

Chapter 4: The cornerstones of Functional Analysis

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4.1 Hahn-Banach Theorem

Recall:

Definition Let M be a set, and $D \subseteq M \times M$. Let $< ("sub")$ be the associated binary relation: $x < y \Leftrightarrow (x, y) \in D$.

(a) $<$ is a partial ordering (or, M is a partially ordered set)

$\Leftrightarrow \forall x, y, z \in M$:

(i) $x < x$ (reflexive)

(ii) $(x < y \ \& \ y < x) \Rightarrow (x = y)$ (antisymmetry)

(iii) $(x < y \ \& \ y < z) \Rightarrow (x < z)$ (transitive)

(b) $x, y \in M$ are comparable $\Leftrightarrow x < y$ or $y < x$

x, y are incomparable $\Leftrightarrow x, y$ are not comparable.

(c) $<$ is a total ordering \Leftrightarrow

(i) $<$ is a partial ordering

(ii) x, y are comparable for all $x, y \in M$

(d) Let $W \subseteq M$. Then $u \in M$ is an upper bound of / for W

$\Leftrightarrow w < u \ \forall w \in W$.

(e) $m \in W$ is a maximal element of W \Leftrightarrow The following implication holds:

$m < w \ \text{for } w \in W \Rightarrow m = w$.

(Note: A maximal element need not be an upper bound and vice versa - see examples below).

Example 1 (a) " \leq " is a total ordering on \mathbb{R} .

$W = [0, 1)$ has no maximal element, but any $u \geq 1$ is an upper bound for W

(b) " \subseteq " is a partial ordering on $\mathcal{P}(X)$, but not a total ordering.

(c) $(x_1, x_2) \leq (y_1, y_2) : \Leftrightarrow x_j \leq y_j, j=1,2$, (82)
 is a partial ordering (but not total) on $M = \mathbb{R}^2$.
 $W := \{(0,0), (1,0), (0,1)\}$ has 2 (!) maximal elements,
 but none of them is an upper bound for W .

The following axiom is equivalent to the axiom of choice:

(Axiom 4.1) (Zorn's Lemma) Let $M \neq \emptyset$ be a partially ordered set. Assume every totally ordered subset of M has an upper bound. Then M has a maximal element.

We now finish the

Proof of Thm. 2.53 (Remains: Every Hilbert space $\mathcal{H} \neq \{0\}$ has an orthonormal basis)

Let $M := \{E \subseteq \mathcal{H} \mid E \text{ orthonormal}\}$. Then:

- (i) $M \neq \emptyset$
- (ii) " \subseteq " is partial ordering on M
- (iii) Let W be a totally ordered subset of M .

Then $U := \bigcup_{E \in W} E$:

If $x_1, x_2 \in U$ then $\exists E_j \in W : x_j \in E_j, j=1,2$.

As W is totally ordered, $E_1 \subseteq E_2$ (wlog.)

Hence, $x_1, x_2 \in E_2 \Rightarrow x_1 \perp x_2 \Rightarrow U$ orthonormal

That is, U is an upper bound for W .

By Zorn: \exists maximal element $M \in M$.

Claim: \mathcal{M} is an orthonormal basis for \mathcal{X}

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(i) Orthonormal clear, since $\mathcal{M} \in \mathcal{M}$

(ii) Assume \mathcal{M} is not complete, i.e., not a basis.

Then (by Def. 2.47(d)), there is some $0 \neq x \in \mathcal{X}$ such that $x \perp m \quad \forall m \in \mathcal{M}$. Hence,

$$\mathcal{M}' := \left\{ \frac{x}{\|x\|} \right\} \cup \mathcal{M} \in \mathcal{M} \quad (\text{since } \mathcal{M}' \text{ is orthonormal})$$

so $\mathcal{M} \subsetneq \mathcal{M}'$ which contradicts \mathcal{M} being maximal \square

Remark 4.2 Similar arguments prove Thm. 2.4
(existence of Hamel basis; see exercise)

Theorem 4.3 (Hahn - Banach (H-B)) Let \mathcal{X} be a vector space, let $p: \mathcal{X} \rightarrow \mathbb{R}$ be convex, i.e. $\forall x, x' \in \mathcal{X}, \forall \alpha \in [0, 1]$:

$$p(\alpha x + (1-\alpha)x') \leq \alpha p(x) + (1-\alpha)p(x'). \quad (*)$$

Let \mathcal{Y} be a subspace, and let $\lambda: \mathcal{Y} \rightarrow \mathbb{K}$ be linear, with

$$\operatorname{Re} \lambda(y) \leq p(y) \quad \forall y \in \mathcal{Y}. \quad (A)$$

Then there exists $\Lambda: \mathcal{X} \rightarrow \mathbb{K}$ linear such that:

(i) $\Lambda|_{\mathcal{Y}} = \lambda$ (i.e. Λ is extension of λ)

(ii) $\operatorname{Re} \Lambda(x) \leq p(x) \quad \forall x \in \mathcal{X}$ (i.e. (A) is preserved)

If, in addition, p also satisfies

$$(**) \quad p(\alpha x) \leq p(x) \quad \forall x \in \mathcal{X}, \forall \alpha \in \mathbb{K} \text{ with } |\alpha| = 1$$

then we even have $|\Lambda(x)| \leq p(x) \quad \forall x \in \mathcal{X}$.

Remark 4.4 (a) Condition (**) is equivalent to:

$$p(\alpha x) = p(x) \quad \forall x \in \mathcal{X} \text{ with } |\alpha| = 1 \quad (\text{check!})$$

(b) If p is a (semi-) norm on \mathcal{X} , then p satisfies (*) & (**).

Pf (of 4.3): 4 Steps: Steps 1 & 2 prove the main part for \mathbb{R} -vector spaces, step 3 for \mathbb{C} -vector spaces, and step 4 proves the addendum under add. condition (**).
 Wlog: $Y \subsetneq X$ (otherwise trivial). (84)

Step 1: Case $K = \mathbb{R}$. Extend by one dimension — a preparation for Step 2.

By assumption: $\exists z \in X \setminus Y$ (so $z \neq 0$). Let $\tilde{Y} := \text{span}(Y, \{z\})$.

For every $\tilde{y} \in \tilde{Y}$, \exists ! decomposition $\tilde{y} = y + \alpha z$, $y \in Y$, $\alpha \in \mathbb{R}$

Candidate for extension $\tilde{\lambda}$ of λ to \tilde{Y} :

$$\tilde{\lambda}(\tilde{y}) := \lambda(y) + \alpha \tilde{z} \quad \text{for some } \tilde{z} \in \mathbb{R} \quad (\text{to be chosen})$$

Interpretation: $\tilde{z} = \tilde{\lambda}(z)$. Clearly, $\tilde{\lambda}$ is (\mathbb{R} -) linear on \tilde{Y} , and $\tilde{\lambda}|_Y = \lambda$. Will choose \tilde{z} s.t. $\tilde{\lambda} \leq p$ on \tilde{Y} (recall (ii)):

Let $\beta_1, \beta_2 > 0$, $y_1, y_2 \in Y$, then

$$\begin{aligned} \beta_1 \lambda(y_1) + \beta_2 \lambda(y_2) &= (\beta_1 + \beta_2) \lambda \left(\frac{\beta_1}{\beta_1 + \beta_2} y_1 + \frac{\beta_2}{\beta_1 + \beta_2} y_2 \right) \quad (\forall) \\ &\stackrel{\text{by (ii)}}{\leq} p \left(\frac{\beta_1}{\beta_1 + \beta_2} y_1 + \frac{\beta_2}{\beta_1 + \beta_2} y_2 \right) \\ &= \frac{\beta_1}{\beta_1 + \beta_2} (y_1 - \beta_2 z) + \frac{\beta_2}{\beta_1 + \beta_2} (y_2 + \beta_1 z) \end{aligned}$$

Hence,

$$(\forall) \quad p^{\text{convex}} \leq \beta_1 p(y_1 - \beta_2 z) + \beta_2 p(y_2 + \beta_1 z).$$

Re-arranging, we get

$$\frac{1}{\beta_2} (\lambda(y_1) - p(y_1 - \beta_2 z)) \leq \frac{1}{\beta_1} (p(y_2 + \beta_1 z) - \lambda(y_2))$$

$$\forall \beta_1, \beta_2 > 0, \forall y_1, y_2 \in Y$$

Hence, there exists $a \in \mathbb{R}$:

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$$(\forall \forall) \sup_{\substack{\beta_2 > 0 \\ \gamma_1 \in Y}} \left[\frac{1}{\beta_2} (\lambda(\gamma_1) - p(\gamma_1 - \beta_2 z)) \right] \leq a \leq \inf_{\substack{\beta_1 > 0 \\ \gamma_2 \in Y}} \left[\frac{1}{\beta_1} (p(\gamma_2 + \beta_1 z) - \lambda(\gamma_2)) \right]$$

Set $\tilde{\lambda}(z) := a$. Then, $\forall \tilde{\gamma} = \gamma + \alpha z \in \tilde{Y}$ with $\alpha > 0$, the right inequality in $(\forall \forall)$ implies

$$\lambda(\tilde{z}) \leq \frac{1}{\alpha} (p(\tilde{\gamma}) - \lambda(\gamma_1)), \text{ so } \tilde{\lambda}(\tilde{\gamma}) \leq p(\tilde{\gamma}).$$

If $\tilde{\gamma} = \gamma - \alpha z$ with $\alpha > 0$, use the left ineq. in $(\forall \forall)$ instead. Therefore, $\tilde{\lambda} \leq p$ on \tilde{Y} .

Step 2: Case $\mathbb{K} = \mathbb{R}$. Idea: Use Zorn to construct the extension.

Let $M := \{ (\mathbb{R}\text{-}) \text{ linear extensions } e \text{ of } \lambda \text{ with } e \leq p \text{ on } \text{dom}(e) \}$

We have

(i) $M \neq \emptyset$ since $\lambda \in M$

(ii) Define partial ordering $<$ on M :

$$e_1 < e_2 \iff \text{dom}(e_1) \subseteq \text{dom}(e_2) \wedge e_2|_{\text{dom}(e_1)} = e_1.$$

(iii) Let $\mathcal{W} \subseteq M$ be totally ordered.

Define $u: \bigcup_{e \in \mathcal{W}} \text{dom}(e) \rightarrow \mathbb{R}$ where \tilde{e} is any element of \mathcal{W} such that $x \in \text{dom}(\tilde{e})$

$$x \mapsto \tilde{e}(x)$$

u is well-defined: (i.e. indep. of the chosen \tilde{e} among allowed elements).

Let $x \in \text{dom}(\tilde{e}_1) \cap \text{dom}(\tilde{e}_2)$. Since \mathcal{W} is totally ordered, we have (wlog.) \tilde{e}_2 is an extension of \tilde{e}_1 . Hence, $\tilde{e}_2(x) = \tilde{e}_1(x)$.

u is $(\mathbb{R}\text{-})$ linear: Let $x_1, x_2 \in \bigcup_{e \in \mathcal{W}} \text{dom}(e)$, $\gamma \in \mathbb{R}$. Hence,

$\exists \tilde{e}_j \in \mathcal{W} : x_j \in \text{dom}(\tilde{e}_j)$, $j=1,2$. Since \mathcal{W} is totally ordered, (wlog.) \tilde{e}_2 is extension of \tilde{e}_1 . Then $x_1, x_2, x_1 + \gamma x_2 \in \text{dom}(\tilde{e}_2)$ & linearity of u follows from that of \tilde{e}_2 .

Lastly: $u(x) = \hat{e}(x) \leq p(x) \quad \forall x \in \bigcup_{e \in W} \text{dom}(e)$. (86)

Hence: $u \in M$, and (check!) u is an upper bound for W .

By Zorn's Lemma: M has a maximal element λ .

Claim: $\text{dom}(\lambda) = \bar{X}$: Suppose $\text{dom}(\lambda) \neq \bar{X}$. Then there

is some $0 \neq z \in \bar{X} \setminus \text{dom}(\lambda)$. Let $\tilde{Y} := \text{span}(\text{dom}(\lambda), \{z\})$.

By Step 1, λ has some (\mathbb{R} -) linear extension $\tilde{\lambda} \in M$ to \tilde{Y} , which contradicts M being maximal.

Hence, the main part of the thm. follows for $\mathbb{K} = \mathbb{R}$.

Step 3: Case $\mathbb{K} = \mathbb{C}$. Reduce to the real case.

Define $\ell(y) := \text{Re} \lambda(y) \quad \forall y \in Y$. Then $\ell: \bar{X} \rightarrow \mathbb{R}$ is \mathbb{R} -linear, and $\ell \leq p$ on Y . So, Step 2 implies: There exists \mathbb{R} -linear functional $L: \bar{X} \rightarrow \mathbb{R}$ with $L|_Y = \ell$ and $L \leq p$ on \bar{X} .

Note: $\lambda(y) = \ell(y) + i \ell(-iy) \quad \forall y \in Y$ since:

$$\ell(-iy) = \text{Re} \lambda(-iy) \stackrel{\lambda \mathbb{C}\text{-lin}}{=} \text{Re} [-i \lambda(y)] = \text{Im} \lambda(y)$$

Define $\Lambda(x) := L(x) + i L(-ix), x \in \bar{X}$.

Then, by Step 2,

(i) Λ is \mathbb{R} -linear on \bar{X}

(ii) $\Lambda|_Y = \lambda$

(iii) $\text{Re } \Lambda \stackrel{L(-ix) \in \mathbb{R}}{=} L \leq p$ on \bar{X}

Also, Λ is \mathbb{C} -linear, since it is \mathbb{R} -linear and

$$\Lambda(ix) = L(ix) + i L(-ix) \stackrel{L \mathbb{R}\text{-lin}}{=} i (L(x) + i L(-ix)) = i \Lambda(x),$$

proving the main part for $\mathbb{K} = \mathbb{C}$.

Step 4: Addendum: We fix $x \in \bar{X}$ and use the polar repr.

$$\Lambda(x) = |\Lambda(x)| e^{i\theta(x)} \quad [\text{If } \mathbb{K} = \mathbb{R} \text{ then } e^{i\theta(x)} \in \{-1, 1\}]$$

$$\text{Then } |\Lambda(x)| = e^{-i\theta(x)} \Lambda(x) \stackrel{\mathbb{K}\text{-lin}}{=} \Lambda(e^{-i\theta(x)} x) \stackrel{LHS \in \mathbb{R}}{=} \text{Re } \Lambda(e^{-i\theta(x)} x)$$

$$\stackrel{\text{main part}}{\leq} p(e^{-i\theta(x)} x) \stackrel{(*)}{\leq} p(x)$$

■

Corollary 4.5 Let X be a normed space, $Y \subseteq X$ a subspace, (87)
and $\varphi \in Y^*$. Then there exists $f \in X^*$ with $f|_Y = \varphi$
and $\|f\|_{X^*} = \|\varphi\|_{Y^*}$.

Pf: Apply Thm. 4.3 with $p(x) := \|\varphi\|_{Y^*} \cdot \|x\|$ for $x \in X$
(fulfills assumptions - check!) to $\lambda = \varphi$. Then $\exists f: X \rightarrow \mathbb{K}$
linear with $|f(x)| \leq \|\varphi\|_{Y^*} \|x\|$, hence $\|f\|_{X^*} \leq \|\varphi\|_{Y^*}$.

But

$$\|f\|_{X^*} = \sup_{0 \neq x \in X} \frac{|f(x)|}{\|x\|} \geq \sup_{0 \neq x \in Y} \frac{|\varphi(x)|}{\|x\|} = \|\varphi\|_{Y^*} \quad \square$$

Corollary 4.6 Let X be a normed space and let $0 \neq x_0 \in X$.
Then there exists $f \in X^*$ with $f(x_0) = \|x_0\|$ and $\|f\|_{X^*} = 1$.

Pf: Let $Y = \text{span}\{x_0\}$. If $y \in Y$ then $y = \alpha x_0$ for some
unique $\alpha \in \mathbb{K}$. Define φ on Y by $\varphi(y) := \alpha \cdot \|x_0\|$.

This implies $\varphi(x_0) = \|x_0\|$ and $\varphi \in Y^*$ with $\|\varphi\|_{Y^*} = 1$,
since $|\varphi(y)| = \|y\|$. By Cor. 4.5, $\exists f: X \rightarrow \mathbb{K}$ linear, with
 $f|_Y = \varphi$ (in particular, $f(x_0) = \|x_0\|$) and $\|f\|_{X^*} = \|\varphi\|_{Y^*} = 1$. \(\square\)

Corollary 4.7 Let X be a normed space, $Z \subseteq X$ a
closed subspace, and $x_0 \in X \setminus Z$ with $0 < \text{dist}(x_0, Z) =: d$.
Then there exists $f \in X^*$ with $f|_Z = 0$, $f(x_0) = d$
and $\|f\|_{X^*} = 1$.

Proof: See exercise.

4.2 Three consequences of Baire's theorem

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Theorem 4.8 (Banach-Steinhaus; Uniform Boundedness Principle)

Let X be a Banach space, Y a normed (!) space, and $F \subseteq BL(X, Y)$.

If
$$\sup_{T \in F} \|Tx\| < \infty \quad \forall x \in X$$

then
$$\sup_{T \in F} \|T\| < \infty.$$

Pf: For $n \in \mathbb{N}$, define

$$A_n := \{x \in X \mid \|Tx\| \leq n \quad \forall T \in F\} = \bigcap_{T \in F} T^{-1}(\overline{B_n^Y(0)}) \quad \left(\begin{array}{l} \text{closed, since} \\ T \text{ cont.} \end{array} \right)$$

Then, by hypothesis, $X = \bigcup_{n \in \mathbb{N}} A_n$. By Cor 1.49 (conseq. Baire)

there exists $n_0 \in \mathbb{N}$ s.t. A_{n_0} is not nowhere dense.

Since A_{n_0} is also closed: $\exists x_0 \in A_{n_0}$ & $r > 0$: $B_r(x_0) \subseteq A_{n_0}$.

Now let $0 \neq x \in X$ and $T \in F$, then

$$\frac{r}{2\|x\|} \|Tx\| \leq \|T(\underbrace{\frac{r}{2\|x\|}x + x_0})\| + \|Tx_0\| \leq n_0 + \|Tx_0\|,$$

$\in B_r(x_0) \subseteq A_{n_0}$

hence,

$$\|T\| \leq \frac{2}{r} (n_0 + \|Tx_0\|).$$

Taking the supremum over all $T \in F$ proves the claim. \blacksquare

Theorem 4.9 (Open mapping theorem) Let X, Y be Banach spaces.

Let $T \in BL(X, Y)$ be onto (surjective). Then T is open (an open map), i.e.:

$$A \subseteq X \text{ open} \Rightarrow \underline{T(A) \subseteq Y \text{ open}}$$

Pf: The thm. follows from 3 claims:

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Claim 1: $\exists r > 0$: $T(B_r^{\mathbb{X}}(0))$ has non-empty interior
 $\Rightarrow T$ is open (i.e. claim is " \Rightarrow " holds).

Pf: Assume $T(B_r^{\mathbb{X}}(0))$ has non-empty interior.

Let $y := Tx$ for some $x \in B_r^{\mathbb{X}}(0)$ be an interior point of $T(B_r^{\mathbb{X}}(0))$, i.e., $\exists r_y$: $B_{r_y}^Y(y) \subseteq T(B_r^{\mathbb{X}}(0))$.

Note: $B_r^{\mathbb{X}}(0) \subseteq B_{\delta}^{\mathbb{X}}(x)$ for some $\delta > 0$ (large enough, e.g. $3r$)

Hence, $B_{r_y}^Y(y) \subseteq T(B_{\delta}^{\mathbb{X}}(x))$. (*)

By scaling, translation and linearity, we get $\forall r' > 0$:

$$\begin{aligned} T(B_{r'}^{\mathbb{X}}(x)) &= T(B_{r'}^{\mathbb{X}}(0) + x) = T\left(\frac{r'}{\delta} B_{\delta}^{\mathbb{X}}(0) + x\right) \\ &= \frac{r'}{\delta} T(B_{\delta}^{\mathbb{X}}(0)) + Tx = \frac{r'}{\delta} T(B_{\delta}^{\mathbb{X}}(x) - x) + Tx \end{aligned}$$

$$(*) \geq \frac{r'}{\delta} (B_{r_y}^Y(y) - \underbrace{Tx}_{=y}) + Tx = \frac{r'}{\delta} B_{r_y}^Y(y) + Tx$$

$$= B_{\frac{r'r_y}{\delta}}^Y(y) + Tx = B_{\frac{r'r_y}{\delta}}^Y(y). \quad (**)$$

Now, let $A \subseteq \mathbb{X}$ be open, let $y' \in T(A)$ be arbitrary, and choose $x' \in A$ s.t. $y' = Tx'$. Since A is open, $\exists r' > 0$:

$B_{r'}^{\mathbb{X}}(x') \subseteq A$, hence

$$T(A) \supseteq T(B_{r'}^{\mathbb{X}}(x')) = T(B_{r'}^{\mathbb{X}}(x) + x' - x) \stackrel{(**)}{\supseteq} B_{\frac{r'r_y}{\delta}}^Y(y) + y' - y = B_{\frac{r'r_y}{\delta}}^Y(y').$$

i.e. $T(A)$ is open, so Claim 1 holds. \checkmark

From now on: center of all balls is 0 (unless otherwise noted) and we drop the center from the notation

Claim 2: $\exists \varepsilon > 0$:

$$B_{\varepsilon}^Y \subseteq \overline{T(B_1^{\mathbb{X}})} \quad (***)$$

Pf: $Y = \overset{\text{onto}}{T}(X) = T\left(\bigcup_{n \in \mathbb{N}} B_n^X\right) = \bigcup_{n \in \mathbb{N}} T(B_n^X).$ (90)

Now, Y is complete, so we can apply Baire's Thm (in form of Cor. 1.49): Then exists $n \in \mathbb{N}$: $T(B_n^X)$ is not nowhere dense, i.e., $\exists y \in \overline{T(B_n^X)}$ and $\varepsilon > 0$: $B_\varepsilon^Y(y) \subseteq \overline{T(B_n^X)}$, or, equivalently, $B_\varepsilon^Y \subseteq \overline{T(B_n^X)} - y$.

We have:

(i) $\exists (x_k)_{k \in \mathbb{N}} \subseteq B_n^X$ s.t. $y = \lim_{k \rightarrow \infty} Tx_k$

(ii) $\forall k \in \mathbb{N}$: $\overline{T(B_n^X)} - Tx_k = \overline{T(B_n^X - x_k)} \subseteq \overline{T(B_{2n}^X)}$

hence $B_\varepsilon^Y \subseteq \overline{T(B_{2n}^X)}$, so $B_{\frac{\varepsilon}{2n}}^Y = \frac{1}{2n} B_\varepsilon^Y \subseteq \frac{1}{2n} \overline{T(B_{2n}^X)} = \overline{T\left(\frac{1}{2n} B_{2n}^X\right)} = \overline{T(B_1^X)}$. ✓

Claim 3: $\overline{T(B_1^X)} \subseteq T(B_2^X)$

Pf: Let ε be as in Claim 2. Let $y \in \overline{T(B_1^X)}$.

Then there exists $x_1 \in B_1^X$ such that

$$y - Tx_1 \in B_{\varepsilon/2}^Y \stackrel{(***)}{\subseteq} \overline{T(B_{1/2}^X)}.$$

Similarly, there exists $x_2 \in B_{1/2}^X$ such that

$$y - Tx_1 - Tx_2 = (y - Tx_1) - Tx_2 \in B_{\varepsilon/4}^Y \stackrel{(***)}{\subseteq} \overline{T(B_{1/4}^X)}$$

Inductively we get: $\forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{2^{n-1}}}^X$ s.t.

$$y - \sum_{j=1}^n Tx_j \in B_{\varepsilon/2^n}^Y \quad (***)$$

Since $\sum_{n \in \mathbb{N}} \underbrace{\|x_n\|}_{< 2^{-(n-1)}} < 2 < \infty$, and X is a Banach space,

$\sum_{j \in \mathbb{N}} x_j := x \in X$ exists by Lemma 2.50(a). As T is cont., we have $Tx = \sum_{j \in \mathbb{N}} Tx_j$. Using (***) and the continuity of $\|\cdot\|$,

$$\|y - Tx\| = \lim_{n \rightarrow \infty} \left\| y - \sum_{j=1}^n Tx_j \right\| = 0, \text{ hence } y = Tx, \|x\| < 2$$

i.e. $y \in T(B_2^X)$ ■

Corollary 4.10 | Inverse mapping theorem

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Let X, Y be Banach spaces, and $T \in BL(X, Y)$ a bijection. Then $T^{-1} \in BL(Y, X)$

Pf: Clearly, T^{-1} exists and is linear (recall 2.25).

By Thm. 4.9, T is open, that is, T^{-1} is continuous.

Definition 4.11 | Let X, Y be normed spaces, and

$T: X \supseteq \text{dom}(T) \rightarrow Y$ a linear operator.

(a) Graph of T :

$$G(T) := \{(x, Tx) \in X \times Y \mid x \in \text{dom}(T)\}$$

We equip $X \times Y$ with the norm

$$\|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y$$

Then: X, Y complete $\Rightarrow X \times Y$ complete.

(b) T is closed operator $\Leftrightarrow G(T) \subseteq X \times Y$ is closed

(in $X \times Y, \|\cdot\|_{X \times Y}$)

Remark 4.12

(a) T is closed if and only if the following implication

holds: $(x_n)_n \subseteq \text{dom}(T)$ with: $x_n \xrightarrow{n \rightarrow \infty} x \in X$ & $Tx_n \xrightarrow{n \rightarrow \infty} y \in Y$

$$\Rightarrow x \in \text{dom}(T) \text{ and } y = Tx$$

(b) Compare (a) with the definition of T (seq.) cont., where convergence of $(Tx_n)_n$ must be proved.

Here, it is given / assumed!

Theorem 4.13 (Closed graph theorem)

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Let X, Y be Banach spaces and $T: X \supseteq \text{dom } T \rightarrow Y$ a closed linear operator. Then

$$\text{dom}(T) \text{ closed} \Leftrightarrow T \text{ bounded.}$$

Pf: " \Leftarrow ": Let $(x_n)_n \in \text{dom } T$ with $x_n \xrightarrow{n \rightarrow \infty} x \in X$. Then $(x_n)_n$ is Cauchy in X . By hypothesis, T is bounded, so $(Tx_n)_n$ is Cauchy in Y . But Y is complete, so $\exists y \in Y: Tx_n \xrightarrow{n \rightarrow \infty} y$ and, since T is closed, it follows that $x = \lim_{n \rightarrow \infty} x_n \in \text{dom}(T)$ (and $Tx = y$).

" \Rightarrow ": Define the projection $P_1: G(T) \rightarrow \text{dom}(T)$
 $(x, Tx) \mapsto x$

(i) $G(T)$ is closed in $X \times Y$, hence $G(T)$ is a Banach space.

(ii) $\text{dom}(T)$ is closed in X (by hypothesis), hence $\text{dom}(T)$ is a Banach space.

(iii) P_1 is a bijection

(iv) P_1 is bounded: Let $z = (x, Tx) \in G(T)$, then

$$\|P_1 z\|_X = \|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|z\|_{X \times Y}.$$

$$\text{So } \|P_1\| \leq 1$$

By the inverse mapping theorem $P_1^{-1}: \text{dom}(T) \rightarrow G(T)$
 $x \mapsto (x, Tx)$
is also bounded, hence:

$$\exists c < \infty: \|P_1^{-1}x\|_{X \times Y} \leq c\|x\|_X. \text{ Since } \|P_1^{-1}x\|_{X \times Y} = \|x\|_X + \|Tx\|_Y,$$

$$\text{this implies } \|Tx\|_Y \leq (c+1)\|x\|_X$$

So T is bounded



Example 4.14 | Let $X = Y = C_0(\mathbb{R})$ with supremum norm (Banach!) (93)

Consider $T := \frac{d}{dx} : \text{dom}(T) \rightarrow C_0(\mathbb{R})$
 $f \mapsto f'$

with $\text{dom}(T) = \{f \in C_0(\mathbb{R}) \mid f \in C^1(\mathbb{R}) \text{ and } f' \in C_0(\mathbb{R})\} \subseteq C_0(\mathbb{R})$.

Claim: T is closed (a closed operator).

Pf: Let $(f_n)_{n \in \mathbb{N}} \subseteq \text{dom}(T)$ be a sequence such that:

$$(i) \exists g \in C_0(\mathbb{R}) : \|f_n - g\|_\infty \xrightarrow{n \rightarrow \infty} 0$$

$$(ii) \exists h \in C_0(\mathbb{R}) : \|f'_n - h\|_\infty \xrightarrow{n \rightarrow \infty} 0$$

i.e., $(f_n, f'_n)_n \in G(T)$, $(f_n, f'_n)_n \xrightarrow{n \rightarrow \infty} (g, h)$ in $X \times Y$; we need to prove that $(g, h) \in G(T)$, equivalently, $g \in \text{dom}(T)$ and $h = g'$.

By uniform convergence, we can exchange limits, so, $\forall x \in \mathbb{R}$,

$$\int_0^x h(t) dt = \int_0^x \lim_{n \rightarrow \infty} f'_n(t) dt = \lim_{n \rightarrow \infty} \underbrace{\int_0^x f'_n(t) dt}_{f_n(x) - f_n(0)} = g(x) - g(0)$$

(since $f_n \rightarrow g$ pointwise, by (i)).

$$\text{Hence, } g(x) = g(0) + \int_0^x h(t) dt \quad \forall x \in \mathbb{R}$$

So, by the Fundamental Theorem of Calculus (HDI): $g \in C^1(\mathbb{R})$ with $g' = h \in C_0(\mathbb{R})$. Since $g \in C_0(\mathbb{R})$, we have $g \in \text{dom}(T)$, and so $(g, h) = (g, g') \in G(T)$. So T is closed.

Also: Since T is unbounded (cf. 2.28) $\xRightarrow{\text{Thm. 4.13}}$ $\text{dom}(T)$ is not a closed subspace of $C_0(\mathbb{R})$ w.r. $\|\cdot\|_\infty$.

4.3 (Bi-) Dual spaces and weak topologies

Theorem 4.15 | Let X be a normed space. Then, for every $x \in X$,

$$\|x\| = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|_*}$$

Pf: See exercise.

(Definition 4.16) Let X be a normed space. We call

(94)

$X^{**} := (X^*)^*$ the bidual space of X (always Banach!)

More generally, introduce the n -fold dual space

$X^{\overbrace{* \dots *}^n} := (X^{\overbrace{* \dots *}^{n-1}})^*$ for $n \in \mathbb{N}$, recursively.

(Theorem (& definition) 4.17) Let X be a normed space.

The canonical embedding $J: X \rightarrow X^{**}$ with $Jx: X^* \rightarrow \mathbb{K}$
 $x \mapsto Jx$ $f \mapsto f(x)$

is well-defined, linear, and isometric.

If J is surjective, we call X reflexive.

(Remarks 4.18) (a) Hilbert spaces are reflexive (Riesz!)

(b) Every finite dimensional normed space is reflexive (use the dual basis)

(c) ℓ^p, L^p are reflexive for $p \in (1, \infty)$; c_0 & ℓ^1 are not refl.

(d) Milman-Pettis Thm: X uniformly convex & Banach $\Rightarrow X$ reflexive

(e) X reflexive $\Rightarrow X$ complete.

Pf (4.17): J is well-defined (i.e. $Jx \in X^{**} \forall x \in X$):

Let $\alpha, \beta \in \mathbb{K}$, $f, g \in X^*$, then

$$(Jx)(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha (Jx)(f) + \beta (Jx)(g)$$

So, Jx is linear. Also, Jx is bounded, since

$$\|Jx\|_{**} = \sup_{0 \neq f \in X^*} \frac{|(Jx)(f)|}{\|f\|_*} = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|_*} \stackrel{\text{Thm. 4.15}}{=} \|x\| < \infty$$

Hence, $Jx \in X^{**}$, so J is well-defined - and, an isometry.

Also, J is linear: Let $\alpha, \beta \in \mathbb{K}$, $x, y \in X$, then, for all $f \in X^*$,

$$(J(\alpha x + \beta y))(f) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha (Jx)(f) + \beta (Jy)(f),$$

$$\text{and so } J(\alpha x + \beta y) = \alpha J(x) + \beta J(y) \quad \square$$

Theorem 4.19 | Let X be a Banach space. Then

$$\underline{X \text{ reflexive}} \Leftrightarrow \underline{X^* \text{ reflexive}}$$

Pf. " \Rightarrow ": $(X^*)^{**} = ((X^*)^*)^* = (X^{**})^* = X^*$

" \Leftarrow ": See f.ex. Werner, "Funktionalanalysis", 8. Aufl., Kap. III 3.4.

Theorem 4.20 | Let X be a normed space. Then

$$X^* \text{ separable} \Rightarrow X \text{ separable.}$$

Remark 4.21 | $X = \ell^1$ shows that " \Leftarrow " does not hold, since $\ell^\infty \simeq (\ell^1)^*$, which is not separable.

Pf (4.20): Let $A := \{f_n \in X^* \mid n \in \mathbb{N}\}$ be dense (exists by hyp.).

For $n \in \mathbb{N}$ choose $x_n \in X$ s.t. $\|x_n\| = 1$ & $|f_n(x_n)| \geq \frac{1}{2} \|f_n\|_*$.

Let $D := \text{span}_{\mathbb{K}} \{x_n \mid n \in \mathbb{N}\} \subseteq X$.

Claim: D is dense in X (note: D is not countable, but a \mathbb{K} -linear subspace).

Pf (claim): Assume (for contradiction) D not dense.

Then there exists $z \in X$ s.t. $\text{dist}(z, D) > 0$. By Cor. 4.7 (H-B!) $\exists f \in X^* : f|_D = 0, f(z) > 0$. Now:

$\exists f \in X^* : f|_D = 0, f(z) > 0$. Now:

(i) A dense in X^* implies: $\exists \text{ seq. } (f_{n_k})_k \subseteq A :$

$$\|f_{n_k} - f\|_* \xrightarrow{k \rightarrow \infty} 0 \quad (\text{A})$$

(ii) We have

$$\|f_{n_k} - f\|_* \geq |f_{n_k}(x_{n_k}) - f(x_{n_k})| \stackrel{f|_D=0}{=} |f_{n_k}(x_{n_k})| \geq \frac{1}{2} \|f_{n_k}\|_*$$

Then $\|f_{n_k}\|_* \xrightarrow{k \rightarrow \infty} 0$, so $f_{n_k} \xrightarrow{k \rightarrow \infty} 0$ in X^* , hence $f = 0$ by (A) ∇ .

So, D is dense.

Define $\tilde{\mathbb{K}} := \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$, resp. $\tilde{\mathbb{K}} := \mathbb{Q} + i\mathbb{Q}$ if $\mathbb{K} = \mathbb{C}$, and set

$\tilde{D} := \text{span}_{\mathbb{K}} \{x_n \mid n \in \mathbb{N}\}$. Then \tilde{D} is countable & dense in X (96)

Definition 4.22 Let X be a \mathbb{K} -vector space, let $\{p_\alpha\}_{\alpha \in I}$ be a family of seminorms on X

(a) $\{p_\alpha\}_{\alpha \in I}$ is separating : $\Leftrightarrow \forall 0 \neq x \in X : \exists \alpha \in I : p_\alpha(x) > 0$

(b) For given $\alpha \in I, r > 0$, let

$$U_{\alpha, r} := \{y \in X \mid p_\alpha(y) < r\} \ni 0.$$

Also, for $x \in X$, let

$$U_{\alpha, r}(x) := x + U_{\alpha, r} = \{y \in X \mid p_\alpha(y - x) < r\} \ni x.$$

Let

$$\mathcal{N}_x := \left\{ \bigcap_{j=1}^n U_{\alpha_j, r_j}(x) \mid n \in \mathbb{N}, \alpha_j \in I, r_j > 0 \text{ for } j = 1, \dots, n \right\}$$

(family of finite intersections of $U_{\alpha, r}(x)$'s)

Define the Locally convex topology (l.c.t.) on X induced by $\{p_\alpha\}_{\alpha \in I}$:

$$\begin{aligned} Y \subseteq X \text{ open in the l.c.t.} &\Leftrightarrow \forall x \in Y \exists V_x \in \mathcal{N}_x : V_x \subseteq Y \\ &\Leftrightarrow \forall x \in Y \exists V_x \in \mathcal{N}_x : Y = \bigcup_{x \in Y} V_x \end{aligned}$$

Remark 4.23 (a) $U_{\alpha, r}(x)$ is open in l.c.t., and \mathcal{N}_x is a neighbourhood base of the l.c.t. at x .

(b) If $\{p_\alpha\}_{\alpha \in I}$ is separating, then the l.c.t. is Hausdorff (see exercise).

(c) The elements of the neighbourhood base are convex sets (hence the name): For $y_1, y_2 \in U_{\alpha, r}$ and $\lambda \in [0, 1]$, we have $\lambda y_1 + (1 - \lambda) y_2 \in U_{\alpha, r}$, since:

$$\begin{aligned} p_\alpha(\lambda y_1 + (1 - \lambda) y_2) &\leq p_\alpha(\lambda y_1) + p_\alpha((1 - \lambda) y_2) \\ &= \underbrace{\lambda p_\alpha(y_1)}_{< r} + \underbrace{(1 - \lambda) p_\alpha(y_2)}_{< r} < r. \end{aligned}$$

(d) \mathcal{X} is a topological vector space wrt. a locally convex topology, i.e., addition of vectors and mult. of a vector by a scalar are continuous wrt. a l.c.t. (97)

This relies on $U_{\alpha, r}(x) = x \in U_{\alpha, r} \ \& \ \lambda U_{\alpha, r}(x) = U_{\alpha, |\lambda| r}(\lambda x)$
 $\forall x \in \mathcal{X}, \lambda \in \mathbb{K}, \alpha \in I, r > 0.$

Notation: vector space topology : \Leftrightarrow a topology making \mathcal{X} into a topological vector space.

The composition of a linear map with a norm gives a seminorm (check!). The following abstract result will be applied several times with different choices of S_α and \mathcal{Z} :

[Lemma 4.24] Let \mathcal{X} be a vector space, $(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ a normed space, and $S_\alpha: \mathcal{X} \rightarrow \mathcal{Z}$ a linear map for every index $\alpha \in I$. Let \mathcal{T}_{loc} be the locally convex topology on \mathcal{X} induced by the family of seminorms $\{x \mapsto \|S_\alpha x\|_{\mathcal{Z}}\}_{\alpha \in I}$. Then:

(a) \mathcal{T}_{loc} is the coarsest vector space topology on \mathcal{X} s.t.

For every $\alpha \in I$ the map S_α is continuous.

(b) $(x_k)_{k \in \mathbb{N}} \subseteq \mathcal{X}$ converges to $x \in \mathcal{X}$ wrt. \mathcal{T}_{loc} \Leftrightarrow

$$\|S_\alpha x_k - S_\alpha x\|_{\mathcal{Z}} \xrightarrow{k \rightarrow \infty} 0 \quad \forall \alpha \in I$$

Pf: See exercise.

A very important example is when $S_\alpha \in \mathcal{X}^*$ and $\mathcal{Z} = \mathbb{K}$:

[Definition 4.25] Let \mathcal{X} be a normed space. The weak topology on \mathcal{X} is the locally convex topology (l.c.t.-) on \mathcal{X} induced by the family of seminorms $\{x \mapsto |f(x)|\}_{f \in \mathcal{X}^*}$.

Lemma 4.26 Let X be a normed space. Then the weak topology on X is

- (a) Hausdorff.
- (b) the coarsest vector-space topology on X such that every $f \in X^*$ is continuous. In particular, the weak topology is coarser than the norm (= strong) topology.
- (c) identical to the norm topology, if $\dim X < \infty$,
- (d) such that $(x_n)_{n \in \mathbb{N}} \subseteq X$ converges to x in the weak topology (or, weakly) iff

$$\lim_{n \rightarrow \infty} f(x_n) = f(x) \quad \forall f \in X^*$$

Notation: $x_n \xrightarrow{w} x$, or $x_n \xrightarrow{w \rightarrow \infty} x$.

Pf: (a) By Remark 4.23(b), it suffices to check that the family of seminorms is separating. This follows from Corollary 4.6 (H-B!): For every $0 \neq x \in X \exists f_x \in X^*: f_x(x) = \|x\|$.

(b) Lemma 4.24(a) (c) See exercise (d) Lemma 4.24(b).

Remark 4.27 (a) Weak limits are unique by Lemma 4.26(a)

(b) If $\dim X = \infty$, then the weak topology is not 1^{st} countable (hence not metrisable!) (for pf., see exercise)

(c) Strong convergence $\xrightarrow{4.26(b)} \Rightarrow$ weak convergence (but in general not vice versa!)

(d) In a Hilbert space X (by Lem. 4.26(d) & Riesz):

$$x_n \xrightarrow{w} x \iff \langle y, x_n \rangle \xrightarrow{n \rightarrow \infty} \langle y, x \rangle \quad \forall y \in X$$

(e) In ℓ^2 : $x_n \xrightarrow{w} x \iff \|x_n - x\|_2 \xrightarrow{n \rightarrow \infty} 0$

(Schr, 1921; See also Conway, "A course in FA", 1990, Prop. 5.2)

Example 4.28 Let $\mathcal{X} = \ell^p$, $p \in [1, \infty]$, $e_n := (\delta_{nk})_{k \in \mathbb{N}}$ for $n \in \mathbb{N}$. (99)

Then (i) $(e_n)_n$ has no $\|\cdot\|_p$ -convergent (i.e., strongly conv.) subsequence.

(ii) If $p = 1$, then $(e_n)_n$ is not weakly convergent:

Recall $(\ell^1)^* \cong \ell^\infty$, and choose $f \leftrightarrow (-1, 1, -1, 1, \dots) \in \ell^\infty$ then $f(e_n) = (-1)^n \forall n \in \mathbb{N}$ which is not convergent as $n \rightarrow \infty$ (This is consistent with Rmk 4.27(e)).

(iii) If $1 < p \leq \infty$, then $(e_n)_n$ is weakly convergent to 0:

In case $1 < p < \infty$, we have $(\ell^p)^* \cong \ell^q$ (Thm. 2.38) with $1 < q < \infty$. Hence, for $f \leftrightarrow \gamma = (\gamma_k)_k \in \ell^q$:

$$f(e_n) = \gamma_n \xrightarrow{n \rightarrow \infty} 0 \quad \left(\text{since } \sum_{k \in \mathbb{N}} |\gamma_k|^q < \infty \right).$$

In the case $p = \infty$, use also exercises Eb.1 & Eb.3.

Lemma 4.29 Let \mathcal{X} be a normed space, $(x_n)_{n \in \mathbb{N}} \subseteq \mathcal{X}$, $x \in \mathcal{X}$, and

$$x_n \xrightarrow{w} x. \text{ Then}$$

$$(a) \sup_{n \in \mathbb{N}} \|x_n\| < \infty, \quad (b) \|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$$

Pf (a): Let $x_n \xrightarrow{w} x$, so $f(x_n) \rightarrow f(x) \forall f \in \mathcal{X}^*$, hence

$$\forall f \in \mathcal{X}^* \text{ (fixed)} : \sup_{n \in \mathbb{N}} \underbrace{|f(x_n)|}_{(Jx_n)(f)} < \infty,$$

with the canonical embedding $J: \mathcal{X} \rightarrow \mathcal{X}^{**}$. Since \mathcal{X}^* is Banach, the Uniform Boundedness Principle (Thm. 4.9) gives

$$\sup_{n \in \mathbb{N}} \|Jx_n\|_{**} < \infty. \quad \checkmark$$

$$= \|x_n\|$$

(b) By Cor. 4.6 (H-B!):

$$\exists f_x \in \mathcal{X}^* : \|f_x\|_* = 1 \text{ and } f_x(x) = \|x\|$$

Then

$$\|x\| = |f_x(x)| = \lim_{n \rightarrow \infty} |f_x(x_n)| = \liminf_{n \rightarrow \infty} |f_x(x_n)| \leq \liminf_{n \rightarrow \infty} \underbrace{\|f_x\|_*}_{=1} \|x_n\|$$

(100)

Theorem 4.30 | Let X be a normed space. Then $x_n \overset{w}{\rightarrow} x$ iff the following two statements hold:

(i) $\sup_{n \in \mathbb{N}} \|x_n\| < \infty$

(ii) $\exists F \subseteq X^*$ with $\text{span}(F)$ dense (wrt. $\|\cdot\|_*$) in X^* s.t.

$\forall f \in F: \lim_{n \rightarrow \infty} f(x_n) = f(x)$. (*)

Pf: " \Rightarrow ": From Lem. 4.29 and by definition of weak convergence

" \Leftarrow ": Use an $\frac{\varepsilon}{3}$ -argument: Let $\varepsilon > 0$ and $g \in X^*$. Let

$K := \frac{1}{2} (\|x\| + \sup_{n \in \mathbb{N}} \|x_n\|) < \infty$. Since $\text{span}(F)$ is dense in X^* ,

there exists $f \in \text{span}(F)$: $\|f - g\|_* < \frac{\varepsilon}{3K}$. Also, for

$f \in \text{span}(F)$, $\exists N \in \mathbb{N}$: $\forall n \geq N$: $|f(x_n) - f(x)| < \frac{\varepsilon}{3}$

(Note: (*) holds also for $f \in \text{span}(F)$ as finite lin. comb.)

Hence, $\forall n \geq N$:

$$|g(x) - g(x_n)| \leq |g(x) - f(x)| + |f(x) - f(x_n)| + |f(x_n) - g(x_n)|$$

$$\leq \underbrace{\|g - f\|_*}_{< \frac{\varepsilon}{3K}} \underbrace{(\|x\| + \|x_n\|)}_{\leq 2K} + \underbrace{|f(x) - f(x_n)|}_{< \frac{\varepsilon}{3}} < \varepsilon$$

□

Theorem 4.31 | (Eberlein - Šmulian) Let X be a Banach space and $A \subseteq X$. Then A weakly compact \Leftrightarrow A weakly sequentially compact.

Pf. Not here; see Whitley, Math. Ann. 172, 116 - 118 (1967). □

Definition 4.32 | Let X be a normed space. Then the weak* topology on X^* ("weak-star") is the locally convex topology on X^* induced by the family of seminorms $\{f \mapsto |f(x)|\}_{x \in X}$

Lemma 4.33 | Let X be a normed space. Then the weak* topology (101)

is (a) Hausdorff

(b) The coarsest vector-space topology on X^* s.t. $\forall x \in X$, the map

$$\begin{aligned} X^* &\rightarrow \mathbb{K} \\ f &\mapsto f(x) \end{aligned} \quad \text{is continuous}$$

(c) coarser than the weak topology on X^* , and the two coincide iff X is reflexive.

(d) such that $(f_n)_n \subseteq X^*$ converges to $f \in X^*$ in the weak* - top.

$$\text{iff.} \quad f_n(x) \xrightarrow{n \rightarrow \infty} f(x) \quad \forall x \in X$$

Notation: $f_n \xrightarrow{w^*} f$.

Pf: (a) $\{f \mapsto |f(x)|\}_{x \in X}$ is a separating family of seminorms:

For $f \neq 0$ there exists $x \in X$ with $f(x) \neq 0$.

(b) Lemma 4.24(a) with $Z = \mathbb{K}$ and $S_x = Jx$ ($J: X \rightarrow X^{**}$ canonical embedding)

(c) $J(X) \subseteq X^{**}$ with equality iff X is reflexive.

(d) Lemma 4.24(b).

The next theorem is the analogue of Lem. 4.29 & Thm. 4.30.

However: Note that here, X must be complete in order to apply the Uniform Boundedness Principle in (a)(i).

Theorem 4.34 | Let X be a normed space, $f \in X^*$, $(f_n)_n \subseteq X^*$

(a) If $f_n \xrightarrow{w^*} f$, then

(i) If X is even a Banach space, then $\sup_{n \in \mathbb{N}} \|f_n\|_* < \infty$

$$(ii) \|f\|_* \leq \liminf_{n \rightarrow \infty} \|f_n\|_*$$

(b) If (i) $\sup_{n \in \mathbb{N}} \|f_n\|_* < \infty$

(ii) $\exists A \subseteq X$ with $\text{span}(A)$ dense in X (w.d. $\|\cdot\|$) s.t.

$$\forall x \in A: f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$$

then $f_n \xrightarrow{w^*} f$. (If X Banach, the reverse holds by (a)(i))

Theorem 4.35 (Banach-Alaoglu) Let X be a normed space. Then the closed unit ball in X^* , $\bar{B}_1^* := \{f \in X^* \mid \|f\|_* \leq 1\}$ is compact in the weak* topology.

Pf. Equip the set of maps $\{f: X \rightarrow \mathbb{K}\} = \mathbb{K}^X$ with the product topology $\mathcal{T}_{\text{prod}} := \prod_{x \in X} \mathcal{T}_{\mathbb{K}}$ of the Euclidean topology $\mathcal{T}_{\mathbb{K}}$ on \mathbb{K} . Then:

(i) $\mathcal{T}_{\text{prod}}$ is the coarsest topology on \mathbb{K}^X such that the projection/evaluation map $\pi_x: \mathbb{K}^X \rightarrow \mathbb{K}$ is continuous for all $x \in X$
 $f \mapsto f(x)$
 (see T2.3, Tut Sheet 2)

(ii) $\mathcal{T}_{\text{prod}}$ is a vector-space topology on \mathbb{K}^X :

Addition: $\mathcal{A}: \mathbb{K}^X \times \mathbb{K}^X \rightarrow \mathbb{K}^X$. It suffices to prove openness of $\mathcal{A}^{-1}(U)$
 $(f, g) \mapsto f+g$

in $\mathbb{K}^X \times \mathbb{K}^X$ for U in a base of $\mathcal{T}_{\text{prod}}$, that is, for $U = \bigcap_{x \in U} U_x$ with $U_x \in \mathcal{T}_{\mathbb{K}}$, and $U_x = \mathbb{K}$ for all but at most finitely many $x \in X$. Addition in the field \mathbb{K} , $\mathcal{A}_{\mathbb{K}}: \mathbb{K} \times \mathbb{K} \rightarrow \mathbb{K}$, is continuous
 $(z, z') \mapsto z+z'$

Hence, $\mathcal{A}^{-1}(U) = \bigcap_{x \in X} \mathcal{A}_{\mathbb{K}}^{-1}(U_x)$ with

$$\mathcal{A}_{\mathbb{K}}^{-1}(U_x) \subseteq \mathcal{T}_{\mathbb{K}} \times \mathcal{T}_{\mathbb{K}} \quad \forall x \in X$$

and

$$\mathcal{A}_{\mathbb{K}}^{-1}(U_x) = \mathbb{K} \times \mathbb{K} \text{ except for at most finitely many } x \in X.$$

Hence, $\mathcal{A}^{-1}(U) \in \mathcal{T}_{\text{prod}} \times \mathcal{T}_{\text{prod}}$, as finite intersection of a union of product sets $V_1 \times V_2$ with each $V_j \in \mathcal{T}_{\text{prod}}$, $j=1,2$, being a product set of factors all of which are equal to \mathbb{K} except for one. \checkmark The case of scalar mult. is analogous & simpler.

(iii) Given a linear subspace $S \subseteq \mathbb{K}^X$, the subspace topology $\mathcal{T}_{\text{prod}}|_S := \{U \cap S \mid U \in \mathcal{T}_{\text{prod}}\}$ is a vector space topology on S w.r. which the restricted eval. maps $\pi_x|_S$ are continuous $\forall x \in X$

Now, choose $S = \Sigma^*$, and let \mathcal{T}_{w*} denote the weak* top. (103)
 on Σ^* . By 4.33(b), $\mathcal{T}_{w*} \leq \mathcal{T}_{\text{prod}}|_{\Sigma^*}$. Therefore, the map

$$\begin{aligned} \sigma: (\Sigma^*, \mathcal{T}_{\text{prod}}|_{\Sigma^*}) &\rightarrow (\Sigma^*, \mathcal{T}_{w*}) \\ f &\mapsto f \end{aligned}$$

is continuous, and the theorem follows if we prove: \bar{B}_1^* is

$\mathcal{T}_{\text{prod}}$ -compact: We claim: Then it is $\mathcal{T}_{\text{prod}}|_{\Sigma^*}$ -compact
 (and hence, by continuity of σ , \mathcal{T}_{w*} -compact).

Pf Claim: Consider a $\mathcal{T}_{\text{prod}}|_{\Sigma^*}$ -open cover $\bigcup_{\alpha} V_{\alpha}$ of \bar{B}_1^* and
 use that, for $V_{\alpha} \in \mathcal{T}_{\text{prod}}|_{\Sigma^*}$, there exists $U_{\alpha} \in \mathcal{T}_{\text{prod}}$ with
 $V_{\alpha} = U_{\alpha} \cap \Sigma^*$. The $\mathcal{T}_{\text{prod}}$ -open cover $\bigcup_{\alpha} U_{\alpha}$ has a finite subcover
 (by assumption), and intersection with Σ^* gives finite sub-
 cover of $\bigcup_{\alpha} V_{\alpha}$. \checkmark (claim).

To prove $\mathcal{T}_{\text{prod}}$ -compactness: Define $A := \bigcap_{x \in \Sigma} A_x$, where

$A_x := \{z \in \mathbb{K} \mid |z| \leq \|x\|\}$ is compact in \mathbb{K} . By Tychonoff (Thm. 1.28),

A is $\mathcal{T}_{\text{prod}}$ -compact. Note that: $f \in A \Leftrightarrow f(x) \in A_x \forall x \in \Sigma$

$\Rightarrow \sup_{0 \neq x \in \Sigma} \frac{|f(x)|}{\|x\|} \leq 1$. For $x, y \in \Sigma, \alpha, \beta \in \mathbb{K}$, let

$$\begin{aligned} L_{x,y,\alpha,\beta} &:= \left\{ f \in \mathbb{K}^{\Sigma} \mid \underbrace{f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)}_{= (\pi_{\alpha x + \beta y} - \alpha \pi_x - \beta \pi_y)(f)} = 0 \right\} \\ &= (\pi_{\alpha x + \beta y} - \alpha \pi_x - \beta \pi_y)^{-1}(\{0\}). \end{aligned}$$

But $\pi_{\alpha x + \beta y} - \alpha \pi_x - \beta \pi_y$ is $\mathcal{T}_{\text{prod}} - \mathcal{T}_{\mathbb{K}}$ -continuous, and $\{0\} \subseteq \mathbb{K}$
 is $\mathcal{T}_{\mathbb{K}}$ -closed, hence $L_{x,y,\alpha,\beta}$ is $\mathcal{T}_{\text{prod}}$ -closed in \mathbb{K}^{Σ} , and

so also $L := \bigcap_{\substack{x,y \in \Sigma \\ \alpha,\beta \in \mathbb{K}}} L_{x,y,\alpha,\beta} = \{f: \Sigma \rightarrow \mathbb{K} \text{ is linear}\}$
 is closed.

Finally, $\bar{B}_1^* = A \cap L$ is $\mathcal{T}_{\text{prod}}$ -compact (as closed subset of
 the compact A) \blacksquare

Theorem 4.36 (Helly; version 2 of Banach-Alaoglu)

(104)

Let X be a separable normed space. Then \bar{B}_1^* is weak* sequentially compact.

Pf: Let $\{x_k | k \in \mathbb{N}\} \subseteq X$ be (countable) dense in X . We need to prove: Any sequence $(f_n)_n \subseteq \bar{B}_1^*$ has a weak*-convergent subsequence: Fix $k \in \mathbb{N}$ arbitrary. Consider the sequence $(f_n(x_k))_{n \in \mathbb{N}} \subseteq \mathbb{K}$. This sequence is bounded, since

$$|f_n(x_k)| \leq \underbrace{\|f_n\|_*}_{\leq 1} \|x_k\| \leq \|x_k\| \quad (\text{A } k \text{ fixed})$$

Hence, by Bolzano-Weierstrass, $(f_n(x_k))_n$ has a convergent subsequence (in \mathbb{K}).

Claim: There exist a common subsequence $(n_j)_j \subseteq \mathbb{N}$:
 $\forall k \in \mathbb{N} : (f_{n_j}(x_k))_{j \in \mathbb{N}}$ is convergent.

Pf: Use Cantor's diagonal sequence trick:

There exists $(n_j^{(1)})_j \subseteq \mathbb{N}$ such that $(f_{n_j^{(1)}}(x_1))_j$ converges.

Then there exists $(n_j^{(2)})_j \subseteq (n_j^{(1)})_j$ s.t. $(f_{n_j^{(2)}}(x_2))_j$ converges.

Continuing this procedure, there exists $(n_j^{(k+1)})_j \subseteq (n_j^{(k)})_j$ such that $(f_{n_j^{(k+1)}}(x_{k+1}))_{j \in \mathbb{N}}$ converges. The claim then holds with $n_j := n_j^{(j)}$.

Now, define $g(x) := \lim_{j \rightarrow \infty} f_{n_j}(x) \quad \forall x \in \text{span}\{x_k | k \in \mathbb{N}\} =: \text{dom}(g)$.

(i) $\text{dom}(g)$ is a dense subspace of X with respect to $\|\cdot\|$

(ii) $g: \text{dom}(g) \rightarrow \mathbb{K}$ is linear

(iii) g is bounded: $|g(x)| = \lim_{j \rightarrow \infty} \underbrace{|f_{n_j}(x)|}_{\leq \|f_{n_j}\|_* \|x\|} \leq \|x\|$

Since \mathbb{K} is complete, we can apply the Bounded Linear Ext. Thm. 2-31:

There exists $\tilde{g}: X \rightarrow \mathbb{K}$ such that $\tilde{g}|_{\text{dom}(g)} = g$ and $\|\tilde{g}\|_X = \|g\|_X \leq 1$

Hence, $\tilde{g} \in B_1^*$. Then, by Thm. 4.34(b), $f_{n_j} \xrightarrow{w^*} \tilde{g}$ as $j \rightarrow \infty$ (b5)

Theorem 4.37 (Kakutani, version 3 of Banach-Alaoglu)

Let X be a Banach space. Then

X reflexive $\Leftrightarrow \overline{B}_X := \{x \in X \mid \|x\| \leq 1\}$ is weakly compact

Pf: " \Rightarrow ": By Thm 4.19, $X^* =: Y$ is reflexive. Applying 4.33(c) to Y , we get: The weak topology on $Y^* = (X^*)^* = X$ coincides with the weak* topology on $Y^* = X$, hence the claim follows from Thm. 4.35 applied to Y . \square

" \Leftarrow ": See f.ex. Werner, "Funktionalanalysis", 8. Aufl, Satz VIII.3.18. \square

Example 4.38 Compactness of \overline{B}_1 in different spaces.

Here, $p, q \in (1, \infty)$ are Hölder conjugate

	weak	weak*	seq. weak	seq. weak*
$\ell^1 (\cong \ell_0^*)$	no	yes	no	yes
$\ell^p (\cong (\ell^q)^*)$	yes	yes	yes	yes
$\ell^\infty (\cong (\ell^1)^*)$	no	yes	no	yes
(*) L^1 (not a dual!)	no	—	no	—
$L^p (\cong (L^q)^*)$	yes	yes	yes	yes
$L^\infty (\cong (L^1)^*)$	no	yes	no	yes
Thm. used	4.37	4.35	4.31	4.36

(*) Cor. to Krein-Milman Theorem, see f.ex.

Werner, Sect. VIII.4.