Example 4.14 Let $X = Y = C_0(\mathbb{R})$ with supremum norm (Banach!)

Consider
$$T := \frac{d}{dx} : \operatorname{dom}(T) \to C_0(\mathbb{R}), \quad f \mapsto f'$$
with $\operatorname{dom}(T) = \{ f \in C_0(\mathbb{R}) \mid f \in C^1(\mathbb{R}) \text{ and } f' \in C_0(\mathbb{R}) \} \subseteq C_0(\mathbb{R})$.

Claim: T is closed (a closed operator).

Pf: Let $(f_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom}(T)$ be a sequence such that:

(i) $\exists g \in C_0(\mathbb{R})$: $\|f_n - g\|_\infty \xrightarrow{n\to\infty} 0$

(ii) $\exists h \in C_0(\mathbb{R})$: $\|f_n' - h\|_\infty \xrightarrow{n\to\infty} 0$

i.e., $(f_n, f_n')_n \in G(T)$, $(f_n, f_n')_n \xrightarrow{n\to\infty} (g, h)$ in $X \times Y$; we need to prove that $(g,h) \in G(T)$, equivalently, $g \in \operatorname{dom}(T)$ and $h = g'$.

By uniform convergence, we can exchange limits, so, $\forall x \in \mathbb{R}$,
$$\int_0^x h(t)\,dt = \int_0^x \lim_{n\to\infty} f_n'(t)\,dt = \lim_{n\to\infty} \underbrace{\int_0^x f_n'(t)\,dt}_{f_n(x) - f_n(0)} = g(x) - g(0)$$
(since $f_n \to g$ pointwise, by (i)).

Hence, $\quad g(x) = g(0) + \int_0^x h(t)\,dt \quad \forall x \in \mathbb{R}$

So, by the Fundamental Theorem of Calculus (HDI): $g \in C^1(\mathbb{R})$ with $g' = h \in C_0(\mathbb{R})$. Since $g \in C_0(\mathbb{R})$, we have $g \in \operatorname{dom}(T)$, and so $(g,h) = (g,g') \in G(T)$. So T is closed.

Also: Since T is unbounded (cf. 2.28) $\overset{\text{Thm. 4.13}}{\Longrightarrow}$ $\operatorname{dom}(T)$ is not a closed subspace of $C_0(\mathbb{R})$ w.r.t. $\|\cdot\|_\infty$.

4.3 (Bi-) Dual spaces and weak topologies

Theorem 4.15 Let X be a normed space. Then, for every $x \in X$,
$$\|x\| = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|_*}$$

Pf: See exercise.

Definition 4.16 Let X be a normed space. We call

$X^{**} := (X^*)^*$ the *bidual space* of X (always Banach!)

More generally, introduce the *n-fold dual space*

$$X^{\overbrace{*\cdots*}^{n \text{ times}}} := \left(X^{\overbrace{*\cdots*}^{n-1 \text{ times}}}\right)^* \text{ for } n \in \mathbb{N}, \text{ recursively.}$$

Theorem (& definition) 4.17 Let X be a normed space.

The *canonical embedding* $J: \begin{array}{l} X \to X^{**} \\ x \mapsto Jx \end{array}$ with $Jx: \begin{array}{l} X^* \to \mathbb{K} \\ f \mapsto f(x) \end{array}$

is well-defined, linear, and isometric.

If J is surjective, we call X *reflexive*.

Remarks 4.18 (a) Hilbert spaces are reflexive (Riesz!)

(b) Every finite dimensional normed space is reflexive (use the dual basis)

(c) ℓ^p, L^p are reflexive for $p \in (1,\infty)$; c_0 & ℓ^1 are not refl.

(d) Milman-Pettis Thm: X uniformly convex & Banach $\Rightarrow$ X reflexive

(e) X reflexive $\Rightarrow$ X complete.

Pf (4.17): J is well-defined (i.e. $Jx \in X^{**}\ \forall x \in X$):

Let $\alpha, \beta \in \mathbb{K}$, $f, g \in X^*$, then

$$(Jx)(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha (Jx)(f) + \beta (Jx)(g)$$

So, Jx is linear. Also, Jx is bounded, since

$$\|Jx\|_{**} = \sup_{0 \neq f \in X^*} \frac{|(Jx)(f)|}{\|f\|_*} = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|_*} \overset{\text{Thm. 4.15}}{=} \|x\| < \infty.$$

Hence, $Jx \in X^{**}$, so J is well-defined – and an isometry.

Also, J is linear: Let $\alpha, \beta \in \mathbb{K}$, $x, y \in X$, then, for all $f \in X^*$,

$$(J(\alpha x + \beta y))(f) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha (Jx)(f) + \beta (Jy)(f),$$

and so $J(\alpha x + \beta y) = \alpha J(x) + \beta J(y)$ ∎

Theorem 4.19 Let X be a Banach space. Then

$$X \text{ reflexive} \iff X^* \text{ reflexive}$$

Pf. "$\Rightarrow$": $(X^*)^{**} = ((X^*)^*)^* = (X^{**})^* = X^*$

"$\Leftarrow$": See f. ex. Werner, "Funktionalanalysis", 8. Aufl., Thm. III 3.4.

Theorem 4.20 Let X be a normed space. Then

$$X^* \text{ separable} \Rightarrow X \text{ separable}.$$

Remark 4.21 $X = \ell^1$ shows that "$\Leftarrow$" does not hold, since $\ell^\infty \simeq (\ell^1)^*$, which is not separable.

Pf (4.20): Let $A := \{f_n \in X^* \mid n \in \mathbb{N}\}$ be dense (exists by hyp.).

For $n \in \mathbb{N}$ choose $x_n \in X$ s.t. $\|x_n\| = 1$ & $|f_n(x_n)| \geq \frac{1}{2} \|f_n\|_*$.

Let $D := \operatorname{span}_{\mathbb{K}} \{x_n \mid n \in \mathbb{N}\} \subseteq X$.

Claim: D is dense in X (Note: D is not countable, but a $\mathbb{K}$-linear subspace).

Pf (claim): Assume (for contradiction) D not dense.

Then there exists $z \in X$ s.t. $\operatorname{dist}(z, D) > 0$. By Cor. 4.7 (H-B!)

$\exists f \in X^*$: $f|_D = 0$, $f(z) > 0$. Now:

(i) A dense in X^* implies: $\exists$ seq. $(f_{n_k})_k \subseteq A$:

$$\|f_{n_k} - f\|_* \xrightarrow{k \to \infty} 0 \quad (\ast)$$

(ii) We have

$$\|f_{n_k} - f\|_* \geq |f_{n_k}(x_{n_k}) - f(x_{n_k})| \overset{f|_D = 0}{=} |f_{n_k}(x_{n_k})| \geq \frac{1}{2} \|f_{n_k}\|_*$$

Then $\|f_{n_k}\|_* \xrightarrow{k \to \infty} 0$, so $f_{n_k} \xrightarrow{k \to \infty} 0$ in X^*, hence $f = 0$ by $(\ast)$ ↯.

So, D is dense.

Define $\tilde{\mathbb{K}} := \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$, resp. $\tilde{\mathbb{K}} := \mathbb{Q} + i\mathbb{Q}$ if $\mathbb{K} = \mathbb{C}$, and set

$\tilde{D} := \mathrm{span}_{\mathbb{K}}\{x_n \mid n \in \mathbb{N}\}$. Then $\tilde{D}$ is countable & dense in X.

Definition 4.22 Let X be a $\mathbb{K}$-vector space, let $\{p_\alpha\}_{\alpha \in I}$ be a family of seminorms on X

(a) $\{p_\alpha\}_{\alpha\in I}$ is separating $:\iff \forall 0 \neq x \in X : \exists \alpha \in I : p_\alpha(x) > 0$

(b) For given $\alpha \in I$, $r > 0$, let

$$U_{\alpha,r} := \{ y \in X \mid p_\alpha(y) < r\} \ni 0.$$

Also, for $x \in X$, let

$$U_{\alpha,r}(x) := x + U_{\alpha,r} = \{ y \in X \mid p_\alpha(y-x) < r\} \ni x.$$

Let

$$\mathcal{N}_x := \left\{ \bigcap_{j=1}^{n} U_{\alpha_j, r_j}(x) \;\middle|\; n \in \mathbb{N}, \alpha_j \in I, r_j > 0 \text{ for } j = 1, \dots, n \right\}$$

(family of finite intersections of $U_{\alpha,r}(x)$'s)

Define the Locally convex topology (l.c.t.) on X induced by $\{p_\alpha\}_{\alpha\in I}$:

$$Y \subseteq X \text{ open in the l.c.t.} :\iff \forall x \in Y\ \exists V_x \in \mathcal{N}_x : V_x \subseteq Y$$

$$\iff \forall x \in Y\ \exists V_x \in \mathcal{N}_x : Y = \bigcup_{x \in Y} V_x$$

Remark 4.23 (a) $U_{\alpha,r}(x)$ is open in l.c.t., and $\mathcal{N}_x$ is a neighbourhood base of the l.c.t. at x.

(b) If $\{p_\alpha\}_{\alpha\in I}$ is separating, then the l.c.t. is Hausdorff (see exercise).

(c) The elements of the neighbourhood base are convex sets (hence the name): For $y_1, y_2 \in U_{\alpha,r}$ and $\lambda \in [0,1]$, we have $\lambda y_1 + (1-\lambda) y_2 \in U_{\alpha,r}$, since:

$$p_\alpha(\lambda y_1 + (1-\lambda) y_2) \le p_\alpha(\lambda y_1) + p_\alpha((1-\lambda) y_2)$$

$$= \lambda \underbrace{p_\alpha(y_1)}_{<r} + (1-\lambda) \underbrace{p_\alpha(y_2)}_{<r} < r.$$

(d) $\mathfrak{X}$ is a topological vector space w.r.t. a locally convex topology, i.e., addition of vectors and mult. of a vector by a scalar are continuous w.r.t. a l.c.t.
This relies on $U_{\alpha,r}(x) = x + U_{\alpha,r}$ & $\lambda U_{\alpha,r}(x) = U_{\alpha,|\lambda|r}(\lambda x)$
$\forall x \in \mathfrak{X}, \lambda \in \mathbb{K}, \alpha \in I, r > 0$.

Notation: vector space topology :$\Leftrightarrow$ a topology making $\mathfrak{X}$ into a topological vector space.

The composition of a linear map with a norm gives a seminorm (check!). The following abstract result will be applied several times with different choices of S_α and Z:

Lemma 4.24 Let $\mathfrak{X}$ be a vector space, $(Z, \|\cdot\|_Z)$ a normed space, and $S_\alpha : \mathfrak{X} \to Z$ a linear map for every index $\alpha \in I$. Let $\mathcal{T}_{l.c.t.}$ be the locally convex topology on $\mathfrak{X}$ induced by the family of seminorms $\{x \mapsto \|S_\alpha x\|_Z\}_{\alpha \in I}$. Then:

(a) $\mathcal{T}_{l.c.t.}$ is the coarsest vector space topology on $\mathfrak{X}$ s.t. for every $\alpha \in I$ the map S_α is continuous.

(b) $(x_k)_{k\in\mathbb{N}} \subseteq \mathfrak{X}$ converges to $x \in \mathfrak{X}$ w.r.t. $\mathcal{T}_{l.c.t.}$ $\Leftrightarrow$
$$\|S_\alpha x_k - S_\alpha x\|_Z \xrightarrow{k\to\infty} 0 \quad \forall \alpha \in I$$

Pf: See exercise.

A very important example is when $S_\alpha \in \mathfrak{X}^*$ and $Z = \mathbb{K}$:

Definition 4.25 Let $\mathfrak{X}$ be a normed space. The weak topology on $\mathfrak{X}$ is the locally convex topology (l.c.t.) on $\mathfrak{X}$ induced by the family of seminorms $\{x \mapsto |f(x)|\}_{f \in \mathfrak{X}^*}$.

$\boxed{\text{Lemma 4.26}}$ Let X be a normed space. Then the weak topology on X is

(a) Hausdorff.

(b) the coarsest vector-space topology on X such that every $f \in X^*$ is continuous. In particular, the weak topology is coarser than the norm (=: strong) topology.

(c) identical to the norm topology, if $\dim X < \infty$,

(d) such that $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges to x in the weak topology (or, weakly) iff

$$\lim_{n\to\infty} f(x_n) = f(x) \quad \forall f \in X^*$$

Notation: $x_n \xrightarrow{w} x$, or $x_n \xrightarrow{n\to\infty} x$.

Pf: (a) By Rmk 4.23(b), it suffices to check that the family of seminorms is separating. This follows from Corollary 4.6 (H-B!): For every $0 \neq x \in X$ $\exists f_x \in X^*: f_x(x) = \|x\|$.

(b) Lemma 4.24(a) (c) See exercise (d) Lemma 4.24(b). ∎

$\boxed{\text{Remark 4.27}}$ (a) Weak limits are unique by Lemma 4.26(a)

(b) If $\dim X = \infty$, then the weak topology is not 1^{st} countable (hence not metrisable!) (for pf., see exercise)

(c) Strong convergence $\overset{4.26(b)}{\Longrightarrow}$ weak convergence (but in general not vice versa!)

(d) In a Hilbert space X (by Lem. 4.26(d) & Riesz):

$$x_n \xrightarrow{w} x \iff \langle y, x_n\rangle \xrightarrow{n\to\infty} \langle y, x\rangle \quad \forall y \in X$$

(e) In ℓ^1: $\quad x_n \xrightarrow{w} x \iff \|x_n - x\|_1 \xrightarrow{n\to\infty} 0$

(Schur, 1921; See also Conway, "A course in FA", 1990, Prop. 5.2)

Example 4.28 Let $X = \ell^p$, $p \in [1,\infty]$, $e_n := (\delta_{nk})_{k\in\mathbb{N}}$ for $n \in \mathbb{N}$.

Then (i) $(e_n)_n$ has no $\|\cdot\|_p$-convergent (i.e., strongly conv.) subsequence.

(ii) If $p=1$, then $(e_n)_n$ is <u>not</u> weakly convergent:
Recall $(\ell^1)^* \cong \ell^\infty$, and choose $f \leftrightarrow (-1,1,-1,1,-1,\dots) \in \ell^\infty$
then $f(e_n) = (-1)^n$ $\forall n\in\mathbb{N}$ which is not convergent as $n\to\infty$ (This is consistent with Rmk 4.27(e)).

(iii) If $1<p\leq\infty$, then $(e_n)_n$ is weakly convergent to 0:
In case $1<p<\infty$, we have $(\ell^p)^* \cong \ell^q$ (Thm. 2.38) with $1<q<\infty$. Hence, for $f \leftrightarrow y=(y_k)_k \in \ell^q$:
$$f(e_n) = y_n \xrightarrow{n\to\infty} 0 \quad (\text{since } \sum_{k\in\mathbb{N}} |y_k|^q < \infty).$$
In the case $p=\infty$, use also exercises E6.1 & E6.3.

Lemma 4.29 Let X be a normed space, $(x_n)_{n\in\mathbb{N}} \subset X$, $x \in X$, and $x_n \xrightarrow{w} x$. Then

(a) $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$, (b) $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$

Pf (a): Let $x_n \xrightarrow{w} x$, so $f(x_n) \xrightarrow{n\to\infty} f(x)$ $\forall f \in X^*$, hence
$$\forall f \in X^* \text{ (fixed)}: \quad \sup_{n\in\mathbb{N}} \underbrace{|f(x_n)|}_{(Jx_n)(f)} < \infty,$$
with the canonical embedding $J: X \to X^{**}$. Since X^* is Banach, the Uniform Boundedness Principle (Thm. 4.9) gives
$$\sup_{n\in\mathbb{N}} \underbrace{\|Jx_n\|_{**}}_{=\|x_n\|} < \infty. \quad \checkmark$$

(b) By Cor. 4.6 (H-B!):
$$\exists f_x \in X^*: \|f_x\|_* = 1 \text{ and } f_x(x) = \|x\|$$

Then

$$\|x\| = |f_x(x)| = \lim_{n\to\infty} |f_x(x_n)| = \liminf_{n\to\infty} |f_x(x_n)| \leq \liminf_{n\to\infty} \underbrace{\|f_x\|_*}_{=1} \|x_n\|$$ ■

Theorem 4.30 Let X be a normed space. Then $x_n \xrightarrow{w} x$ iff the following two statements hold:

(i) $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$

(ii) $\exists F \subseteq X^*$ with $\mathrm{span}(F)$ dense (wrt. $\|\cdot\|_*$) in X^* s.t.

$$\forall f \in F: \qquad \lim_{n\to\infty} f(x_n) = f(x). \qquad (*)$$

Pf: "$\Rightarrow$": From Lem. 4.29 and by definition of weak convergence

"$\Leftarrow$": Use an $\frac{\varepsilon}{3}$-argument: Let $\varepsilon > 0$ and $g \in X^*$. Let

$K := \frac{1}{2}(\|x\| + \sup_{n\in\mathbb{N}} \|x_n\|) < \infty$. Since $\mathrm{span}(F)$ is dense in X^*, there exists $f \in \mathrm{span}(F)$: $\|f-g\|_* < \frac{\varepsilon}{3K}$. Also, for $f \in \mathrm{span}(F)$, $\exists N \in \mathbb{N}: \forall n \geq N: \quad |f(x_n) - f(x)| < \frac{\varepsilon}{3}$

(Note: $(*)$ holds also for $f \in \mathrm{span}(F)$ as finite lin. comb.)

Hence, $\forall n \geq N$:

$$|g(x) - g(x_n)| \leq |g(x) - f(x)| + |f(x) - f(x_n)| + |f(x_n) - g(x_n)|$$

$$\leq \underbrace{\|g-f\|_*}_{< \frac{\varepsilon}{3K}} \underbrace{(\|x\| + \|x_n\|)}_{\leq 2K} + \underbrace{|f(x) - f(x_n)|}_{< \frac{\varepsilon}{3}} < \varepsilon$$ ■

Theorem 4.31 (Eberlein–Šmulian) Let X be a Banach space and $A \subseteq X$. Then A weakly compact $\Leftrightarrow$ A weakly sequentially compact.

Pf. Not here; see Whitley, Math. Ann. 172, 116–118 (1967). ■

Definition 4.32 Let X be a normed space. Then the weak* topology on X^* ("weak-star") is the locally convex topology on X^* induced by the family of seminorms $\{f \mapsto |f(x)|\}_{x \in X}$

Lemma 4.33 Let X be a normed space. Then the weak* topology is

(a) Hausdorff

(b) The coarsest vector-space topology on X^* s.t. $\forall x \in X$, the map
$$X^* \to \mathbb{K}, \quad f \mapsto f(x) \quad \text{is continuous}$$

(c) coarser than the weak topology on X^*, and the two coincide iff X is reflexive.

(d) such that $(f_n)_n \subseteq X^*$ converges to $f \in X^*$ in the weak*-top. iff
$$f_n(x) \xrightarrow{n\to\infty} f(x) \quad \forall x \in X$$
Notation: $f_n \xrightarrow{w^*} f$.

Pf: (a) $\{f \mapsto |f(x)|\}_{x \in X}$ is a separating family of seminorms:
For $f \neq 0$ there exists $x \in X$ with $f(x) \neq 0$.

(b) Lemma 4.24(a) with $Z = \mathbb{K}$ and $S_\alpha = Jx$ ($J: X \to X^{**}$ canonical embedding)

(c) $J(X) \subseteq X^{**}$ with equality iff X is reflexive.

(d) Lemma 4.24 (b).

The next theorem is the analogue of Lem. 4.29 & Thm. 4.30. However: Note that here, X must be complete in order to apply the Uniform Boundedness Principle in (a) (i).

Theorem 4.34 Let X be a normed space, $f \in X^*$, $(f_n)_n \subseteq X^*$

(a) If $f_n \xrightarrow{w^*} f$, then

(i) If X is even a Banach space, then $\sup_{n \in \mathbb{N}} \|f_n\|_* < \infty$

(ii) $\|f\|_* \le \liminf_{n\to\infty} \|f_n\|_*$

(b) If (i) $\sup_{n \in \mathbb{N}} \|f_n\|_* < \infty$

(ii) $\exists A \subseteq X$ with $\mathrm{span}(A)$ dense in X (w.r.t. $\|\cdot\|$) s.t.
$$\forall x \in A: \quad f_n(x) \xrightarrow{n\to\infty} f(x)$$

then $f_n \xrightarrow{w^*} f$. (If X Banach, the reverse holds by (a)(i))

Pf: See exercise ☒

$\boxed{\text{Theorem 4.35}}$ (Banach-Alaoglu) Let $\mathbb{X}$ be a normed space. Then the closed unit ball in $\mathbb{X}^*$, $\bar{B}_1^* := \{f \in \mathbb{X}^* \mid \|f\|_* \leq 1\}$ is compact in the weak* topology.

Pf: Equip the set of maps $\{f : \mathbb{X} \to \mathbb{K}\} = \mathbb{K}^{\mathbb{X}}$ with the product topology $\mathcal{T}_{prod} := \bigtimes_{x \in \mathbb{X}} \mathcal{T}_{\mathbb{K}}$ of the Euclidean topology $\mathcal{T}_{\mathbb{K}}$ on $\mathbb{K}$. Then:

(i) $\mathcal{T}_{prod}$ is the coarsest topology on $\mathbb{K}^{\mathbb{X}}$ such that the projection/evaluation map $\Pi_x : \mathbb{K}^{\mathbb{X}} \to \mathbb{K}$, $f \mapsto f(x)$ is continuous for all $x \in \mathbb{X}$ (see T2.3, Tut Sheet 2)

(ii) $\mathcal{T}_{prod}$ is a vector-space topology on $\mathbb{K}^{\mathbb{X}}$:

Addition: $\mathcal{A} : \mathbb{K}^{\mathbb{X}} \times \mathbb{K}^{\mathbb{X}} \to \mathbb{K}^{\mathbb{X}}$, $(f,g) \mapsto f+g$. It suffices to prove openness of $\mathcal{A}^{-1}(U)$ in $\mathbb{K}^{\mathbb{X}} \times \mathbb{K}^{\mathbb{X}}$ for U in a base of $\mathcal{T}_{prod}$, that is, for $U = \bigtimes_{x \in U} U_x$ with $U_x \in \mathcal{T}_{\mathbb{K}}$, and $U_x = \mathbb{K}$ for all but at most finitely many $x \in \mathbb{X}$. Addition in the field $\mathbb{K}$, $\mathcal{A}_{\mathbb{K}} : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, $(z, z') \mapsto z + z'$ is continuous.

Hence, $\mathcal{A}^{-1}(U) = \bigtimes_{x \in \mathbb{X}} \mathcal{A}_{\mathbb{K}}^{-1}(U_x)$ with

$$\mathcal{A}_{\mathbb{K}}^{-1}(U_x) \subseteq \mathcal{T}_{\mathbb{K}} \times \mathcal{T}_{\mathbb{K}} \quad \forall x \in \mathbb{X}$$

and

$$\mathcal{A}_{\mathbb{K}}^{-1}(U_x) = \mathbb{K} \times \mathbb{K} \text{ except for at most finitely many } x \in \mathbb{X}.$$

Hence, $\mathcal{A}^{-1}(U) \in \mathcal{T}_{prod} \times \mathcal{T}_{prod}$, as finite intersection of a union of product sets $V_1 \times V_2$ with each $V_j \in \mathcal{T}_{prod}$, $j = 1, 2$, being a product set of factors all of which are equal to $\mathbb{K}$ except for one. ✓ The case of scalar mult. is analogous & simpler.

(iii) Given a linear subspace $S \subseteq \mathbb{K}^{\mathbb{X}}$, the subspace topology $\mathcal{T}_{prod}|_S := \{U \cap S \mid U \in \mathcal{T}_{prod}\}$ is a vector space topology on S wrt. which the restricted eval. maps $\Pi_x|_S$ are continuous $\forall x \in \mathbb{X}$

Now, choose $S = X^*$, and let $\mathcal{T}_{w*}$ denote the weak* top. on X^*. By 4.33(b), $\mathcal{T}_{w*} \subseteq \mathcal{T}_{prod}|_{X^*}$. Therefore, the map

$$\sigma : (X^*, \mathcal{T}_{prod}|_{X^*}) \longrightarrow (X^*, \mathcal{T}_{w*})$$
$$f \longmapsto f$$

is continuous, and the theorem follows if we prove: $\bar{B}_1^*$ is $\mathcal{T}_{prod}$-compact: We claim: Then it is $\mathcal{T}_{prod}|_{X^*}$-compact (and hence, by continuity of σ, $\mathcal{T}_{w*}$-compact).

Pf Claim: Consider a $\mathcal{T}_{prod}|_{X^*}$-open cover $\bigcup_\alpha V_\alpha$ of $\bar{B}_1^*$ and use that, for $V_\alpha \in \mathcal{T}_{prod}|_{X^*}$, there exists $U_\alpha \in \mathcal{T}_{prod}$ with $V_\alpha = U_\alpha \cap X^*$. The $\mathcal{T}_{prod}$-open cover $\bigcup_\alpha U_\alpha$ has a finite subcover (by assumption), and intersection with X^* gives finite subcover of $\bigcup_\alpha V_\alpha$. ✓ (claim).

To prove $\mathcal{T}_{prod}$-compactness: Define $A := \bigtimes_{x \in X} A_x$, where $A_x := \{ z \in \mathbb{K} \mid |z| \leq \|x\| \}$ is compact in $\mathbb{K}$. By Tychonoff (Thm. 7.28), A is $\mathcal{T}_{prod}$-compact. Note that: $f \in A \Leftrightarrow f(x) \in A_x \ \forall x \in X$

$\Leftrightarrow \sup_{0 \neq x \in X} \frac{|f(x)|}{\|x\|} \leq 1$. For $x, y \in X$, $\alpha, \beta \in \mathbb{K}$, let

$$L_{x,y,\alpha,\beta} := \{ f \in \mathbb{K}^X \mid \underbrace{f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)}_{= (\Pi_{\alpha x + \beta y} - \alpha \Pi_x - \beta \Pi_y)(f)} = 0 \}$$

$$= (\Pi_{\alpha x + \beta y} - \alpha \Pi_x - \beta \Pi_y)^{-1}(\{0\}).$$

But $\Pi_{\alpha x + \beta y} - \alpha \Pi_x - \beta \Pi_y$ is $\mathcal{T}_{prod}$-$\mathcal{T}_{\mathbb{K}}$-continuous, and $\{0\} \subseteq \mathbb{K}$ is $\mathcal{T}_{\mathbb{K}}$-closed, hence $L_{x,y,\alpha,\beta}$ is $\mathcal{T}_{prod}$-closed in $\mathbb{K}^X$, and so also

$$L := \bigcap_{\substack{x,y \in X \\ \alpha,\beta \in \mathbb{K}}} L_{x,y,\alpha,\beta} = \{ f : X \to \mathbb{K} \text{ is linear} \}$$

is closed.

Finally, $\bar{B}_1^* = A \cap L$ is $\mathcal{T}_{prod}$-compact (as closed subset of the compact A) ∎

Theorem 4.36 (Helly; version 2 of Banach-Alaoglu)

Let X be a separable normed space. Then $\overline{B}_1^*$ is weak* sequentially compact.

Pf: Let $\{x_k \mid k \in \mathbb{N}\} \subseteq X$ be (countable) dense in X. We need to prove: Any sequence $(f_n)_n \subseteq \overline{B}_1^*$ has a weak*-convergent subsequence: Fix $k \in \mathbb{N}$ arbitrary. Consider the sequence $(f_n(x_k))_{n \in \mathbb{N}} \subseteq \mathbb{K}$. This sequence is bounded, since

$$|f_n(x_k)| \le \underbrace{\|f_n\|_*}_{\le 1} \|x_k\| \le \|x_k\| \quad (\& \ k \text{ fix})$$

Hence, by Bolzano-Weierstraß, for all k fixed, $(f_n(x_k))_n$ has a convergent subsequence (in $\mathbb{K}$).

Claim: Then exist a common subsequence $(n_j)_j \subseteq \mathbb{N}$:

$$\forall k \in \mathbb{N}: \quad (f_{n_j}(x_k))_{j \in \mathbb{N}} \text{ is convergent.}$$

Pf: Use Cantor's diagonal sequence trick:

There exists $(n_j^{(1)})_j \subseteq \mathbb{N}$ such that $(f_{n_j^{(1)}}(x_1))_j$ converges.

Then there exists $(n_j^{(2)})_j \subseteq (n_j^{(1)})_j$ s.t. $(f_{n_j^{(2)}}(x_2))_j$ converges.

Continuing this procedure, there exists $(n_j^{(\ell+1)})_j \subseteq (n_j^{(\ell)})_j$ such that $(f_{n_j^{(\ell+1)}}(x_{\ell+1}))_{j \in \mathbb{N}}$ converges. The claim then holds with $n_j := n_j^{(j)}$.

Now, define $g(x) := \lim_{j \to \infty} f_{n_j}(x) \quad \forall x \in \operatorname{span}\{x_k \mid k \in \mathbb{N}\} =: \operatorname{dom}(g)$.

(i) $\operatorname{dom}(g)$ is a dense subspace of X with respect to $\|\cdot\|$

(ii) $g: \operatorname{dom}(g) \to \mathbb{K}$ is linear

(iii) g is bounded: $|g(x)| = \lim_{j \to \infty} \underbrace{|f_{n_j}(x)|}_{\le \underbrace{\|f_{n_j}\|_*}_{\le 1} \|x\|} \le \|x\|$

Since $\mathbb{K}$ is complete, we can apply the Bounded Linear Ext. Thm. 2.31:

There exists $\tilde{g}: X \to \mathbb{K}$ such that $\tilde{g}|_{\operatorname{dom}(g)} = g$ & $\|\tilde{g}\|_* = \underbrace{\|g\|_*}_{\le 1}$

Hence, $\tilde{g} \in \bar{B}_1^*$. Then, by Thm. 4.34(b), $f_{n_j} \xrightarrow{w^*} \tilde{g}$ as $j \to \infty$ (05) ∎

Theorem 4.37 (Kakutani, version 3 of Banach–Alaoglu)

Let X be a Banach space. Then

$$X \text{ reflexive} \iff \bar{B}_1 := \{x \in X \mid \|x\| \leq 1\} \text{ is weakly compact}$$

Pf: "$\Rightarrow$": By Thm 4.19, $X^* =: Y$ is reflexive. Applying 4.33(c) to Y, we get: The weak topology on $Y^* = (X^*)^* = X$ coincides with the weak* topology on $Y^* = X$, hence the claim follows from Thm. 4.35 applied to Y. ∎

"$\Leftarrow$": See f.ex. Werner, "Funktionalanalysis", 8. Aufl, Satz VIII.3.18. ∎

Example 4.38 Compactness of $\bar{B}_1$ in different spaces.

Here, $p, q \in (1, \infty)$ are Hölder conjugate

	weak	weak*	seq. weak	seq. weak*
$\ell^1 \; (\cong c_0^*)$	no	yes	no	yes
$\ell^p \; (\cong (\ell^q)^*)$	yes	yes	yes	yes
$\ell^\infty \; (\cong (\ell^1)^*)$	no	yes	no	yes
(*) L^1 (not a dual!)	no	—	no	—
$L^p \; (\cong (L^q)^*)$	yes	yes	yes	yes
$L^\infty \; (\cong (L^1)^*)$	no	yes	no	yes
Thm. used	4.37	4.35	4.31	4.36

(*) Cor. to Krein-Milman Theorem, see f.ex. Werner, Sect. VIII.4.