \underbrace{g(\nabla_X \nabla\varphi, X)}_{\mathrm{Hess}_\varphi(X,X)} + \underbrace{g(X,Y)}_{=0}\cdot g(\nabla_Y \nabla\varphi, X)$$

$$\boxed{\cancel{+ X(\varphi)\cdot g(\nabla_Y Y, X)}}\ \boxed{\cancel{- Y(\varphi)\cdot g(\nabla_X Y, X)}}$$

$$+ g(\nabla_Y Y(\varphi)\cdot X, X) - g\big(\underbrace{(\nabla_X Y)(\varphi)\cdot Y, X}_{=0}\big)$$

$$+ g\big(\underbrace{Y(\varphi)\cdot Y(\varphi)\cdot X, X}_{=g(Y(\varphi), Y(\varphi)) = |Y(\varphi)|^2}\big) - g\big(\underbrace{Y(\varphi)\cdot X(\varphi)\cdot Y, X}_{=0}\big)$$

$$= \underbrace{g(Y,Y)}_{=1}\cdot g\big(\underbrace{(\nabla\varphi)(\varphi)\cdot X, X}_{=g(\nabla\varphi,\nabla\varphi)}\big) + \underbrace{g(X,Y)}_{=0}\cdot g((\nabla\varphi)(\varphi)\cdot Y, X)$$

$$- \underbrace{g(X,Y)}_{=0}\cdot Y(\varphi)\cdot g(\nabla\varphi, X) + \underbrace{g(Y,Y)}_{=1}\cdot \underbrace{X(\varphi)\, g(\nabla\varphi, X)}_{|X(\varphi)|^2}$$

$$= - \underbrace{Y(Y(\varphi))}_{= Y(g(\nabla\varphi, Y)) = g(\nabla_Y \nabla\varphi, Y) + \cancel{g(\nabla\varphi, \nabla_Y Y)}}$$

$$- \mathrm{Hess}_\varphi(X,X) + \cancel{(\nabla_Y Y)(\varphi)} + |Y(\varphi)|^2 + |X(\varphi)|^2 + g(\nabla\varphi, \nabla\varphi)$$

$$= -\mathrm{Hess}_\varphi(Y,Y) - \mathrm{Hess}_\varphi(X,X) + |Y(\varphi)|^2 + |X(\varphi)|^2 + g(\nabla\varphi,\nabla\varphi)$$

$\Rightarrow$ Claim by using $e^{2\varphi}K^\varphi(X,Y) - K(X,Y) = g(R^\varphi(X,Y)Y,X) - g(R(X,Y)Y,X)$

(on the initial sheet, there was a bracket missing indeed in the statement!)