Remark 5.14 (a) self-adjoint or unitary $\Rightarrow$ normal.

(b) T normal $\Leftrightarrow \langle Tx, Ty\rangle = \langle T^*x, T^*y\rangle \quad \forall x,y \in H.$

Pf: LHS $= \langle T^*Tx, y\rangle$, RHS $= \langle (T^*)^*T^*x, y\rangle = \langle TT^*x, y\rangle$.

Hence: T normal $\Rightarrow \|Tx\| = \|T^*x\| \quad \forall x \in H$.

In particular: $\ker(T) = \ker(T^*)$. (if T normal!)

(c) T self-adjoint $\Leftrightarrow \quad \langle x, Ty\rangle = \langle Tx, y\rangle \quad \forall x,y \in H \quad (*)$.

Setting $x=y$ gives: $\quad \langle x, Tx\rangle \in \mathbb{R} \quad \forall x \in H$.

[The property $(*)$ is called symmetry (T is symmetric). The notions "symmetric" and "selfadjoint" agree for bounded linear operators, but their generalisations to unbounded operators do NOT!]

(d) T unitary $\Leftrightarrow \langle Tx, Ty\rangle = \langle x, y\rangle = \langle T^*x, T^*y\rangle \quad \forall x,y \in H$.

Pf: $\langle Tx, Ty\rangle = \langle T^*Tx, y\rangle$ and $\langle T^*x, T^*y\rangle = \langle (T^*)^*T^*x, y\rangle = \langle TT^*x, y\rangle$

Example 5.15 (a) $T = z\mathbb{1}$, $z \in \mathbb{K}$. Then $T^* = \bar{z}\mathbb{1}$. So, T is self-adjoint iff. $z \in \mathbb{R}$.

(b) Let $H = L^2([0,1])$, $k \in C([0,1]^2)$ and

$$(Tf)(x) := \int_0^1 k(x,y) f(y)\, dy \quad \forall f \in H, \ \forall x \in [0,1]$$

Then $T \in BL(H)$ and, if $k(x,y) = \overline{k(y,x)} \ \forall x,y \in [0,1]$, then T is selfadjoint

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5.3 The spectrum

In this subsection: X Banach over $\mathbb{C}$ (i.e. $\mathbb{K} = \mathbb{C}$)

Definition 5.16 Let $T \in BL(X)$

(i) The resolvent set (of T) is $\quad \rho(T) := \{ z \in \mathbb{C} \mid T - z\mathbb{1} \text{ is bijective} \}$

(ii) The resolvent of T is $\quad R_z := R(z,T) := (T - z\mathbb{1})^{-1} =: (T-z)^{-1}$

for which ever $z \in \mathbb{C}$ this exists as a (possibly unbounded) operator.

(a) If $z \in \rho(T)$, then $R_z \in BL(X)$ (bounded inverse thm!)

(b) R_z need not exist for $z \notin \rho(T)$.

(iii) The spectrum of T is $\quad \mathrm{spec}(T) := \sigma(T) := \mathbb{C} \setminus \rho(T)$

(iv) If there exists $0 \neq x \in X$ s.t. $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then λ is an eigenvalue and x the corresponding eigenvector.

(v) The point spectrum of T is
$$\operatorname{spec}_p(T) := \sigma_p(T) := \{z \in \mathbb{C} \mid T - z \text{ not injective}\} = \{\text{eigenvalues of } T\}.$$
(vi) The continuous spectrum of T is
$$\operatorname{spec}_c(T) := \sigma_c(T) := \{z \in \mathbb{C} \mid T - z \text{ injective and } \operatorname{ran}(T-z) \neq \mathfrak{X}, \text{ but dense}\}$$
(vii) The residual spectrum of T is
$$\operatorname{spec}_r(T) := \sigma_r(T) := \{z \in \mathbb{C} \mid T - z \text{ injective and } \operatorname{ran}(T-z) \text{ not dense}\}.$$
[$\sigma_r(T) = \emptyset$ for most T of interest in applications]

Lemma 5.17 Let $T \in BL(\mathfrak{X})$. Then

(a) $\mathbb{C} = \sigma(T) \,\dot\cup\, \rho(T)$ (disjoint union)

(b) $\sigma(T) = \sigma_p(T) \,\dot\cup\, \sigma_c(T) \,\dot\cup\, \sigma_r(T)$

(c) If $\dim \mathfrak{X} < \infty$, then $\sigma_c(T) = \emptyset = \sigma_r(T)$ (so $\sigma(T) = \sigma_p(T)$).

Pf: (a), (b) clear by definition.

(c) follows from Linear Algebra: $T - z$ injective $\Rightarrow \dim\ker(T-z) = 0$
$\overset{\dim \mathfrak{X} < \infty}{\Longrightarrow} \dim \operatorname{ran}(T-z) = \dim(\mathfrak{X}) \Rightarrow \operatorname{ran}(T-z) = \mathfrak{X}$ ∎

Lemma 5.18 (Neumann series). Let $T \in BL(\mathfrak{X})$, with $\|T\| < 1$.
Then $(\mathbb{1} - T)^{-1} \in BL(\mathfrak{X})$ and $(\mathbb{1} - T)^{-1} = \sum_{j=0}^{\infty} T^j$.

Pf: See exercise (T6.3 Sheet 6). Here details for completeness:
From $\|T^j\| \leq \|T\|^j$, $\|T\| < 1$, and Lemma 2.50 (a) follows: $S := \sum_{j=0}^{\infty} T^j \in BL(\mathfrak{X})$
For $N \in \mathbb{N}$, we have
$$(\mathbb{1} - T)\sum_{j=0}^{N} T^j = \mathbb{1} - T^{N+1} = \Big(\sum_{j=0}^{N} T^j\Big)(\mathbb{1} - T).$$
Now, $\sum_{j=0}^{N} T^j \xrightarrow{N\to\infty} S$ and $T^{N+1} \xrightarrow{N\to\infty} 0$ (since $\|T^{N+1}\| \leq \|T\|^{N+1} \xrightarrow{N\to\infty} 0$). Hence,
$$(\mathbb{1} - T)S = \mathbb{1} = S(\mathbb{1} - T) \quad \text{so } S = (\mathbb{1} - T)^{-1}$$ ∎

Lemma 5.19 Let $T \in BL(\mathfrak{X})$. Then $\sigma(T) \subseteq \{z \in \mathbb{C} \mid |z| \leq \|T\|\}$

Pf: For $|z| > \|T\|$ we have $\|\frac{T}{z}\| < 1$, hence 5.18 gives existence of
$$BL(\mathfrak{X}) \ni \frac{1}{z}\Big(\frac{T}{z} - \mathbb{1}\Big)^{-1} = (T - z)^{-1} = R_z.$$
Hence, $z \in \rho(T) = \mathbb{C} \setminus \sigma(T)$ ∎

Definition 5.20 Let $D \subseteq \mathbb{C}$ be open, $(\mathcal{Z}, \|\cdot\|)$ a Banach space over $\mathbb{C}$, and
$$\xi : D \to \mathcal{Z}$$
$$z \mapsto \xi(z)$$

(i) The map ξ is norm-differentiable (or strongly differentiable) at $z_0 \in D$
$$:\Leftrightarrow \exists\, \zeta \in \mathcal{Z} : \lim_{\mathbb{C} \ni h \to 0} \left\| \frac{\xi(z_0+h) - \xi(z_0)}{h} - \zeta \right\| = 0$$

Notation: $\frac{d\xi}{dz}(z_0) := \zeta$.

(ii) ξ is weakly differentiable at $z_0 \in D$
$:\Leftrightarrow$ $z \mapsto l(\xi(z)) \in \mathbb{C}$ is complex differentiable at z_0 $\forall\, l \in \mathfrak{X}^*$

(iii) ξ is norm-analytic (resp. weakly analytic) in D
$:\Leftrightarrow$ ξ is norm-differentiable (resp. weakly differentiable) at z_0 for all $z_0 \in D$.

Remark 5.21 (a) $\mathcal{Z}$-valued norm-analytic functions have properties analogous to those of $\mathbb{C}$-valued analytic functions (f.ex. a power-series expansion converging wrt. $\|\cdot\|$, etc.).
Moral: Replace $|\cdot|$ (abs. value in $\mathbb{C}$) by $\|\cdot\|$
(For a discussion, see Hille-Philips, "FA and semigroups", or Dunford-Schwartz, "Linear operators" vol 1, Sect. III.14)
(b) Norm-differentiable $\Leftrightarrow$ weakly differentiable (For "$\Leftarrow$", see Reed-Simon Thm. VI.4)

Theorem 5.22 Let $T \in BL(\mathfrak{X})$. Then

(a) $\rho(T) \subseteq \mathbb{C}$ is open in $\mathbb{C}$, and the map $R(\cdot, T) : \rho(T) \to BL(\mathfrak{X})$, $z \mapsto R(z,T) =: R_z$
is norm-analytic on $\rho(T)$, with derivative
$$\frac{dR_z}{dz} = R_z^2 \qquad (*)$$

[Note: $\mathcal{Z} = BL(\mathfrak{X})$ in Def. 5.20 in this case - i.e., complex diff. holds wrt. the operator norm]

(b) For $z_1, z_2 \in \rho(T)$, we have
$$(**) \qquad R_{z_1} - R_{z_2} = (z_1 - z_2) R_{z_1} R_{z_2} \qquad \text{(First resolvent identity)}$$

In particular: $R_{z_1} R_{z_2} = R_{z_2} R_{z_1}$ (they commute!)

Pf: (b) Equation (**) follows from

$$(T-z_1)(R_{z_1}-R_{z_2})(T-z_2) = (\mathbb{1}-(T-z_1)R_{z_2})(T-z_2)$$
$$= T-z_2-(T-z_1) = z_1-z_2$$

and from multiplication by R_{z_1} from the left and R_{z_2} from the right.

Commutativity: Exchange $z_1 \leftrightarrow z_2$ in (**) & compare with (**).

(a) Openness of $\rho(T)$: Let $z_0 \in \rho(T)$ ($\neq \phi$ by Lemma 5.19) and $z \in B_{1/\|R_{z_0}\|}(z_0)$ (open ball in $\mathbb{C}$). Then

$$T-z = T-z_0-(z-z_0) = (T-z_0)(\mathbb{1}-\underbrace{(z-z_0)R_{z_0}}_{=:V_z})$$

By definition, $\|V_z\|<1$, hence by Lemma 5.18: $(\mathbb{1}-V_z)^{-1} \in BL(X)$. So,

$$R_z = (T-z)^{-1} = (\mathbb{1}-V_z)^{-1}R_{z_0} \in BL(X) \qquad (\text{II})$$

and so $z \in \rho(T)$, i.e. $\rho(T)$ is open.

Analytic: Let $z_0 \in \rho(T)$. By the above: $\forall 0 \neq h \in \mathbb{C}, |h| < \|R_{z_0}\|^{-1}$: $z_0+h \in \rho(T)$. Hence

$$\frac{R_{z_0+h}-R_{z_0}}{h} \overset{(**)}{=} R_{z_0+h}R_{z_0} \overset{(\text{II})\,\&\,5.18}{=} \sum_{n\in\mathbb{N}_0} V_{z_0+h}^n R_{z_0}^2 = \sum_{n\in\mathbb{N}_0} h^n R_{z_0}^{n+2}$$

and so

$$\left\| \frac{R_{z_0+h}-R_{z_0}}{h} - R_{z_0}^2 \right\| \le \sum_{n\in\underline{\underline{\mathbb{N}}}} |h|^n \|R_{z_0}\|^{n+2} = \|R_{z_0}\|^2 \left\{ \frac{1}{1-|h|\|R_z\|} - 1 \right\} \xrightarrow{|h|\to 0} 0.$$

Hence, $z \mapsto R_z \in BL(X)$ is norm-differentiable at z_0, with derivative $R_{z_0}^2$. As $z_0 \in \rho(T)$ was arbitrary, $R(\cdot,T)$ is norm-analytic in $\rho(T)$ (and (*) holds) ∎

$\boxed{\text{Lemma 5.23}}$ Let $T \in BL(X)$. Then $\sigma(T) \neq \phi$.

Pf: Let $|z| > \|T\|$. Then (see pf Lemma 5.19)

$$R_z = (T-z)^{-1} = \frac{1}{z}\left(\frac{T}{z}-\mathbb{1}\right)^{-1} = -\frac{1}{z}\sum_{j=0}^{\infty}\frac{T^j}{z^j}.$$

This expansion shows that

$$\|R_z\| \le \frac{|z|^{-1}}{1 - \|T\|/|z|} = \frac{1}{|z| - \|T\|} \xrightarrow{|z| \to \infty} 0 \qquad (\nabla)$$

Assume $\sigma(T) = \emptyset$ (for contradiction). Then $\rho(T) = \mathbb{C}$ and, by Thm 5.22, $\mathbb{C} \to BL(X)$, $z \mapsto R_z$, is <u>entire</u> (i.e. analytic on all of $\mathbb{C}$). Also, $\mathbb{C} \ni z \mapsto \|R_z\|$ is <u>bounded</u> (using (∇) & continuity). By Liouville's Thm, (for $\mathbb{C}$-valued fct's, see any book on Complex Analysis) $z \mapsto \|R_z\|$ is constant, and by (∇), this constant has to be 0, i.e. $R_z = 0$ $\forall z \in \mathbb{C}$ $\lightning$ ∎

Definition 5.24 For $T \in BL(X)$, the <u>spectral radius of T</u> is

$$r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| \underset{\text{Lemma 5.19}}{\le} \|T\|$$

Theorem 5.25 Let $T \in BL(X)$. Then

(a) $$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|T^n\|^{1/n}$$

(b) If, in addition, X is a Hilbert space, and T is <u>normal</u>, then: $\lim_{n \to \infty} \|T^n\|^{1/n} = \|T\|$. Hence: $r(T) = \|T\|$.

<u>Pf:</u> (a) We first prove: $\lim_{n \to \infty} \|T^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|T^n\|^{1/n}$:

W.l.o.g. assume $T^n \neq 0$ $\forall n \in \mathbb{N}$ (otherwise, if $\exists n_0 : T^{n_0} = 0$, then $T^n = 0$ $\forall n \ge n_0$ and claim is clear).

Let $a_n := \ln \|T^n\|$. Then $a_{n_1 + n_2} \le a_{n_1} + a_{n_2}$ $\forall n_1, n_2 \in \mathbb{N}$ (check!)

Fix $m \in \mathbb{N}$ (arbitrary), and let $n(q,r) := qm + r$ for $q \in \mathbb{N}$ and $r \in \{0, 1, \dots, m-1\}$. Then

$$\frac{a_{n(q,r)}}{n(q,r)} \le \frac{q a_m + a_r}{qm + r} = \frac{a_m}{m} \frac{1}{1 + r/(qm)} + \frac{a_r}{qm + r}$$

Hence,

$$\limsup_{n\to\infty} \frac{a_n}{n} \le \limsup_{q\to\infty} \max_{r\in\{0,\dots,m-1\}} \frac{a_{n(q,r)}}{n(q,r)} \le \frac{a_m}{m} \quad \underline{\forall m\in\mathbb{N}}$$

so
$$\limsup_{n\to\infty} \frac{a_n}{n} \le \inf_{m\in\mathbb{N}} \frac{a_m}{m}.$$

But (trivially) $\liminf_{n\to\infty} \frac{a_n}{n} \ge \inf_{m\in\mathbb{N}} \frac{a_m}{m}$. Taken together,

$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}$ exists, and $\lim_{n\to\infty} e^{a_n/n} = \inf_{n\in\mathbb{N}} e^{a_n/n}$

because the exp. function is continuous and increasing. ✓

Now, by Thm. 5.22, $z\mapsto R_z$ is strongly analytic in $D := \mathbb{C}\setminus\{z\in\mathbb{C} \mid |z|\le r(T)\}$, hence it has a Laurent series expansion about the origin (in $\mathbb{C}$):

$$R_z = \sum_{j\in\mathbb{Z}} z^j A_j, \quad A_j \in BL(X) \text{ for } j\in\mathbb{Z},$$

which is norm-convergent $\forall z\in D$. [For $\mathbb{C}$-valued fct's, see Conway, "Fct.'s of one complex variable", 2nd. ed. (1978), Thm V.1.11]

On the other hand, on $\{z\in\mathbb{C} \mid |z| > \|T\|\} \subseteq D$, the expansion

$$(*) \quad R_z = -\frac{1}{z}\sum_{j=0}^{\infty} \frac{T^j}{z^j}$$

holds (see pf. Lemma 5.23).

Hence, by uniqueness of the Laurent series, $(*)$ <u>is</u> the Laurent series of the resolvent, hence norm convergent $\forall z\in \underline{D}$.

Equivalently, the power series

$$\xi \mapsto \sum_{j\in\mathbb{N}_0} \xi^j T^j \qquad (**)$$

converges on $B_{1/r(T)}(0)$.

On the other hand: For any $\varepsilon>0$, the power series in $(**)$ is <u>not</u> norm-convergent on $B_{\varepsilon+\frac{1}{r(T)}}(0)$, since <u>if</u>, then $(*)$ would converge for some $z\in\sigma(T)$ (recall: $r(T) = \sup_{\lambda\in\sigma(T)}|\lambda|$). Therefore $\frac{1}{r(T)}$ <u>is</u> the radius of convergence of $(**)$, and as such, it is given by Hadamard's root criterion:

$$r(T) = \limsup_{j\to\infty} \|T^j\|^{1/j} = \lim_{j\to\infty} \|T^j\|^{1/j}$$

where the last equality was proved in the first part. ✓

(b) Let "C^*" refer to the C^*-property $\|TT^*\| = \|T\|^2$ from Thm. 5.12(e).

Then

$$\|T^2\|^2 \overset{C^* \text{ for } T^2}{=} \|T^2(T^2)^*\| \overset{T \text{ normal}}{=} \|(TT^*)(TT^*)^*\| \overset{C^* \text{ for } TT^*}{=} \|TT^*\|^2 \overset{C^* \text{ for } T}{=} \|T\|^4,$$

so $\|T^2\| = \|T\|^2$. By induction on $k \in \mathbb{N}$:

$$\|T^{2^k}\| = \|T\|^{2^k} \qquad (\nabla)$$

hence $\displaystyle\lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{k\to\infty} \|T^{2^k}\|^{1/2^k} \overset{(\nabla)}{=} \|T\|$ ▨

$\boxed{\text{Example 5.26}}$ Right-shift & left-shift operator on ℓ^2:

For $x = (x_1, x_2, \dots) \in \ell^2$, let $Rx := (0, x_1, x_2, \dots)$ & $Lx := (x_2, x_3, \dots)$

Then $\|R\| = 1 = \|L\|$, R is isometric and $R^* = L$ (check!)

Claims: (i) $\sigma_p(L) = B_1(0)$ (open ball in $\mathbb{C}$) (ii) $\sigma_c(L) = \partial B_1(0)$

(iii) $\sigma_r(L) = \emptyset$ (iv) $\sigma_p(R) = \emptyset$ (v) $\sigma_c(R) = \partial B_1(0)$

(vi) $\sigma_r(R) = B_1(0)$.

Proofs: (i) Let $\lambda \in \mathbb{C}$, then $Lx = \lambda x \iff (x_2, x_3, \dots) = (\lambda x_1, \lambda x_2, \dots)$

$\iff x_2 = \lambda x_1,\ x_3 = \lambda x_2 = \lambda^2 x_1, \dots \iff x = x_1(1, \lambda, \lambda^2, \lambda^3, \dots)$

Hence, $x \in \ell^2$ & $Lx = \lambda x \iff \sum_{n\in\mathbb{N}} (|\lambda|^2)^n < \infty \iff |\lambda| < 1$ ✓

(iv) R isometric $\Rightarrow \sigma_p(R) \subseteq \partial B_1(0)$ (check!). Let $|\lambda| = 1$,

then $Rx = \lambda x \iff (0, x_1, \dots) = (\lambda x_1, \lambda x_2, \dots)$

$\iff 0 = \lambda x_1,\ x_1 = \lambda x_2 \overset{\lambda \neq 0}{\iff} x_1 = 0,\ x_2 = 0, \dots$ ✓

(iii) Lemma (pf: see exercise). Let X be a Hilb. sp., $T \in BL(X)$

Then (a) $z \in \sigma_r(T) \Rightarrow \bar{z} \in \sigma_p(T^*)$

(b) $z \in \sigma_p(T) \Rightarrow \bar{z} \in \sigma_p(T^*) \cup \sigma_r(T^*)$

Hence from (a) with $T = L$: (iv) $\Rightarrow$ (iii).

(ii), (v) and (vi): See exercises.

5.4 Compact operators

General assumption: X and Y are both Banach spaces

Recall: In metric spaces: compactness $\overset{1.27}{\Longleftrightarrow}$ sequential cpt.ness

Hence the equivalence in

Definition 5.27 An operator $T \in BL(X,Y)$ is compact

$:\Longleftrightarrow$ $\forall A \subset X$: A bounded $\Rightarrow$ $\overline{T(A)} \subseteq Y$ compact (i.e. $T(A)$ rel. cpt. in Y)

$\Longleftrightarrow$ $\forall (x_n)_n \subseteq X$: $(x_n)_n$ bounded $\Rightarrow$ $(Tx_n)_n$ has a convergent subsequence

Example 5.28 (a) For $k \in C([0,1]^2)$, let $T: L^2([0,1]) \to L^2([0,1])$,

$$(Tf)(x) := \int_0^x k(x,y) f(y)\, dy \quad , \quad f \in L^2([0,1]).$$

Then T is compact (see Exercise E13.2)

(b) Finite-rank operators:

$T \in BL(X,Y)$ is of finite rank $:\Longleftrightarrow \dim \operatorname{ran}(T) < \infty$.

Then T is compact:

$(x_n)_n \subseteq X$ bounded $\overset{T \text{ bdd.}}{\Longrightarrow}$ $(Tx_n)_n \subseteq \operatorname{ran}(T)$ bounded,

hence, $(Tx_n)_n$ is a bounded seq. in finite-dim. space, so, by Bolzano-Weierstraß, it has convergent subsequence.

Theorem 5.29 Let $T \in BL(X,Y)$ be compact, and $(x_n)_n \subseteq X$ s.t. $x_n \overset{w}{\to} x \in X$, $n \to \infty$. Then $Tx_n \overset{n\to\infty}{\longrightarrow} Tx$ strongly, i.e. w.r.t. the norm in Y ($\|Tx_n - Tx\|_Y \to 0$, $n \to \infty$)