Hence from (a) with $T = L$: (iv) $\Rightarrow$ (iii).

(ii), (v) and (vi): See exercises.

5.4 Compact operators

General assumption: X and Y are both Banach spaces

Recall: In metric spaces: compactness $\overset{1.27}{\Longleftrightarrow}$ sequential cpt.ness

Hence the equivalence in

Definition 5.27 An operator $T \in BL(X,Y)$ is compact

$:\Longleftrightarrow$ $\forall A \subset X$: A bounded $\Rightarrow$ $\overline{T(A)} \subseteq Y$ compact (i.e. $T(A)$ rel. cpt. in Y)

$\Longleftrightarrow$ $\forall (x_n)_n \subseteq X$: $(x_n)_n$ bounded $\Rightarrow$ $(Tx_n)_n$ has a convergent subsequence

Example 5.28 (a) For $k \in C([0,1]^2)$, let $T: L^2([0,1]) \to L^2([0,1])$,

$$(Tf)(x) = \int_0^x k(x,y) f(y)\, dy \quad , \quad f \in L^2([0,1]).$$

Then T is compact (see Exercise E13.2)

(b) Finite-rank operators:

$T \in BL(X,Y)$ is of finite rank $:\Longleftrightarrow$ $\dim \operatorname{ran}(T) < \infty$.

Then T is compact:

$(x_n)_n \subseteq X$ bounded $\overset{T \text{ bdd.}}{\Longrightarrow}$ $(Tx_n)_n \subseteq \operatorname{ran}(T)$ bounded,

hence, $(Tx_n)_n$ is a bounded seq. in finite-dim. space, so, by Bolzano-Weierstraß, it has convergent subsequence.

Theorem 5.29 Let $T \in BL(X,Y)$ be compact, and $(x_n)_n \subseteq X$ s.t. $x_n \overset{w}{\rightarrow} x \in X$, $n \to \infty$. Then $Tx_n \overset{n\to\infty}{\longrightarrow} Tx$ strongly, i.e. wrt. the norm in Y ($\|Tx_n - Tx\|_Y \to 0$, $n \to \infty$)

Pf: Let $y_n := Tx_n$ and $y := Tx$. Since $x_n \xrightarrow{w} x$, we have (by Lem. 5.9): $y_n \xrightarrow{w} y$. Assume, for contradiction, that $y_n \overset{s}{\not\to} y$ (i.e. not conv. in norm).

Then $\exists \varepsilon > 0$, $\exists (y_{n_k})_k$: $\|y_{n_k} - y\| \geq \varepsilon \quad \forall k \in \mathbb{N}$.

But $(x_{n_k})_{k\in\mathbb{N}}$ is weakly conv., so (by 4.29) bounded.

Since T is compact, $\exists (y_{n_{k_\ell}})_\ell$: $y_{n_{k_\ell}} \xrightarrow{\ell\to\infty} \tilde{y} \in Y$, $\tilde{y} \neq y$.

Since $y_{n_{k_\ell}} \underset{\ell\to\infty}{\xrightarrow{w}} y \neq \tilde{y}$ (4.27(c)) this is a contradiction, so $y_n \xrightarrow{n\to\infty} y$ strongly / in norm ∎

Theorem 5.30 Let $T \in BL(X,Y)$.

(a) Assume $(T_n)_{n\in\mathbb{N}} \subseteq BL(X,Y)$, T_n compact $\forall n \in \mathbb{N}$, and

$$\|T_n - T\| \xrightarrow{n\to\infty} 0 \quad , \quad T \in BL(X,Y)$$

Then T is compact.

(b) T is compact iff $T^\times$ is compact (Schauder's theorem)

(c) Let Z be a Banach space and $S \in BL(Y,Z)$.
If S or T is compact, then $ST \in BL(X,Z)$ is compact.

(d) In particular: Denote $K(X,Y) := \{T \in BL(X,Y) \mid T \text{ compact}\}$, $K(X) := K(X,X)$. Then $K(X) \subseteq BL(X)$ is a two-sided norm-closed ideal.

Pf: (d) follows from (a) & (c) ($Y = Z = X$).

(c) Clear, since bounded linear operators preserve convergence & boundedness.

(a) Let $(x_m)_{m\in\mathbb{N}} \subseteq X$ be bounded, wlog. $\|x_m\| \leq 1 \ \forall m \in \mathbb{N}$.
As T_n is compact, $(T_n x_m)_m$ has a convergent subseq (dep. on n) with limit $y_n \in Y \ \forall n \in \mathbb{N}$. By Cantor's diagonal sequence trick, $\exists$ common subseq. $(x_{m_k})_{k\in\mathbb{N}}$ (i.e. indep. of n !) s.t. $T_n x_{m_k} \xrightarrow{k\to\infty} y_n \quad \forall n \in \mathbb{N}$ (check!). (*)

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\|T_n - T_{n'}\| < \varepsilon \ \forall n, n' \geq N$.

Then

$$\|y_n - y_{n'}\| \leq \underbrace{\|y_n - T_n x_{m_k}\|}_{(1)} + \underbrace{\|T_n x_{m_k} - T_{n'} x_{m_k}\|}_{\leq \|T_n - T_{n'}\| < \varepsilon} + \underbrace{\|T_{n'} x_{m_k} - y_{n'}\|}_{(2)}$$

$$\overset{k \to \infty}{<} \varepsilon, \text{ since (1) \& (2) tend to 0 as } k \to \infty \text{ (by (*))}$$

So $(y_n)_{n \in \mathbb{N}}$ is Cauchy, hence (Y Banach) convergent, i.e.

$y_n \xrightarrow{n \to \infty} y \in Y$

Claim: $\|T x_{m_k} - y\| \xrightarrow{k \to \infty} 0$.

Indeed, let $\varepsilon > 0$. $\exists n \in \mathbb{N}$ s.t.

$$\|T - T_n\| < \frac{\varepsilon}{3} \quad \text{and} \quad \|y_n - y\| < \frac{\varepsilon}{3}$$

Fix this. Then there is $K \in \mathbb{N}$ s.t. $\|T_n x_{m_k} - y_n\| < \frac{\varepsilon}{3} \ \forall k \geq K$ (by (*))

Hence, for all $k \geq K$,

$$\|T x_{m_k} - y\| \leq \underbrace{\|T x_{m_k} - T_n x_{m_k}\|}_{\leq \|T - T_n\| \cdot \underbrace{\|x_{m_k}\|}_{\leq r} < \frac{\varepsilon}{3}} + \underbrace{\|T_n x_{m_k} - y_n\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|y_n - y\|}_{< \frac{\varepsilon}{3}}$$

$$< \varepsilon$$

So, claim holds, and T is compact.

(b) "$\Rightarrow$": Let $(l_n)_{n \in \mathbb{N}} \subseteq Y^*$ be a bounded sequence.

Then $K := T(\overline{B_1^X(0)}) = \overline{T(B_1^X(0))} \subseteq Y$ is a compact metric space. The restrictions $f_n := l_n|_K \in C(K)$ (equipped with $\|\cdot\|_\infty$-norm) form a bounded and equicontinuous sequence, because

$$|f_n(y_1) - f_n(y_2)| \leq \underbrace{\left(\sup_{k \in \mathbb{N}} \|l_k\|\right)}_{< \infty} \cdot \|y_1 - y_2\|_Y \quad \forall y_1, y_2 \in K$$

By the theorem of Arzelà-Ascoli (1.40) there exists a uniformly convergent subsequence $(f_{n_k})_{k\in\mathbb{N}}$, hence Cauchy, i.e. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}: \forall k, l \geq N:$
$$\|f_{n_k} - f_{n_l}\|_{C(K)} < \varepsilon$$
So, for all $k, l \geq N$:

$$\|T^\times l_{n_k} - T^\times l_{n_l}\|_{X^*} = \sup_{x\in B_1^X(0)} |T^\times l_{n_k} x - T^\times l_{n_l} x|$$
$$= \sup_{x\in B_1^X(0)} |l_{n_k}(Tx) - l_{n_l}(Tx)| \overset{*}{\leq} \|f_{n_k} - f_{n_l}\|_{C(K)} < \varepsilon.$$
(*in fact "=", since $T(B_1^X(0))$ dense in K).

So $(T^\times l_{n_k})_{k\in\mathbb{N}}$ converges in X^*, hence $T^\times : Y^* \to X^*$ is cpt.

"$\Leftarrow$": If $T^\times$ compact, then $T^{\times\times}: X^{**} \to Y^{**}$ is compact by "$\Rightarrow$". So, from (c), $T^{\times\times} J_X$ is compact, where $J^X : X \to X^{**}$ is the canonical embedding, $(J_X x)(f) = f(x)$ $\forall x \in X, \forall f \in X^*$. Now, for $x \in X$ and $l \in Y^*$, we have

$$(T^{\times\times} J_X x)(l) = J_X(x)(T^\times l) = (T^\times l)(x) = l(Tx) = (J_Y(Tx))(l)$$

This implies $T^{\times\times} J_X = J_Y T$ being compact. Since Y is closed in Y^{**} (i.e. $J_Y(Y) \subseteq Y^{**}$ is closed), T is also compact. ∎

Theorem 5.31 Let H be a separable Hilbert space. Then $K(H) = \{T \in BL(H) \mid T \text{ compact}\} = \overline{\{T \in BL(H) \mid T \text{ has finite rank}\}}^{\|\cdot\|}$ where the closure is wrt. the operator norm.

Pf: "$\supseteq$" holds because of Ex. 5.28(b) and Thm. 5.30(a) (this does not need separability – or that H is Hilbert space (Banach ok)).

"$\subseteq$": Need to prove: Any compact $T \in BL(H)$ is the norm-limit of finite-rang operators:

If $\dim(H) < \infty$, there is nothing to prove. So let $\dim(H) = \infty$ and let $\{\varphi_j\}_{j\in\mathbb{N}} \subseteq H$ be an ONB. For $n \in \mathbb{N}$, define the orthogonal rank-n-projection $P_n := \sum_{j=1}^{n} \langle \varphi_j, \cdot \rangle \varphi_j$.

By Thm. 2.51: $P_n \xrightarrow{s} \mathbb{1}$, hence $\mathbb{1} - P_n = \sum_{j=n+1}^{\infty} \langle \varphi_j, \cdot \rangle \varphi_j$ (where the series converges in the strong operator sense).

Let $R_n := TP_n \in BL(H)$ for $n \in \mathbb{N}$. Then R_n has rank $\leq n$, and converges to T strongly ($R_n \xrightarrow{s} T$).

Strategy: Use compactness of T to prove convergence even uniformly.

Let $\psi \in H$, $\|\psi\| = 1$. Then

$$\|(\mathbb{1} - P_n)\psi\|^2 = \sum_{j=n+1}^{\infty} |\langle \varphi_j, \psi \rangle|^2 \overset{\text{Bessel}}{\leq} \|\psi\|^2 = 1$$

so that

$$\|T - R_n\| = \sup_{\substack{\psi \in H, \\ \|\psi\|=1}} \|T(\mathbb{1} - P_n)\psi\| \leq \sup_{\substack{\psi\in H \\ \|\psi\|=1}} \underbrace{\sup_{\substack{\psi \in \operatorname{ran}(\mathbb{1}-P_n) \\ \|\psi\| \leq 1}} \|T\psi\|}_{=:\lambda_n} = \lambda_n .$$

Note: $(\lambda_n)_{n\in\mathbb{N}}$ is non-negative and decreasing, so $\lim_{n\to\infty} \lambda_n =: \lambda \geq 0$ exists. The theorem now follows from:

Claim: $\lambda = 0$: For $n \in \mathbb{N}$, $\exists\, \psi_n \in \operatorname{ran}(\mathbb{1} - P_n)$ with $\|\psi_n\| \leq 1$ and $\|T\psi_n\| \geq \frac{\lambda_n}{2} \geq \frac{\lambda}{2}$. Now, for all $\gamma \in H$,

$$|\langle \gamma, \psi_n \rangle|^2 = |\langle \gamma, (\mathbb{1} - P_n)\psi_n \rangle|^2 = |\langle (\mathbb{1} - P_n)\gamma, \psi_n \rangle|^2$$

$$\overset{\text{C.S.}}{\leq} \underbrace{\|(\mathbb{1} - P_n)\gamma\|^2}_{\sum_{j=n+1}^{\infty} |\langle \varphi_j, \gamma \rangle|^2} \cdot \underbrace{\|\psi_n\|^2}_{\leq 1} \xrightarrow{n\to\infty} 0$$

So $\psi_n \xrightarrow{w} 0$ and (since T cpt.) $T\psi_n \xrightarrow{n\to\infty} 0$ (by 5.29). So $\lambda = 0$ $\blacksquare$