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HANDOUT

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This is an overview of material on METRIC AND TOPOLOGICAL SPACES (Ana2).

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METRIC AND TOPOLOGICAL SPACES

Let X be a (non-empty) set (for example $X = \mathbb{R}^d$ or any subset of $\mathbb{R}^d$), and let $\mathcal{P}(X)$ be the family of all subsets of X (its power set).

Definition 1 (Topology; topological space). *A family of subsets of X, $\mathcal{T} \subseteq \mathcal{P}(X)$, is called a* topology *(on X) if and only if ('iff')*

(i) $\emptyset, X \in \mathcal{T}$.

(ii) $A_1, A_2 \in \mathcal{T} \Rightarrow A_1 \cap A_2 \in \mathcal{T}$.

(iii) For any index set I: $\Big(A_j \in \mathcal{T} \ \text{ for all } j \in I\Big) \Rightarrow \bigcup_{j\in I} A_j \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space *(often just X), and $A \subseteq X$ is* open $:\Leftrightarrow A \in \mathcal{T}$.

Remark 1. *By induction, (ii) is equivalent to:*

(ii') For any $n \in \mathbb{N} : \Big(A_j \in \mathcal{T} \ \text{ for all } j = 1, \ldots, n\Big) \Rightarrow \bigcap_{j=1}^{n} A_j \in \mathcal{T}$.

Definition 2 (Coarser/finer topologies). *Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X. We say that $\mathcal{T}_1$ is* finer *than $\mathcal{T}_2$, and $\mathcal{T}_2$ is* coarser *than $\mathcal{T}_1$, iff $\mathcal{T}_2 \subseteq \mathcal{T}_1$.*

Example 1.

1. *Indiscrete* topology: $\mathcal{T} := \{\emptyset, X\}$.
2. *Discrete* topology: $\mathcal{T} := \mathcal{P}(X)$.
3. All topologies are finer than the indiscrete, and coarser than the discrete topology.
4. *Euclidean* (or *standard*) topology $\mathcal{T}_E$ on $\mathbb{R}^d, d \in \mathbb{N}$: $A \subseteq \mathbb{R}^d$ is open (in $(\mathbb{R}^d, \mathcal{T}_E)$) iff $\forall x \in A\, \exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^d \,|\, |x - y| < \varepsilon\}$ is the Euclidean ball of radius $\varepsilon > 0$ about $x \in \mathbb{R}^d$.

Definition 3 (Induced / relative topology). *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$ (not necessarily open!). The* relative *(or* induced*) topology $\mathcal{T}_A$ on A is*

$$\mathcal{T}_A := \Big\{B \subseteq A \,\Big|\, \exists C \in \mathcal{T} \text{ with } B = C \cap A\Big\} \subseteq \mathcal{P}(A)\,.$$

Remark 2. *1. $\mathcal{T}_A$ is a topology on A.*

2. If $A \notin \mathcal{T}$ and $B \in \mathcal{T}_A$, then it may happen that $B \notin \mathcal{T}$. Example: Let $X = \mathbb{R}$ with the standard topology $\mathcal{T}_E$, $A = [0,1]$. Then $B := [0, \frac{1}{2}[\in \mathcal{T}_A$ but $B \notin \mathcal{T}_E$.

Definition 4 (Closed set; neighbourhood; Hausdorff space; limit/accumulation point; closure; interior point; interior; boundary point; boundary; dense). *Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$, $x \in X$.*

(i) The set A is called closed *(in X, or, correctly, in $(X, \mathcal{T})$) iff $A^c := X \setminus A \in \mathcal{T}$ (that is, the* complement *A^c of A is open (in X)).*

(ii) A set $U \subseteq X$ (not necessarily open) is called a neighbourhood *of x iff $\exists A \in \mathcal{T}$ such that $x \in A$ and $A \subseteq U$.*

(iii) The (topological) space X (or, correctly, $(X, \mathcal{T})$) is called a Hausdorff space *iff For all $x, y \in X, x \neq y$, there exist neigbourhoods U_x of x and U_y of y such that $U_x \cap U_y = \emptyset$.*

(iv) The point x is called a limit point *of A (or* accumulation point*) iff For all neighbourhoods U of x: $U \cap A \neq \emptyset$.*

(v) The closure *$\overline{A}$ of A is $\overline{A} := \{x \in X \,|\, x$ limit point of $A\}$.*

(vi) The point x is called a boundary point *of A iff For all neighbourhoods U of x: $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$.*

(vii) The boundary *∂A of A is $\partial A := \{x \in X \,|\, x$ boundary point of $A\}$.*

(viii) The point x is called an interior *point of A iff There exists a neigbourhood U of x such that $U \subseteq A$.*

(ix) The interior *A° of A is $A^\circ := \{x \in X \,|\, x$ interior point of $A\}$.*

(x) The set A is dense *in X (correctly, in $(X, \mathcal{T})$) iff $\overline{A} = X$.*

Remark 3.

1. According to this definition, every point of A is a limit point of A. Hence, $A \subseteq \overline{A}$.

2. Note that $\overline{A} = A \cup \partial A$ and $A^\circ = A \setminus \partial A$.

Lemma 1. *Let X be a topological space, $A \subseteq X$.*

(i) A is open iff $\forall x \in A : x$ is an interior point of A.

(ii) A is closed iff $A = \overline{A}$.

(iii) $\overline{A}$ and ∂A are closed.

Definition 5 (Base; subbase; neighbourhood base). *Let $(X, \mathcal{T})$ be a topological space.*

(i) A family $\mathcal{B} \subseteq \mathcal{T}$ is called a base *(or* basis*) for (the topology) $\mathcal{T}$ iff $\mathcal{T}$ consists of unions of sets from $\mathcal{B}$.*

(ii) A family $\mathcal{S} \subseteq \mathcal{T}$ is called a subbase *(or* subbasis*) for (the topology) $\mathcal{T}$ iff finite intersections of sets from $\mathcal{S}$ form a base for $\mathcal{T}$.*

(iii) A family $\mathcal{N} \subseteq \mathcal{T}$ is called a neighbourhood base *(or* neighbourhood basis*) at $x \in X$ iff Every $N \in \mathcal{N}$ is a neighbourhood of x and for every neighbourhood U of x there exists $N \in \mathcal{N}$ with $N \subseteq U$.*

Example 2. Consider $\mathbb{R}^d$ with the standard topology. Let $x \in \mathbb{R}^d$.

1. The family $\{B_{1/n}(x) \,|\, n \in \mathbb{N}\}$ is a neighbourhood base at x.
2. The family $\{B_{1/n}(q) \,|\, n \in \mathbb{N}, q \in \mathbb{Q}^d\}$ is a base for the standard topology.

Lemma 2. *Let $\mathcal{S} \subseteq \mathcal{P}(X)$. Then there exists a topology $\mathcal{T}(\mathcal{S})$ on X such that*

(i) $\mathcal{S}$ is a subbase for $\mathcal{T}(\mathcal{S})$.

(ii) $\mathcal{T}(\mathcal{S})$ is the coarsest topology containing $\mathcal{S}$. That is,

$$\mathcal{T}(\mathcal{S}) = \bigcap_{\mathcal{T} \supseteq \mathcal{S};\, \mathcal{T} \text{ topology}} \mathcal{T}.$$

The topology $\mathcal{T}(\mathcal{S})$ is called the topology generated *by $\mathcal{S}$.*

Definition 6 (Product topology). *Let $I \neq \emptyset$ be an arbitrary (!) index set. For every $\alpha \in I$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space. The* product topology *on the* Cartesian product space*,*

$$\bigtimes_{\alpha \in I} X_\alpha := \Big\{ f : I \to \bigcup_{\alpha \in I} X_\alpha \,\Big|\, \forall \alpha \in I : f(\alpha) \in X_\alpha \Big\} \tag{1}$$

is the one generated by the family

$$\Big\{ \bigtimes_{\alpha \in I} A_\alpha \,\Big|\, \forall \alpha \in I : A_\alpha \in \mathcal{T}_\alpha, \text{ and } A_\alpha \neq X_\alpha \text{ for at most finitely many } \alpha\text{'s} \Big\}.$$

Remark 4. *If I is finite, then the condition '$A_\alpha \neq X_\alpha$ for at most finitely many α's' is always fulfilled.*

Definition 7 (Metric; metric space). *A map $d : X \times X \to [0, \infty[$ is called a* metric *(on X) iff*

(i) $d(x, y) \geq 0 \quad \forall x, y \in X$, with $d(x, y) = 0$ iff $x = y$.

(ii) $d(x, y) = d(y, x) \quad \forall x, y \in X$.

(iii) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$.

If d is a metric on X, then (X, d) is called a metric space *(often just X).*

Definition 8 (Induced metric; open (metric) ball; open set; Cauchy sequence; convergent sequence; limit; complete (metric) space).
Let (X, d) be a metric space.

(i) Let $Y \subseteq X$. The map $d_Y := d|_{Y \times Y} : Y \times Y \to [0, \infty[$ is called the induced metric *(on Y).*

(ii) *The set* $B_\varepsilon(x) := \{y \in X \,|\, d(x,y) < \varepsilon\}$ *is called the* open (metric) ball *of* radius $\varepsilon > 0$ *about* x. *(More correctly:* $B_\varepsilon(x; d)$*.)*

(iii) *A set* $A \subseteq X$ *is called* open *(in the metric space* (X, d)*) iff For every* $x \in A$ *there exists* $\varepsilon > 0$ *such that* $B_\varepsilon(x) \subseteq A$.

(iv) *A sequence* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *is called a* Cauchy sequence *(or, is said to* be Cauchy *(!)) (in* X*, or, in* (X, d)*) iff*

For every $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that for every* $n, m \geq N : d(x_n, x_m) < \varepsilon$.

(v) *A sequence* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *is called* convergent *(or, is said to* converge*) (in* X*, or, in* (X, d)*) iff There exists* $x \in X$ *such that* $\lim_{n\to\infty} d(x_n, x) = 0$ *(which is equivalent to:* $\forall\varepsilon > 0 \exists n_0 \in \mathbb{N} : x_n \in B_\varepsilon(x)$*). In this case,* $(x_n)_{n\in\mathbb{N}}$ *is said to* converge to x*, and* x *is called its* limit. *We write* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x, n \to \infty$.

(vi) *The space* X *(more correctly,* (X, d)*) is called* complete *(is a* complete metric space*) iff every Cauchy sequence in* X *converges. That is: For all Cauchy sequences* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *there exists* $x \in X$ *(!) such that* $\lim_{n\to\infty} x_n = x$.

Remark 5. *Completeness is not a topological notion! See Exercise (??)*

Definition 9. *Let* (X, d) *be a metric space, let* $A \subseteq X, x \in X$.

(i) *The (extended) number* $\operatorname{diam}(A) := \sup_{a,b\in A} d(a, b) \in [0, \infty]$ *is called the* diameter *of* A.

(ii) *The number* $\operatorname{dist}(x, A) := \inf_{a\in A} d(x, a)$ *is called the* distance *of* x *to* A.

Lemma 3. *Let* X *be a complete metric space, and let* $A \subseteq X$*. Then* A *is closed iff* A *is complete.*
(More precisely/correctly: Assume (X, d) *is a complete metric space,* $A \subseteq X$*. Then* A *is closed in* (X, d) *iff* (A, d_Y) *is a complete metric space.*