

2.5 Hilbert spaces

(47)

The main new feature is the "geometry" from the scalar product

Definition 2.40 Let $\mathcal{X}$ be a $(K-)$ vector space.

A map $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow K$ is a scalar product (or inner product) : $\Leftrightarrow$

(i) $\langle x, x \rangle \geq 0 \quad \forall x \in \mathcal{X}$, and $\langle x, x \rangle = 0 \Rightarrow x = 0$
(positive definite) (non-degenerate)

(ii) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle \quad \forall \alpha, \beta \in K, \forall x, y, z \in \mathcal{X}$
(linearity)

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in \mathcal{X}$

$(\mathcal{X}, \langle \cdot, \cdot \rangle)$ is an inner product space (or pre-Hilbert space)

Note: If $K = \mathbb{R}$, (ii) & (iii) imply that $\langle \cdot, \cdot \rangle$ is bi-linear

If, however, $K = \mathbb{C}$, (ii) & (iii) give that

$$\langle \alpha x, y \rangle = \overline{\alpha} \langle x, y \rangle \quad \forall \alpha \in \mathbb{C}, x, y \in \mathcal{X} \quad (\text{conjugate linear})$$

Then $\langle \cdot, \cdot \rangle$ is called sesqui-linear ("1 $\frac{1}{2}$ -linear")

Warning: In literature also: $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ & $\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$

Lemma 2.41 (Cauchy-Schwarz (Bunyakovsky) inequality)

Let $\mathcal{X}$ be an inner product space. Then

$$(C-S) \quad |\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} \quad \forall x, y \in \mathcal{X}$$

with equality iff x & y are linearly dependent

Proof: See Linear Algebra (and/or Axi 1-3).

■

Lemma 2.42 | Let $\mathcal{X}$ be an inner product space.

Then $\mathcal{X}$ is a normed space with norm $\|x\| := \langle x, x \rangle^{1/2}$.

All notions & results from topological, metric, and normed spaces are available. In particular, the scalar product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{K}$ is continuous.

Pf: (i) $\|\cdot\|$ fulfills all axioms of a norm, see LA (or do!)

(ii) Continuity of $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{K}$:

Note that $\mathcal{X} \times \mathcal{X}$ is a normed space (f. ex. with norm $\|(x, y)\|_1 := \|x\| + \|y\|$), hence seq. cont. enough.

So let $x_n \xrightarrow{n \rightarrow \infty} x$, $y_n \xrightarrow{n \rightarrow \infty} y$ be two conv. seq.'s in $\mathcal{X}$.

Then

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &\leq |\langle x_n, y_n \rangle - \langle x_n, y \rangle| + |\langle x_n, y \rangle - \langle x, y \rangle| \\ &= |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle| \\ &\stackrel{C-S}{\leq} \underbrace{\|x_n\|}_{\xrightarrow{n \rightarrow \infty} 0} \cdot \underbrace{\|y_n - y\|}_{\xrightarrow{n \rightarrow \infty} 0} + \underbrace{\|y\|}_{< \infty} \cdot \underbrace{\|x_n - x\|}_{\xrightarrow{n \rightarrow \infty} 0} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

since $\sup_{n \in \mathbb{N}} \|x_n\| < \infty$ (as $(x_n)_n$ converges & $\|\cdot\|$ cont.). $\square$

Definition 2.43 | A complete inner product space is called a Hilbert space.

Note: Hence, any Hilbert space is also a Banach space (for the other direction, see 2.46 below).

Theorem 2.44 | Let $\mathcal{X}$ be an inner product space.

Then there exists a Hilbert space $\tilde{\mathcal{X}}$, a dense subspace $\mathcal{W} \subseteq \tilde{\mathcal{X}}$, and a unitary map $U: \mathcal{X} \rightarrow \mathcal{W}$.

(U is unitary : $\Leftrightarrow U$ is an isomorphism with $\langle x, y \rangle_{\mathcal{X}} = \langle Ux, Uy \rangle_{\tilde{\mathcal{X}}} \quad \forall x, y \in \mathcal{X}$)

Idea of pf: See the pf's of Thm's 2.13 & 1.14.

Additional aspect here: Define a scalar product on

$\tilde{\mathcal{X}} := \{ \text{equivalence classes } \tilde{x} \text{ of Cauchy sequences in } \mathcal{X} \}$:

$$\langle \tilde{x}, \tilde{y} \rangle_{\tilde{\mathcal{X}}} := \lim_{n \rightarrow \infty} \langle x_n, y_n \rangle_{\mathcal{X}}$$

with $(x_n)_n, (y_n)_n \subseteq \mathcal{X}$ being representatives of the equivalence classes $\tilde{x}$ and $\tilde{y}$, and $U: \mathcal{X} \rightarrow \mathcal{W}$
 $x \mapsto \tilde{x}$
where $\tilde{x}$ is the equivalence class of the constant representative $(x, x, x, \dots)$ $\square$

Examples 2.45 | (a) $\ell^2 = \ell^2(\mathbb{N})$ is a Hilbert space

with scalar product $\langle x, y \rangle := \sum_{n \in \mathbb{N}} \overline{x_n} y_n$, where $x = (x_n)_n, y = (y_n)_n$.

Of course, $\ell^2(\{1, \dots, N\}) \cong \mathbb{C}^N$ is also a Hilbert space.

(b) $C([0, 1])$ is an inner product space with the scalar product $\langle f, g \rangle := \int_0^1 \overline{f(x)} g(x) dx$

but not a Hilbert space (proof analogous to Ex. 1.11 (b)).

(Applying Thm. 2.44 to this example gives $L^2([0, 1])$, see next chapter).

(c) ℓ^p for $p \neq 2$ is not an inner product space, because (50)
of the following theorem:

(Theorem 2.46) (Jordan-von Neumann) Let $(X, \|\cdot\|)$ be a normed space. Then X is an inner product space with $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ iff $\|\cdot\|$ satisfies the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x, y \in X.$$

Pf: " $\Rightarrow$ ": Elementary computation (do!)

" $\Leftarrow$ ": Define inner product (!) by polarisation:

$$\langle x, y \rangle := \begin{cases} \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2) & \mathbb{K} = \mathbb{R} \\ \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2)) & \mathbb{K} = \mathbb{C} \end{cases}$$

We verify that this definition satisfies axioms of an inner product:

(i) Symmetry and definiteness are obvious (check!)

(ii) Bi-/sesqui-linearity follows from parallelogram identity (here only $\mathbb{K} = \mathbb{R}$)

$$\begin{aligned} \langle x, y \rangle + \langle x, z \rangle &= \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2 + \|x+z\|^2 - \|x-z\|^2) \\ &= \frac{1}{2} (\|x + \frac{y+z}{2}\|^2 - \|x - \frac{y+z}{2}\|^2) \\ &= 2 \langle x, \frac{y+z}{2} \rangle \end{aligned} \quad (1)$$

$$z=0 \text{ in (1)} \Rightarrow \langle x, y \rangle = 2 \langle x, \frac{y}{2} \rangle \quad (2)$$

$$(1) \text{ \& (2) } \Rightarrow \langle x, y \rangle + \langle x, z \rangle = \langle x, y+z \rangle \quad (3)$$

$$(2) \text{ with } 2y \text{ instead of } y: \langle x, 2y \rangle = 2 \langle x, y \rangle \quad (4)$$

$$\text{By induction from (3) \& (4): } \langle x, ny \rangle = n \langle x, y \rangle \quad \forall n \in \mathbb{N}_0 \quad (5)$$

$$z = -y \text{ in (3)} \Rightarrow \langle x, y \rangle = -\langle x, -y \rangle \Rightarrow (5) \text{ holds } \forall n \in \mathbb{Z}$$

Let $m \in \mathbb{Z} \setminus \{0\}$ and use (5) with $\frac{y}{m}$ instead of y (writing $\lambda := \frac{n}{m}$):

$$\langle x, \lambda y \rangle = n \langle x, \frac{y}{m} \rangle = \lambda m \langle x, \frac{y}{m} \rangle = \lambda \langle x, y \rangle.$$

This holds for all $\lambda \in \mathbb{Q}$, and thus for all $\lambda \in \mathbb{R}$ by continuity (2.10)

Definition 2.47 | Let $\mathcal{X}$ be an inner product space, and $A \subseteq \mathcal{X}$ (not necessarily subspace)

(a) $x, y \in \mathcal{X}$ are orthogonal : $\Leftrightarrow \langle x, y \rangle = 0$. In symbols: $x \perp y$

(b) Orthogonal complement of A : $A^\perp := \{x \in \mathcal{X} \mid x \perp a \quad \forall a \in A\}$

(c) Let $J \neq \emptyset$ be an index set, and for all $\alpha \in J$, let $e_\alpha \in \mathcal{X}$.

We call $\{e_\alpha\}_{\alpha \in J}$ an orthonormal set (or family) : $\Leftrightarrow$

$$\langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta} \quad \forall \alpha, \beta \in J$$

(and orthogonal set if $\langle e_\alpha, e_\beta \rangle = 0 \quad \forall \alpha \neq \beta$)

(d) $\{e_\alpha\}_{\alpha \in J}$ is an orthonormal basis (ONB) or complete orthonormal set

: $\Leftrightarrow$ (i) $\{e_\alpha\}_{\alpha \in J}$ is an orthonormal set.

(ii) "Completeness": If $\langle e_\alpha, x \rangle = 0$ for some $x \in \mathcal{X}$ and all $\alpha \in J$, then $x = 0$ (I.e.: The zero vector is the only vector orthogonal to all e_α 's).

Lemma 2.48 | Let $\mathcal{X}$ be an inner product space.

(a) Let $A \subseteq \mathcal{X}$ be a subset (not necessarily subspace). Then $A^\perp$ is a closed (linear) subspace in $\mathcal{X}$.

(b) Every orthonormal set $\{e_\alpha\}_{\alpha \in J}$ is linearly independent.

Pf: (a) See exercises.

(b) Assume $\sum_{j=1}^n \lambda_j e_{\alpha_j} = 0$ for some $n \in \mathbb{N}$, some $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, and some $\alpha_1, \dots, \alpha_n \in J$ with $\alpha_j \neq \alpha_k \quad \forall j \neq k$. Then we have

$$0 = \langle e_{\alpha_k}, 0 \rangle = \langle e_{\alpha_k}, \sum_{j=1}^n \lambda_j e_{\alpha_j} \rangle = \sum_{j=1}^n \lambda_j \underbrace{\langle e_{\alpha_k}, e_{\alpha_j} \rangle}_{\delta_{kj}} = \lambda_k \quad \forall k = 1, \dots, n \quad \square$$

Next we prove two lemmas/lemmata as preparation for Theorem 2.51:

Lemma 2.49 (Uncountable series) Let $\phi \neq J$ be an index set (S_2)

and $0 \leq c_\alpha < \infty \quad \forall \alpha \in J$. Then

$$\sum_{\alpha \in J} c_\alpha := \sup_{\substack{J_0 \subseteq J \\ |J_0| < \infty}} \sum_{\alpha \in J_0} c_\alpha < \infty$$

implies $c_\alpha = 0$ for all but countably many α 's.

Note: The definition agrees with the usual one, if J is countable.

Proof: Define the sets $S_0 := \{\alpha \in J \mid c_\alpha \geq 1\}$

$$S_n := \{\alpha \in J \mid \frac{1}{n} > c_\alpha \geq \frac{1}{n+1}\}, \quad n \in \mathbb{N}$$

Claim: $|S_n| < \infty \quad \forall n \in \mathbb{N}_0$

True, since, if this was violated for some $n \in \mathbb{N}$, then

$$\sum_{\alpha \in J} c_\alpha \geq \sum_{\alpha \in S_n} c_\alpha = \infty. \text{ But then } \{\alpha \in J \mid c_\alpha > 0\} = \bigcup_{n \in \mathbb{N}_0} S_n \text{ is}$$

a countable union of finite sets, hence countable $\square$

Lemma 2.50 (Series in Banach & Hilbert spaces)

(a) Let X be a Banach space and consider a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$.

Then

$$\sum_{n \in \mathbb{N}} \|x_n\| < \infty \Rightarrow \sum_{n \in \mathbb{N}} x_n := \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N x_n \right) \text{ exists in } X.$$

(b) Let X be a Hilbert space, and let $(x_n)_{n \in \mathbb{N}} \subseteq X$ be a sequence of pairwise orthogonal vectors (i.e. an orthogonal family).

$$\text{Then } \sum_{n \in \mathbb{N}} \|x_n\|^2 < \infty \Leftrightarrow \sum_{n \in \mathbb{N}} x_n \text{ exists in } X$$

(Note: Recall that $\ell^1 \subsetneq \ell^2$).

Pf: Let $S_n := \sum_{j=1}^n x_j$ for $n \in \mathbb{N}$. Since X is complete, we have:

$$\sum_{n \in \mathbb{N}} x_n \text{ exists} \Leftrightarrow (S_n)_n \text{ is Cauchy (in } X).$$

(a) Let $m \geq n$. Then

$$\|S_m - S_n\| = \left\| \sum_{j=n+1}^m x_j \right\| \leq \sum_{j=n+1}^m \|x_j\| = \sigma_m - \sigma_n$$

with $\sigma_n := \sum_{j=1}^n \|x_j\|$. But by the assumption, $(\sigma_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ (53) converges in $\mathbb{R}$, and hence is Cauchy.

(b) Consider again the sequence of partial sums $(S_n)_n$. Let $m \geq n$, then

$$\|S_m - S_n\|^2 = \left\| \sum_{j=n+1}^m x_j \right\|^2 = \sum_{j,k=n+1}^m \langle x_j, x_k \rangle = \sum_{j=n+1}^m \|x_j\|^2 = T_m - T_n$$

with $T_n := \sum_{j=1}^n \|x_j\|^2$. Hence, $(S_n)_n$ is Cauchy (in $\mathbb{X}$) iff.

$(T_n)_{n \in \mathbb{N}}$ is Cauchy (in $\mathbb{R}$) $\blacksquare$

Theorem 2.51 | Let $\mathbb{X}$ be an inner product space, $x \in \mathbb{X}$, and $\{e_\alpha\}_{\alpha \in J}$ an orthonormal set. Then:

$$\|x\|^2 \geq \sum_{\alpha \in J} |\langle e_\alpha, x \rangle|^2 \quad (\text{Bessel's inequality}).$$

If $\mathbb{X}$ is even a Hilbert space, and if $\{e_\alpha\}_{\alpha \in J}$ is even an ONB, then

$$x = \sum_{\alpha \in J} \langle e_\alpha, x \rangle e_\alpha$$

where at most countably many terms are $\neq 0$, and

$$\|x\|^2 = \sum_{\alpha \in J} |\langle e_\alpha, x \rangle|^2 \quad (\text{Parseval's equality}).$$

Pf. Let $J_0 \subseteq J$ be any finite subset. Then we can write x as follows:

$$x = \underbrace{\sum_{\alpha \in J_0} \langle e_\alpha, x \rangle e_\alpha}_{=: u} + \underbrace{\left(x - \sum_{\alpha \in J_0} \langle e_\alpha, x \rangle e_\alpha \right)}_{=: v}$$

Using that $\{e_\alpha\}_{\alpha \in J_0}$ is an orthonormal set, we get:

$$\begin{aligned} \langle u, u \rangle &= \left\langle \sum_{\alpha \in J_0} \langle e_\alpha, x \rangle e_\alpha, \sum_{\beta \in J_0} \langle e_\beta, x \rangle e_\beta \right\rangle = \sum_{\alpha, \beta \in J_0} \overline{\langle e_\alpha, x \rangle} \langle e_\beta, x \rangle \langle e_\alpha, e_\beta \rangle \\ &= \sum_{\alpha \in J_0} \overline{\langle e_\alpha, x \rangle} \langle e_\alpha, x \rangle = \sum_{\alpha \in J_0} |\langle e_\alpha, x \rangle|^2, \end{aligned}$$

$$\langle u, v \rangle = \langle u, x - u \rangle = \langle u, x \rangle - \langle u, u \rangle$$

(54)

$$= \sum_{\alpha \in J_0} \overline{\langle e_\alpha, x \rangle} \langle e_\alpha, x \rangle - \langle u, u \rangle = 0.$$

Hence, we can estimate $\|x\|^2$:

$$\|x\|^2 = \langle u+v, u+v \rangle = \langle u, u \rangle + \underbrace{\langle v, v \rangle}_{\geq 0} \geq \sum_{\alpha \in J_0} |\langle e_\alpha, x \rangle|^2 \quad \forall J_0 \subseteq J, \quad |J_0| < \infty$$

Taking sup over all finite subsets $J_0 \subseteq J$ gives Bessel's inequality.

Now, Lemma 2.49 $\Rightarrow \exists N \in \mathbb{N} \cup \{\infty\}$ and $\alpha_n \in J \quad \forall n \in \{1, \dots, N\}$ s.t.

$$(*) \quad \sum_{\alpha \in J} |\langle e_\alpha, x \rangle|^2 = \sum_{n=1}^N |\langle e_{\alpha_n}, x \rangle|^2 \leq \|x\|^2 < \infty \quad (\text{using Bessel's ineq.})$$

Then $(*)$ and Lemma 2.50(b) (which requires completeness) imply that

$$x' := \sum_{n=1}^N \langle e_{\alpha_n}, x \rangle e_{\alpha_n} \text{ exists in } X.$$

We need to prove that $x = x'$:

We have $x - x' \perp e_{\alpha_n} \quad \forall n \in \{1, \dots, N\}$:

$$\begin{aligned} \langle e_{\alpha_m}, x - x' \rangle &= \langle e_{\alpha_m}, x \rangle - \langle e_{\alpha_m}, x' \rangle = \langle e_{\alpha_m}, x \rangle - \sum_{n=1}^N \langle e_{\alpha_n}, x \rangle \underbrace{\langle e_{\alpha_m}, e_{\alpha_n} \rangle}_{\delta_{mn}} \\ &= 0 \quad \forall m \in \{1, \dots, N\} \end{aligned}$$

If $\alpha \notin \{\alpha_n \mid n \in \{1, \dots, N\}\}$ then we have:

$$\langle e_\alpha, x - x' \rangle = \langle e_\alpha, x \rangle - \underbrace{\sum_{n=1}^N \langle e_{\alpha_n}, x \rangle \underbrace{\langle e_\alpha, e_{\alpha_n} \rangle}_{=0}}_{=0} = \langle e_\alpha, x \rangle \stackrel{(*)}{=} 0$$

Taken together, we have $x - x' \perp e_\alpha \quad \forall \alpha \in J \quad \overset{\text{ONB}}{\Rightarrow} x = x'$

Moreover (if $N < \infty$, no limits are needed):

$$\|x\|^2 = \left\| \lim_{n \rightarrow \infty} \sum_{j=1}^n \langle e_{\alpha_j}, x \rangle e_{\alpha_j} \right\|^2 \stackrel{\text{B.I. cont.}}{=} \lim_{n \rightarrow \infty} \left\| \sum_{j=1}^n \langle e_{\alpha_j}, x \rangle e_{\alpha_j} \right\|^2$$

$$= \lim_{n \rightarrow \infty} \left\langle \sum_{j=1}^n \langle e_{\alpha_j}, x \rangle e_{\alpha_j}, \sum_{k=1}^n \langle e_{\alpha_k}, x \rangle e_{\alpha_k} \right\rangle$$

$$= \lim_{n \rightarrow \infty} \sum_{j=1}^n \sum_{k=1}^n \overline{\langle e_{\alpha_j}, x \rangle} \langle e_{\alpha_k}, x \rangle \underbrace{\langle e_{\alpha_j}, e_{\alpha_k} \rangle}_{\delta_{jk}} = \sum_{j=1}^N |\langle e_{\alpha_j}, x \rangle|^2$$

$$\stackrel{(*)}{=} \sum_{\alpha \in J} |\langle e_\alpha, x \rangle|^2$$

Lemma 2.50 (b) and Theorem 2.51 gives

(55)

Corollary 2.52 | Let $\mathcal{X}$ be a Hilbert space and $\{e_\alpha\}_{\alpha \in J}$ an orthonormal set in $\mathcal{X}$.
(a) If $(c_\alpha)_{\alpha \in J} \subseteq \mathbb{K}$ with $\sum_{\alpha \in J} |c_\alpha|^2 < \infty$, then $\sum_{\alpha \in J} c_\alpha e_\alpha$ is well-defined in $\mathcal{X}$.
(b) If J is countable, and $\{e_\alpha\}_{\alpha \in J}$ is even an orthonormal basis (ONB), then $\{e_\alpha\}_{\alpha \in J}$ is a Schauder basis.

Theorem 2.53 | Every Hilbert space $\mathcal{X} \neq \{0\}$ has an ONB. Moreover,
 $\mathcal{X}$ separable $\Leftrightarrow$ there exists a countable ONB.

Pf: The existence of an ONB in the non-separable case will be proven later (when Zorn's Lemma is available).

" $\Leftarrow$ ": Suppose there exists a countable ONB. By Cor. 2.52(b), this is a Schauder basis. Thus, $\mathcal{X}$ is separable by Lemma 2.16. $\checkmark$

" $\Rightarrow$ ": Let $\{x_n\}_{n \in \mathbb{N}}$ be dense in $\mathcal{X}$. We construct an ONB using the Gram-Schmidt procedure:

(i) If $x_1 \neq 0$, define $e_1 := \frac{x_1}{\|x_1\|}$ (otherwise, use x_{n_0} with $n_0 := \min \{k \in \mathbb{N} \mid x_k \neq 0\}$)

(ii) Throw away all x_n 's that are linearly dependent of e_1 .

Let $n_1 := \min \{k \in \mathbb{N} \mid x_k \notin \text{span}\{e_1\}\} > n_0$ be the smallest index of the remaining elements. Set

$$\tilde{e}_2 := x_{n_1} - \langle e_1, x_{n_1} \rangle e_1 \quad \text{and} \quad e_2 := \frac{\tilde{e}_2}{\|\tilde{e}_2\|}.$$

(iii) Throw away all x_n 's that are linearly dependent of $\{e_1, e_2\}$. Let $n_2 := \min \{k \in \mathbb{N} \mid x_k \notin \text{span}\{e_1, e_2\}\} > n_1$ be the smallest index of the remaining elements. Set

$$\tilde{e}_3 := x_{n_2} - \langle e_1, x_{n_2} \rangle e_1 - \langle e_2, x_{n_2} \rangle e_2 \quad \text{and} \quad e_3 := \frac{\tilde{e}_3}{\|\tilde{e}_3\|}.$$

(iv) Continue this procedure.

This terminates iff $\dim \mathcal{X} < \infty$. It is clear, that $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal set. Also, $\text{span}\{e_n \mid n \in \mathbb{N}\}$ is dense in $\mathcal{X}$ because

each x_n is a finite linear combination of e_n 's. It remains (56) to prove that this is a basis: Assume there is $y \in \mathbb{X}$ such that $y \perp e_n \forall n$. By denseness, $\exists (y_k)_k \subseteq \text{span}\{e_n | n \in \mathbb{N}\}$ such that $y_k \xrightarrow{k \rightarrow \infty} y$ in $\mathbb{X}$. Thus $\langle y, y_k \rangle = 0 \forall k$ and

$$\|y\|^2 = \langle y, y \rangle = \lim_{k \rightarrow \infty} \underbrace{\langle y, y_k \rangle}_{=0} = 0 \quad \blacksquare$$

Definition Let $\mathbb{X}, \mathbb{X}'$ be inner product spaces.

$\mathbb{X}, \mathbb{X}'$ are unitarily equivalent : $\Leftrightarrow \exists$ isometric isomorphism $U: \mathbb{X} \rightarrow \mathbb{X}'$

Note: U isometric & linear $\Rightarrow U$ preserves scalar products:

$$\langle x, y \rangle_{\mathbb{X}} = \langle Ux, Uy \rangle_{\mathbb{X}'}, \quad \forall x, y \in \mathbb{X} \quad \left(\begin{array}{l} \text{follows from polarisation,} \\ \text{see proof of Thm. 2.46} \end{array} \right)$$

Theorem 2.54 Let $\mathbb{X}$ be a Hilbert space. If $n := \dim \mathbb{X} < \infty$ then $\mathbb{X}$ is unitarily equivalent to $\mathbb{K}^n$ (with Euclidean scalar product). If $\dim \mathbb{X} = \infty$ and $\mathbb{X}$ is separable, then $\mathbb{X}$ is unit. equiv. to ℓ^2

Pf: 2nd case: Fix a countable ONB $\{e_n\}_{n \in \mathbb{N}}$ (exists by Thm. 2.53 b/c $\mathbb{X}$ is separable) and consider the map

$$U: \mathbb{X} \rightarrow \ell^2 \\ x \mapsto (\langle e_n, x \rangle)_{n \in \mathbb{N}}$$

(i) U is well-defined by Bessel's inequality, and linear.

(ii) U is surjective by Cor. 2.52 (a)

(iii) U is isometric ($\Rightarrow$ injective) by Parseval's equality.

1st case: Analogous $\blacksquare$

Theorem 2.55 (Projection theorem) Let $\mathbb{X}$ be a Hilbert space and $A \subseteq \mathbb{X}$ a closed subspace. Then: For all $x \in \mathbb{X}$ there exists a unique $a \in A$ and a unique $w \in A^\perp$ such that

$$x = a + w$$

The proof relies on:

(57)

[Lemma 2.56] Let X, A be as in Thm. 2.55.

For every $x \in X$ there exists a unique $a \in A$ such that
 $\text{dist}(x, A) = \|x - a\|$. That is, a is the "closest element"
 to x in A .

Pf(2.56): Existence: Let $d := d(x, A) = \inf_{y \in A} \|x - y\|$.

There exists a sequence $(y_n)_{n \in \mathbb{N}} \subseteq A$ s.t.

$$(*) \quad d = \lim_{n \rightarrow \infty} \underbrace{\|x - y_n\|}_{\geq d} \quad (\text{by def. of "inf"; } (y_n)_n \text{ is a minimizing sequence})$$

Using the parallelogram identity

$$\|u + v\|^2 = 2(\|u\|^2 + \|v\|^2) - \|u - v\|^2 \quad (\text{Thm. 2.46}) \text{ we get}$$

$$\begin{aligned} \|y_n - y_m\|^2 &= \|(y_n - x) + (x - y_m)\|^2 \\ &= 2(\|y_n - x\|^2 + \|x - y_m\|^2) - \|2x - y_m - y_n\|^2 \\ &= 2(\|y_n - x\|^2 + \|x - y_m\|^2) - 4\left\|x - \underbrace{\frac{y_m + y_n}{2}}_{\in A \text{ (subspace)}}\right\|^2 \end{aligned}$$

$$\begin{aligned} (**) \quad &\leq 2(\|y_n - x\|^2 + \|y_m - x\|^2) - 4d^2 \\ &\rightarrow 0, \quad m, n \rightarrow \infty \quad (\text{by } (*)). \end{aligned}$$

Hence, $(y_n)_n$ is Cauchy, so (X complete) $\exists a \in X: y_n \rightarrow a, n \rightarrow \infty$

But $(y_n)_n \subseteq A$ and A is closed, so $a \in A$, and $d = \|x - a\|$
 by $(*)$ and continuity of the norm

Uniqueness: Assume there exists $(y'_n)_n \subseteq A$ with

$$\lim_{n \rightarrow \infty} \|y'_n - x\| = d \quad \text{and} \quad a' = \lim_{n \rightarrow \infty} y'_n$$

Replacing y_m by y'_m in $(**)$ gives

$$\begin{aligned} \|a - a'\|^2 &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \|y_n - y'_m\|^2 \\ &\leq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} 2(\|y_n - x\|^2 + \|y'_m - x\|^2) - 4d^2 = 0 \\ &\Rightarrow a = a' \end{aligned}$$

Remark: Note that the proof works if A is closed and convex (not necessarily a subspace).

(58)

Pf (of 2.55): Existence: For given $x \in \bar{X}$, let $a \in A$ be defined by Lemma 2.56, and set $w := x - a$. Need to prove: $w \in A^\perp$.
Let $\xi \in \mathbb{K}$ and $y \in A$. Then

$$\|w\|^2 \leq \|x - \underbrace{(a + \xi y)}_{\substack{\in A \\ w - \xi y}}\|^2 = \|w\|^2 + \|\xi y\|^2 - 2 \operatorname{Re} \langle w, \xi y \rangle.$$

So for $0 \neq y \in A$ we have $|\xi|^2 - \frac{2 \operatorname{Re} \langle w, \xi y \rangle}{\|y\|^2} \geq 0$

(i) For $\xi = t \in \mathbb{R}$ this gives

$$t^2 - \frac{2 \operatorname{Re} \langle w, y \rangle}{\|y\|^2} t \geq 0 \quad \forall t \in \mathbb{R}$$

This implies $\operatorname{Re} \langle w, y \rangle = 0$. If $\mathbb{K} = \mathbb{R}$ we are done, since $y \in A$ was arbitrary.

For $\mathbb{K} = \mathbb{C}$ we also consider the following case:

(ii) For $\xi = it, t \in \mathbb{R}$:

$$t^2 + \frac{2 \operatorname{Im} \langle w, y \rangle}{\|y\|^2} t \geq 0 \quad \forall t \in \mathbb{R}$$

So $\operatorname{Im} \langle w, y \rangle = 0$ for every $y \in A$. Together with (i) this gives $\langle w, y \rangle = 0$ and so $w \in A^\perp$.

Uniqueness: Assume there are $a, a' \in A$ and $w, w' \in A^\perp$

with $a + w = x = a' + w'$. Then $\underbrace{a - a'}_{\in A} = \underbrace{w' - w}_{\in A^\perp}$. Then $a = a'$ and $w = w'$ as $A \cap A^\perp = \{0\}$. □

Theorem 2.57 (Riesz' Representation Thm.) Let X be a Hilbert space, and let $\ell \in X^*$. Then there exists a unique $y_\ell \in X$ such that

$$\ell(x) = \langle y_\ell, x \rangle \quad \forall x \in X \quad (*)$$

and $\|\ell\|_* = \|y_\ell\|$.

Pf: If $\ker l = \mathbb{X} \Rightarrow l = 0$ and true with $y_l = 0$. (59)

Assume therefore $\ker l \subsetneq \mathbb{X}$. Note that $\ker l$ is closed.

Existence of y_l : From Thm. 2.55 we know $(\ker l)^\perp \neq \{0\}$ and this allows to choose $0 \neq x_0 \in (\ker l)^\perp$. Define

$$y_l := \frac{\overline{l(x_0)}}{\|x_0\|^2} \cdot x_0 \in \mathbb{X}$$

(i) So (*) holds for every $x \in \ker l$ ($x_0 \perp \ker l$)

(ii) Let $x = \alpha x_0$, $\alpha \in \mathbb{K}$. Then $l(x) = \alpha l(x_0)$ and

$$\langle y_l, x \rangle = \left\langle \frac{\overline{l(x_0)}}{\|x_0\|^2} \cdot x_0, \alpha x_0 \right\rangle = \frac{l(x_0)}{\|x_0\|^2} \alpha \langle x_0, x_0 \rangle = \alpha l(x_0),$$

so l and $\langle y_l, \cdot \rangle$ agree on $\text{span}\{\ker l, x_0\}$.

(iii) But $\text{span}\{\ker l, x_0\} = \mathbb{X}$ because, for all $x \in \mathbb{X}$,

$$x = \underbrace{\left(x - \frac{l(x)}{l(x_0)} \cdot x_0 \right)}_{\in \ker l} + \underbrace{\frac{l(x)}{l(x_0)} \cdot x_0}_{\in \text{span}\{x_0\}}$$

Since

$$l\left(x - \frac{l(x)}{l(x_0)} \cdot x_0\right) = l(x) - \frac{l(x)}{l(x_0)} l(x_0) = 0.$$

So $l = \langle y_l, \cdot \rangle$ on $\mathbb{X}$ by (i), (ii), and the linearity of l and $\langle y_l, \cdot \rangle$.

Uniqueness of y_l : Assume there exists $y_{l'} \in \mathbb{X}$ with

$l = \langle y_{l'}, \cdot \rangle$. Then, for every $x \in \mathbb{X}$,

$$0 = \underbrace{l(x)}_{\langle y_l, x \rangle} - \underbrace{l(x)}_{\langle y_{l'}, x \rangle} = \langle y_l - y_{l'}, x \rangle$$

Choose $x = y_l - y_{l'}$, so that $0 = \|y_l - y_{l'}\|^2$, so $y_l = y_{l'}$.

Norm: We have

$$\|l\|_0 = \sup_{0 \neq x \in \mathbb{X}} \frac{|l(x)|}{\|x\|} \stackrel{C-S}{\leq} \|y_l\|$$

Note that $y_1 \neq 0$ since $\ker l \neq \bar{X}$. Hence, we may

choose $x = y_1$ in sup above:

$$\|l\|_* \geq \frac{|l(y_1)|}{\|y_1\|} = \frac{|\langle y_1, y_1 \rangle|}{\|y_1\|} = \|y_1\|$$

$$\text{hence, } \|l\|_* = \|y_1\| \quad \square$$

Covollary 2.58 / Let $\bar{X}$ be a Hilbert space. Then the map

$$J: \bar{X}^* \rightarrow \bar{X}$$

$$l \mapsto y_l$$

defined by Thm. 2.57 is a semi-linear, isometric bijection
(linear for $K = \mathbb{R}$, conjugate linear for $K = \mathbb{C}$).

Pf: Let $\alpha, \beta \in K$, and $l_1, l_2 \in \bar{X}^*$ with $l_j = \langle y_{l_j}, \cdot \rangle$, i.e.

$J(l_j) = y_{l_j}$ for $j=1,2$. Then

$$\alpha l_1 + \beta l_2 = \alpha \langle y_{l_1}, \cdot \rangle + \beta \langle y_{l_2}, \cdot \rangle = \langle \bar{\alpha} y_{l_1} + \bar{\beta} y_{l_2}, \cdot \rangle$$

So $J(\alpha l_1 + \beta l_2) = \bar{\alpha} J(l_1) + \bar{\beta} J(l_2)$ (by uniqueness in 2.57)

Hence, J is semi-linear. That J is an isometry was proved in 2.57 ($\Rightarrow J$ injective). Also, J is onto / surjective:

For $y \in \bar{X}$, $l_y := \langle y, \cdot \rangle \in \bar{X}^*$ (use C-S, or Lem. 2.42, to prove continuity) and $J(l_y) = y$ (by uniqueness in 2.57) $\square$