LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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This is an overview of material on METRIC AND TOPOLOGICAL SPACES (Ana2) needed for the course.

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METRIC AND TOPOLOGICAL SPACES

Let X be a (non-empty) set (for example $X = \mathbb{R}^d$ or any subset of $\mathbb{R}^d$), and let $\mathcal{P}(X)$ be the family of all subsets of X (its `power set`).

Definition 1 (Topology; topological space). *A family $\mathcal{T}$ of subsets of X, $\mathcal{T} \subseteq \mathcal{P}(X)$, is called a* `topology` *(*`on` *X) if and only if ('iff')*

(i) $\emptyset, X \in \mathcal{T}$.

(ii) $A_1, A_2 \in \mathcal{T} \Rightarrow A_1 \cap A_2 \in \mathcal{T}$.

(iii) For any index set I: $\Big(A_j \in \mathcal{T}$ *for all* $j \in I\Big) \Rightarrow \bigcup_{j\in I} A_j \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ (or often just X) is called a `topological space`*, and $A \subseteq X$ is* `open` $:\Leftrightarrow A \in \mathcal{T}$.

Remark 1. *By induction, (ii) is equivalent to:*

(ii') For any $n \in \mathbb{N} : \Big(A_j \in \mathcal{T}$ *for all* $j = 1, \ldots, n\Big) \Rightarrow \bigcap_{j=1}^{n} A_j \in \mathcal{T}$.

Definition 2 (Coarser/finer topologies). *Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X. We say that $\mathcal{T}_1$ is* `finer` *than $\mathcal{T}_2$, and $\mathcal{T}_2$ is* `coarser` *than $\mathcal{T}_1$, iff $\mathcal{T}_2 \subseteq \mathcal{T}_1$.*

Example 1.

1. `Indiscrete` topology: $\mathcal{T} := \{\emptyset, X\}$.
2. `Discrete` topology: $\mathcal{T} := \mathcal{P}(X)$.
3. All topologies are finer than the indiscrete, and coarser than the discrete topology.
4. `Euclidean` (or `standard`) topology $\mathcal{T}_E$ on $\mathbb{R}^d, d \in \mathbb{N}$: $A \subseteq \mathbb{R}^d$ is open (in $(\mathbb{R}^d, \mathcal{T}_E)$) iff $\forall x \in A\, \exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^d \,|\, |x - y| < \varepsilon\}$ is the `Euclidean ball` of `radius` $\varepsilon > 0$ about/around/with `centre` $x \in \mathbb{R}^d$.

Definition 3 (Induced/relative topology). *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ (not necessarily open!). The* `relative` *(or* `induced`*) topology $\mathcal{T}_A$ on A is*

$$\mathcal{T}_A := \Big\{B \subseteq A \,\Big|\, \exists C \in \mathcal{T} \text{ with } B = C \cap A\Big\} \subseteq \mathcal{P}(A)\,.$$

Remark 2.

1. *$\mathcal{T}_A$ is a topology on A.*

2. *If $A \notin \mathcal{T}$ and $B \in \mathcal{T}_A$, then it may happen that $B \notin \mathcal{T}$. Example: Let $X = \mathbb{R}$ with the standard topology $\mathcal{T}_E$, $A = [0, 1]$. Then $B := [0, \frac{1}{2}[\in \mathcal{T}_A$ but $B \notin \mathcal{T}_E$.*

Definition 4 (**Closed set; neighbourhood; Hausdorff space; adherent point/point of closure; limit/accumulation point; closure; interior point; interior; boundary point; boundary; dense).**
Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$, $x \in X$.

(i) The set A is called `closed` *(*`in` *X, or, correctly,* `in` *$(X, \mathcal{T})$) iff $A^c := X \setminus A \in \mathcal{T}$ (that is, the* `complement` *A^c of A is open (in X)).*

(ii) A set $U \subseteq X$ (not necessarily open) is called a `neighbourhood` *of x iff $\exists A \in \mathcal{T}$ such that $x \in A$ and $A \subseteq U$.*

(iii) The (topological) space X (or, correctly, $(X, \mathcal{T})$) is called a `Hausdorff space` *(or,* `is Hausdorff`*) iff For all $x, y \in X, x \neq y$, there exist neigbourhoods U_x of x and U_y of y such that $U_x \cap U_y = \emptyset$.*

(iv) The point x is called an `adherent point` *(or, a* `point of closure`*)* `of` *A iff For all neighbourhoods U of x: $U \cap A \neq \emptyset$.*

(v) The point x is called a `limit point` *(or, an* `accumulation point`*)* `of` *A iff For all neighbourhoods U of x: $(U \setminus \{x\}) \cap A \neq \emptyset$.*

(vi) The `closure` *$\overline{A}$* `of` *A is $\overline{A} := \{x \in X \,|\, x \text{ adherent point of } A\}$.*

(vii) The point x is called a `boundary point of` *A iff For all neighbourhoods U of x: $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$.*

(viii) The `boundary` *∂A* `of` *A is $\partial A := \{x \in X \,|\, x \text{ boundary point of } A\}$.*

(ix) The point x is called an `interior point of` *A iff There exists a neigbourhood U of x such that $U \subseteq A$.*

(x) The `interior` *A°* `of` *A is $A^\circ := \{x \in X \,|\, x \text{ interior point of } A\}$.*

(xi) The set A is `dense in` *X (correctly,* `in` *$(X, \mathcal{T})$) iff $\overline{A} = X$.*

Remark 3.

1. *Every point of A is an adherent point of A. Hence, $A \subseteq \overline{A}$.*

2. *Note that $\overline{A} = A \cup \partial A$ and $A^\circ = A \setminus \partial A$.*

Lemma 1. *Let X be a topological space, $A \subseteq X$.*

(i) A is open iff $\forall x \in A : x$ is an interior point of A.

(ii) A is closed iff $A = \overline{A}$.

(iii) $\overline{A}$ and ∂A are closed.

Definition 5 (Base; subbase; neighbourhood base).
Let $(X, \mathcal{T})$ be a topological space.

(i) A family $\mathcal{B} \subseteq \mathcal{T}$ is called a `base` *(or* `basis`*)* `for (the topology)` *$\mathcal{T}$ iff $\mathcal{T}$ consists of unions of sets from $\mathcal{B}$.*

(ii) A family $\mathcal{S} \subseteq \mathcal{T}$ is called a `subbase` *(or* `subbasis`*)* `for (the topology)` *$\mathcal{T}$ iff finite intersections of sets from $\mathcal{S}$ form a base for $\mathcal{T}$.*

(iii) A family $\mathcal{N} \subseteq \mathcal{T}$ is called a `neighbourhood base` *(or* `neighbourhood basis`*)* `at` *$x \in X$ iff Every $N \in \mathcal{N}$ is a neighbourhood of x and for every neighbourhood U of x there exists $N \in \mathcal{N}$ with $N \subseteq U$.*

Example 2. Consider $\mathbb{R}^d$ with the standard topology. Let $x \in \mathbb{R}^d$.

1. The family $\{B_{1/n}(x) \,|\, n \in \mathbb{N}\}$ is a neighbourhood base at x.
2. The family $\{B_{1/n}(q) \,|\, n \in \mathbb{N}, q \in \mathbb{Q}^d\}$ is a base for the standard topology.

Lemma 2. *Let $\mathcal{S} \subseteq \mathcal{P}(X)$. Then there exists a topology $\mathcal{T}(\mathcal{S})$ on X such that*

(i) $\mathcal{S}$ is a subbase for $\mathcal{T}(\mathcal{S})$.

(ii) $\mathcal{T}(\mathcal{S})$ is the coarsest (that is, "smallest") topology containing $\mathcal{S}$. That is,

$$\mathcal{T}(\mathcal{S}) = \bigcap_{\mathcal{T} \supseteq \mathcal{S};\, \mathcal{T} \text{ topology}} \mathcal{T}.$$

The topology $\mathcal{T}(\mathcal{S})$ is called the `topology generated by` *$\mathcal{S}$.*

Definition 6 (Product topology). *Let $I \neq \emptyset$ be an arbitrary (!) index set. For every $\alpha \in I$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space. The* `product topology` *on the* `Cartesian product space`*,*

$$\bigtimes_{\alpha \in I} X_\alpha := \left\{ f : I \to \bigcup_{\alpha \in I} X_\alpha \,\middle|\, \forall \alpha \in I : f(\alpha) \in X_\alpha \right\}, \tag{1}$$

is the one generated by the family

$$\left\{ \bigtimes_{\alpha \in I} A_\alpha \,\middle|\, \forall \alpha \in I : A_\alpha \in \mathcal{T}_\alpha, \text{ and } A_\alpha \neq X_\alpha \text{ for at most finitely many } \alpha\text{'s} \right\}.$$

Remark 4. *If I is finite, then the condition "$A_\alpha \neq X_\alpha$ for at most finitely many α's" is always fulfilled.*

Definition 7 (Metric; metric space). *A map $d : X \times X \to [0, \infty[$ is called a* `metric (on` *X*`)` *iff*

(i) $d(x, y) \geq 0 \quad \forall x, y \in X$, with $d(x, y) = 0$ iff $x = y$. (`positiv definit`*)*

(ii) $d(x, y) = d(y, x) \quad \forall x, y \in X$. (`symmetric`*)*

(iii) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$. (`triangle inequality`*)*

If d is a metric on X, then (X, d) (or, often just X) is called a `metric space`*.*

Definition 8 (Induced metric; open (metric) ball; open set; Cauchy sequence; convergent sequence; limit; complete (metric) space).
Let (X, d) be a metric space.

(i) *Let $Y \subseteq X$. The map $d_Y := d|_{Y\times Y} : Y \times Y \to [0, \infty[$ is called the* `induced metric` *(*`on` *Y).*

(ii) *For $x \in X, \varepsilon > 0$, the set $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$ is called the* `open (metric) ball` *of* `radius` *$\varepsilon > 0$ about/around/with* `centre` *x. (More correctly: $B_\varepsilon(x; d)$.)*

(iii) *A set $A \subseteq X$ is called* `open` *(in the metric space (X, d)) iff For every $x \in A$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$.*

(iv) *The family $\mathcal{T}_d := \{A \subseteq X \mid A \text{ is open in } (X, d)\}$ is a topology on X, called the* `metric topology`*. Hence, any metric space is a topological space.*

(v) *A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called a* `Cauchy sequence` *(or, is said to* `be Cauchy` *(!)) (*`in` *X, or,* `in` *(X, d)) iff*

For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n, m \geq N : d(x_n, x_m) < \varepsilon$.

(vi) *A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called* `convergent` *(or, is said to* `converge`*) (*`in` *X, or,* `in` *(X, d)) iff There exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$ (which is equivalent to: $\forall\varepsilon > 0 \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : x_n \in B_\varepsilon(x)$). In this case, $(x_n)_{n\in\mathbb{N}}$ is said to* `converge to` *x, and x is called its* `limit`*. We write $\lim_{n\to\infty} x_n = x$ or $x_n \to x, n \to \infty$.*

(vii) *The space X (more correctly, (X, d)) is called* `complete` *(or, is a* `complete metric space`*) iff every Cauchy sequence in X converges. That is: For all Cauchy sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ there exists $x \in X$ (!) such that $\lim_{n\to\infty} x_n = x$.*

Remark 5. *Completeness is not a topological notion!*

Definition 9. *Let (X, d) be a metric space, let $A \subseteq X, x \in X$.*

(i) *The (extended) number $\operatorname{diam}(A) := \sup_{a,b\in A} d(a, b) \in [0, \infty]$ is called the* `diameter of` *A.*

(ii) *The number $\operatorname{dist}(x, A) := \inf_{a\in A} d(x, a)$ is called the* `distance of` *x* `to` *A.*

Lemma 3. *Let X be a complete metric space, and let $A \subseteq X$. Then A is closed iff A is complete.*
(More precisely/correctly: Assume (X, d) is a complete metric space, $A \subseteq X$. Then A is closed in (X, d) iff (A, d_A) is a complete metric space.)