LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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This is an overview of material on Measure and Integration Theory (Ana3).

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Measure and Integration Theory

Let X be a non-empty set (for example $X = \mathbb{R}^d$ or any subset of $\mathbb{R}^d$), and let $\mathcal{P}(X)$ be the family of all subsets of X (its *power set*).

Definition 1 (**σ-algebra**). *A family of subsets of X, $\mathcal{A} \subset \mathcal{P}(X)$, is called a* σ-algebra *(on X) if and only if ('iff')*

(i) $X \in \mathcal{A}$.

(ii) $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$.

(iii) $\Big(A_j \in \mathcal{A} \;\; \text{for all } j \in \mathbb{N}\Big) \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a measurable space*, and $A \subset X$ is* measurable $:\Leftrightarrow A \in \mathcal{A}$.

Proposition 1 (**Generated σ-algebras; Borel-(σ-)algebra**).

(i) For any family $\mathcal{B} \subset \mathcal{P}(X)$ there exists a smallest σ-algebra $\sigma(\mathcal{B})$ containing $\mathcal{B}$ (that is, $\sigma(\mathcal{B}) \supset \mathcal{B}$, and if $\mathcal{C}$ is a σ-algebra with $\mathcal{C} \supset \mathcal{B}$, then $\mathcal{C} \supset \sigma(\mathcal{B})$), given by

$$\sigma(\mathcal{B}) := \bigcap_{\mathcal{A} \subset \mathcal{P}(X), \mathcal{A} \ \sigma\text{-algebra}, \mathcal{A} \supset \mathcal{B}} \mathcal{A}\,. \tag{1}$$

We call $\sigma(\mathcal{B})$ the σ-algebra generated by $\mathcal{B}$.

(ii) Let $(X, \mathcal{T})$ be a topological space (for example, a metric space (X, d) with the topology $\mathcal{T}_d$ generated by the metric d). The σ-algebra $\sigma(\mathcal{T})$ is called the Borel-σ-algebra *(or* Borel-algebra*) (on $(X, \mathcal{T})$), denoted $\mathcal{B}(X)$ (more correct would be: $\mathcal{B}(X, \mathcal{T})$), and $B \subset X$ is* a Borel-set *(or* Borel *or* Borel-measurable*)* $:\Leftrightarrow B \in \mathcal{B}(X)$.

(iii) For a measurable space $(X, \mathcal{A})$ and a subset $B \subset X$ (not necessarily measurable), the induced σ-algebra *(or* trace-σ-algebra*) on B is defined by $\mathcal{A}_B := \{B \cap A \,|\, A \in \mathcal{A}\}$. If $B \in \mathcal{A}$, then $\mathcal{A}_B \subset \mathcal{A}$.*

Example 1 (**Borel-algebra on $\mathbb{R}^d$, $\mathbb{R}_{\geq 0}$, $\overline{\mathbb{R}}$, $\overline{\mathbb{R}}_{\geq 0}$**). Let $X := \mathbb{R}^d$ with the usual topology $\mathcal{T}_{\text{Eucl}}$, generated by the Euclidean metric $|\cdot|$. We denote $\mathcal{B}^d := \mathcal{B}(\mathbb{R}^d) := \sigma(\mathcal{T}_{\text{Eucl}})$ the Borel-algebra on $\mathbb{R}^d$, and write $\mathcal{B} := \mathcal{B}^1$ when there is no risk of confusion. We denote $\mathcal{B}_{\geq 0} := \mathcal{B}^1{}_{\mathbb{R}_{\geq 0}} := \{\mathbb{R}_{\geq 0} \cap A \,|\, A \in \mathcal{B}^1\}$. It is the Borel-algebra of $\mathbb{R}_{\geq 0}$ (with the topology on $\mathbb{R}_{\geq 0}$ the one induced from $\mathbb{R}$). For $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, we denote $\mathcal{B}(\overline{\mathbb{R}}) := \sigma(\mathcal{B}^1 \cup \{\{-\infty\}\} \cup \{\{+\infty\}\})$. It is the Borel-algebra on $\overline{\mathbb{R}}$ for the usual topology on $\overline{\mathbb{R}}$. Finally, $\overline{\mathcal{B}}_{\geq 0} := \{\overline{\mathbb{R}}_{\geq 0} \cap A \,|, A \in \mathcal{B}(\overline{\mathbb{R}})\}$ ($= \mathcal{B}(\overline{\mathbb{R}}_{\geq 0})$).

Definition 2 (**(Positive) measure**).

(i) Let $\mathcal{A}$ be a σ-algebra on X. A map $\mu : \mathcal{A} \to [0, \infty]$ is called a (positive) measure (on X, or on $(X, \mathcal{A})$) *iff*

  (i) $\mu(\emptyset) = 0$.

  (ii) For all $A_j \in \mathcal{A}$, $j \in \mathbb{N}$, with $A_j \cap A_k = \emptyset$ for $j \neq k$:

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j) \quad (\sigma\text{-additivity})\,. \tag{2}$$

  The triple $(X, \mathcal{A}, \mu)$ is called a measure space*.*

(ii) A measure μ (or, more correctly, a measure space $(X, \mathcal{A}, \mu)$) is called finite *(or* bounded*) iff $\mu(X) < \infty$, and σ-*finite *iff there exists $(X_j)_{j\in\mathbb{N}}$, $X_j \in \mathcal{A}$, $X = \bigcup_{j=1}^{\infty} X_j$, with $\mu(X_j) < \infty$ for all $j \in \mathbb{N}$.*

Example 2 (**Lebesgue-Borel measure on $\mathbb{R}^d$**). There exists a unique measure, called the *Lebesgue-Borel measure* λ^d, on $\mathcal{B}^d$ so that for all rectangles $Q := \bigtimes_{j=1}^{d} [a_j, b_j) \subset \mathbb{R}^d$, $-\infty \leq a_j \leq b_j \leq \infty$, holds

$$\lambda^d(Q) = \prod_{j=1}^{d} (b_j - a_j)\,. \tag{3}$$

Furthermore, λ^d is translation and rotation invariant, and σ-finite.

Definition 3 (**(μ-)Null sets; complete measure**). *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

(i) A subset $N \subset X$ is called a (μ-)null set *iff $N \in \mathcal{A}$ and $\mu(N) = 0$.*

(ii) $(X, \mathcal{A}, \mu)$ (or just μ) is called complete *iff all subsets of null sets are null sets.*

Theorem 1 (**Completion of measure**). *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then there exists a smallest complete measure space $(X, \overline{\mathcal{A}}, \overline{\mu})$ (called* the completion of $(X, \mathcal{A}, \mu)$*) containing $(X, \mathcal{A}, \mu)$ (that is, $\overline{\mathcal{A}} \supset \mathcal{A}$, $\overline{\mu}|_{\mathcal{A}} = \mu$, and $\overline{\mu}$ is complete).*

Example 3 (**Lebesgue measure on $\mathbb{R}^d$**). The completion of $(\mathbb{R}^d, \mathcal{B}^d, \lambda^d)$ (which is *not* in itself complete) is denoted $(\mathbb{R}^d, \overline{\mathcal{B}^d}, \overline{\lambda^d})$. Elements of $\overline{\mathcal{B}^d}$ are called *Lebesgue-measurable (subsets of $\mathbb{R}^d$)*, and $\overline{\lambda^d}$ is called *($d-$dimensional) Lebesgue measure*. One has

$$\overline{\mathcal{B}^d} = \Big\{ B \cup \widetilde{N} \,\Big|\, B \in \mathcal{B}^d, \exists N \in \mathcal{B}^d \text{ with } \lambda^d(N) = 0, \widetilde{N} \subset N \Big\}\,. \tag{4}$$

Furthermore, $A \in \overline{\mathcal{B}^d}$ iff for all $\varepsilon > 0$ there exists $U \subset \mathbb{R}^d$ open and $C \subset \mathbb{R}^d$ closed, with $C \subset A \subset U$, such that $\lambda^d(U \setminus C) < \varepsilon$.
Note, in particular, that $\mathcal{B}^d \subsetneq \overline{\mathcal{B}^d} \subsetneq \mathcal{P}(\mathbb{R}^d)$.

Definition 4 (Measurable maps and functions).

(i) *Let* $(X,\mathcal{A})$*,* $(Y,\mathcal{C})$ *be measurable spaces. A map* $f : X \to Y$ *is called (*$\mathcal{A}$*-*$\mathcal{C}$*-)*measurable *iff* $f^{-1}(C) \in \mathcal{A}$ *for all* $C \in \mathcal{C}$*. We denote by* $\mathcal{M}(X,Y)$ *the set of all* $\mathcal{A}$*-*$\mathcal{C}$*-measurable maps. (More correct would be* $\mathcal{M}((X,\mathcal{A}),(Y,\mathcal{C}))$*.)*

(ii) *In the special case* $(Y,\mathcal{C}) = (\mathbb{R},\mathcal{B}^1)$*, we denote* $\mathcal{M}(X) := \mathcal{M}(X,\mathbb{R})$ *the set of all measurable* functions $f : X \to \mathbb{R}$*, and* $\mathcal{M}_+(X) := \{ f \in \mathcal{M}(X) \mid f \geq 0 \}$*. Note that* $\mathcal{M}_+(X) = \mathcal{M}(X,\mathbb{R}_{\geq 0}) = \mathcal{M}((X,\mathcal{A}),(\mathbb{R}_{\geq 0},\mathcal{B}_{\geq 0}))$*.*

(iii) *In the special case* $(Y,\mathcal{C}) = (\overline{\mathbb{R}},\mathcal{B}(\overline{\mathbb{R}}))$*, we denote* $\overline{\mathcal{M}}(X) := \mathcal{M}(X,\overline{\mathbb{R}})$ *the set of all measurable* numerical *(or* extended real-valued*)* functions $f : X \to \overline{\mathbb{R}}$*, and* $\overline{\mathcal{M}}_+(X) := \{ f \in \overline{\mathcal{M}}(X) \mid f \geq 0 \}$*. Again,* $\overline{\mathcal{M}}_+(X) = \mathcal{M}(X,\overline{\mathbb{R}}_{\geq 0})$*.*

(iv) *We denote by* $\mathcal{M}(X,\mathbb{C})$ *the set of all complex functions* $f : X \to \mathbb{C}$ *such that* $\Re(f), \Im(f) \in \mathcal{M}(X,\mathbb{R})$*.*

Remark 1. *One has*

$$\mathcal{M}(X) = \Big\{ f : X \to \mathbb{R} \,\Big|\, f^{-1}\Big((-\infty,a)\Big) \in \mathcal{A} \text{ for all } a \in \mathbb{R} \Big\}, \tag{5}$$

and similarly with $(-\infty,a)$ *replaced with* $(-\infty,a]$*,* (a,∞)*, or* $[a,\infty)$*. Analogous statements hold for* $\overline{\mathcal{M}}(X)$*.*
We denote, for $a \in \overline{\mathbb{R}}$*,*

$$\{f < a\} := f^{-1}\Big((-\infty,a)\Big) = \{ x \in X \mid f(x) \in (-\infty,a) \}, \tag{6}$$

and similarly for other types of intervals.

Definition 5 (Distribution function).
Let $(X,\mathcal{A},\mu)$ *be a measure space. For* $f \in \overline{\mathcal{M}}(X)$*, we call the function* $\mu_f : \mathbb{R} \to \mathbb{R}$ *given by*

$$\mu_f(t) := \mu(\{f > t\}) = \mu\Big(\{ x \in X \mid f(x) > t \}\Big) \tag{7}$$

the distribution function of f (relative to μ).

Definition 6 (Almost everywhere (a.e.); $f = g$ **a.e.).**
Let $(X,\mathcal{A},\mu)$ *be a measure space.*

(i) *A mathematical statement* $Q = Q(x)$ *(which is assumed to make sense for all* $x \in X$*) is said to* hold (μ-) almost everywhere (a.e., *or* μ-a.e.) *iff there exists a (*μ*-)null set* N *such that* $Q(x)$ *is true/holds for all* $x \in X \setminus N$*.*

(ii) *Let* $f, g : X \to \mathbb{M}$ *(*$\mathbb{M} \in \{\mathbb{R},\mathbb{R}_{\geq 0},\overline{\mathbb{R}},\overline{\mathbb{R}}_{\geq 0},\mathbb{C}\}$*) be measurable, then* $f = g$ a.e. *iff* $f(x) = g(x)$ *a.e. This defines an equivalence relation* $\sim_{a.e.}$ *on* $\mathcal{M}(X,\mathbb{M})$*.*

Definition 7 (Step functions).
Let $(X,\mathcal{A})$ *be a measurable space. A function* $f : X \to \mathbb{R}$ *is called a* step function *iff there exists* $N \in \mathbb{N}$*,* $A_1,\ldots,A_N \in \mathcal{A}$*, and* $a_1,\ldots,a_N \in \mathbb{R}$ *such that*

$$f = \sum_{n=1}^{N} a_n \mathbb{1}_{A_n} . \tag{8}$$

Here, $\mathbb{1}_B$ *is the* characteristic *(or* indicator*)* function of (the set) B*, given by* $\mathbb{1}_B(x) = 1$ *if* $x \in B$*, and equal* 0 *otherwise* ($x \notin B$)*.*
Note that step functions are measurable by definition. We denote the set of all non-negative step functions by

$$E_+ := \Big\{ f : X \to \mathbb{R} \,\Big|\, f \geq 0\,,\ f \text{ step function } \Big\}\,. \tag{9}$$

Theorem 2 (**Approximating measurable functions by step functions).** *Let* $(X, \mathcal{A})$ *be a measurable space. Then* $f \in \mathcal{M}(X, \overline{\mathbb{R}})$ *iff there exists a sequence* $(f_n)_{n\in\mathbb{N}}$ *of step functions* $f_n : X \to \mathbb{R}$ *with* $f = \lim_{n\to\infty} f_n$ *(pointwise on* X*). If* $f \in \mathcal{M}+(X)$*, then the sequence can be chosen monotone* ($f_n \nearrow f$)*, and if* f *is a bounded function, then the sequence can be chosen such that the convergence is uniform on* X*.*

Definition 8 (**Definition and properties of Lebesgue integral).**
Let $(X, \mathcal{A}, \mu)$ *be a measure space.*

1. *Let* $f \in E_+$ ($f \geq 0$*,* f *step function), with* $f = \sum_{n=1}^{N} a_n \mathbb{1}_{A_n}$*,* $A_n \in \mathcal{A}, a_n \in \mathbb{R}$*. Then*

$$\int_X f \,\mathrm{d}\mu := \int_X f(x) \,\mathrm{d}\mu(x) := \sum_{n=1}^{N} a_n \mu(A_n) \in [0, \infty] \tag{10}$$

   is the (μ-)integral of f over X*. It is independent of the representation in* (8)*.*

2. *Let* $f \in \overline{\mathcal{M}}_+(X)$ ($f : X \to [0, \infty]$*, measurable), and let* $(f_n)_{n\in\mathbb{N}} \subset E_+$ *be an approximating sequence as in Theorem 2. Then*

$$\int_X f \,\mathrm{d}\mu := \int_X f(x) \,\mathrm{d}\mu(x) := \lim_{n\to\infty} \Big(\int_X f_n(x) \,\mathrm{d}\mu(x) \Big) \in [0, \infty] \tag{11}$$

   is the (μ-)integral of f over X*. The limit is well-defined, since the sequence* $(\int_X f_n \,\mathrm{d}\mu)_{n\in\mathbb{N}} \subset [0, \infty]$ *is non-decreasing. The limit is independent of the chosen sequence* $(f_n)_{n\in\mathbb{N}}$*.*

3. *For* $f : X \to \overline{\mathbb{R}}$*, let* $f_\pm := \max\{\pm f, 0\}$ *(so* $f = f_+ - f_-, |f| = f_+ + f_-$*). Then* f *is* (μ-)integrable over X $:\Leftrightarrow f \in \overline{\mathcal{M}}(X)$ *and* $\int_X f_+ \,\mathrm{d}\mu < \infty$*,* $\int_X f_- \,\mathrm{d}\mu < \infty$*. In this case,*

$$\int_X f \,\mathrm{d}\mu := \int_X f(x) \,\mathrm{d}\mu(x) := \int_X f_+ \,\mathrm{d}\mu - \int_X f_- \,\mathrm{d}\mu \in \mathbb{R} \tag{12}$$

   is the (μ-)integral of f over X*. We denote the set of integrable functions by*

$$\mathcal{L}^1 := \mathcal{L}^1(X) := \mathcal{L}^1(\mu) := \mathcal{L}^1(X, \mu) := \mathcal{L}^1(X, \mathcal{A}, \mu) := \Big\{ f : X \to \mathbb{R} \,\Big|\, f \ \mu\text{-integrable } \Big\}\,,$$
$$\overline{\mathcal{L}^1} := \overline{\mathcal{L}^1}(X) := \overline{\mathcal{L}^1}(\mu) := \overline{\mathcal{L}^1}(X, \mu) := \overline{\mathcal{L}^1}(X, \mathcal{A}, \mu) := \{ f : X \to \overline{\mathbb{R}} \,|\, f \ \mu\text{-integrable } \}\,.$$

4. *For* $A \in \mathcal{A}$*, and* $f \in \mathcal{L}^1$ *(or* $f \in \overline{\mathcal{L}^1}$*), let* $\int_A f \,\mathrm{d}\mu := \int_X f \mathbb{1}_A \,\mathrm{d}\mu$*.*

5. *Properties of the integral:*

   (a) *For* $f \in \overline{\mathcal{L}^1}$*,* $f \geq 0$*, one has:* $\int_X f \,\mathrm{d}\mu = 0 \Leftrightarrow f = 0$ μ*-a.e.*

   (b) *The map* $f \mapsto \int_X f \,\mathrm{d}\mu$ *from* $\overline{\mathcal{L}^1}$ *to* $\mathbb{R}$ *is linear and monotone.*

(c) *For* $f \in \overline{\mathcal{L}^1}$,

$$\Big|\int_X f\,\mathrm{d}\mu\Big| \le \int_X |f|\,\mathrm{d}\mu \quad \text{(triangle inequality)}\,. \tag{13}$$

(d) *For* $f \in \overline{\mathcal{L}^1}$, $f \ge 0$, *and all* $\varepsilon > 0$,

$$\mu(\{f \ge \varepsilon\}) \le \frac{1}{\varepsilon}\int_X f\,\mathrm{d}\mu \quad \text{(Chebyshev's inequality)}\,. \tag{14}$$

Proposition 2 (Riemann versus Lebesgue interal in $\mathbb{R}$).
For $f : [a,b] \to \mathbb{R}$ $(a,b \in \mathbb{R}, a < b)$ *Riemann-integrable, denote* $\int_a^b f(x)\,dx$ *the Riemann-integral of* f *over* $[a,b]$.

1. *For* $f : [a,b] \to \mathbb{R}$ *Riemann-integrable there exists* $g : \mathbb{R} \to \mathbb{R}$ *measurable, with* $f = g$ *a.e. on* $[a,b]$ *such that*

$$\int_a^b f(x)\,dx = \int_{[a,b]} g(x)\,\mathrm{d}\lambda^1(x)\,. \tag{15}$$

2. *Let* $f : [0,\infty) \to \mathbb{R}$ *be measurable, and continuous on* $(0,\infty)$. *Then*

$$\int_{\mathbb{R}} f\mathbb{1}_{[1,\infty)}\,\mathrm{d}\lambda^1 = \lim_{n\to\infty}\int_1^n f(x)\,dx\,, \tag{16}$$

$$\int_{\mathbb{R}} f\mathbb{1}_{[0,1]}\,\mathrm{d}\lambda^1 = \lim_{n\to\infty}\int_{1/n}^1 f(x)\,dx\,. \tag{17}$$

In particular,

$$\int_{[0,1]} x^a\,\mathrm{d}\lambda^1(x) < \infty \;\Leftrightarrow\; a > -1\,, \tag{18}$$

$$\int_{[1,\infty)} x^b\,\mathrm{d}\lambda^1(x) < \infty \;\Leftrightarrow\; b < -1\,. \tag{19}$$

Also,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} = \lim_{R\to\infty}\Big(\int_{[0,R]}\frac{\sin x}{x}\,\mathrm{d}\lambda^1(x)\Big)\,, \tag{20}$$

$$\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}\lambda^1(x) = \sqrt{\pi}\,. \tag{21}$$

In what follows, let $(X,\mathcal{A},\mu)$ be any measure space.

Definition 9 (Essential supremum). *For a measurable function* $f : X \to \overline{\mathbb{R}}$ *the* essential supremum of f *is*

$$\begin{aligned}\operatorname{ess\,sup} f = \operatorname{ess\,sup}_X f &= \inf\{s \in \overline{\mathbb{R}} \,|\, f(x) \le s \quad \mu\text{-a.e}\,\} \\ &= \inf\Big\{\sup_{x\in X\setminus N} f(x)\,\Big|\, N \subset X, N \;\mu\text{-null set}\,\Big\}\,.\end{aligned} \tag{22}$$

Definition 10 (The semi-normed spaces $\mathcal{L}^p(X)$, $p \in [1,\infty]$).

(i) *For* $p \in [1, \infty)$*, let*

$$\begin{aligned}\mathcal{L}^p &:= \mathcal{L}^p(X) := \mathcal{L}^p(\mu) := \mathcal{L}^p(X,\mu) := \mathcal{L}^p(X,\mathcal{A},\mu)\\ &:= \Big\{ f : X \to \mathbb{C} \,\Big|\, f \in \mathcal{M}(X,\mathbb{C})\,,\ \int_X |f|^p \,\mathrm{d}\mu < \infty \Big\}\end{aligned} \tag{23}$$

and, for $f \in \mathcal{L}^p(X)$*, let*

$$\|f\|_p := \Big(\int_X |f|^p \,\mathrm{d}\mu < \infty \Big)^{1/p}. \tag{24}$$

(ii) *For* $p = \infty$*, let*

$$\begin{aligned}\mathcal{L}^\infty &:= \mathcal{L}^\infty(X) := \mathcal{L}^\infty(\mu) := \mathcal{L}^\infty(X,\mu) := \mathcal{L}^\infty(X,\mathcal{A},\mu)\\ &:= \Big\{ f : X \to \mathbb{C} \,\Big|\, f \in \mathcal{M}(X,\mathbb{C})\,,\ \operatorname{ess\,sup}_X |f| < \infty \Big\}\,,\end{aligned} \tag{25}$$

and, for $f \in \mathcal{L}^\infty(X)$*, let*

$$\|f\|_\infty := \operatorname{ess\,sup}_X |f|\,. \tag{26}$$

Then, for all $p \in [1, \infty]$*,* $\|\cdot\|_p$ *is a* semi-*norm on* $\mathcal{L}^p(X)$*:* $\|f\|_p = 0 \Leftrightarrow f \sim_{a.e.} 0$ *(which does not mean* $f = 0$*).*

Theorem 3 (**Minkowski and (generalised) Hölder inequalities**)**.**

(i) (Minkowski) *Let* $p \in [1, \infty]$*, then* $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ *for all* $f, g \in \mathcal{L}^p(X)$*.*

(ii) (Hölder) *Let* $p, q \in [1, \infty]$*, with* $\frac{1}{p} + \frac{1}{q} = 1$*. Then, for all* $f \in \mathcal{L}^p(X), g \in \mathcal{L}^q(X)$*,*

$$\int_X |fg| \,\mathrm{d}\mu \leq \|f\|_p \|g\|_q\,. \tag{27}$$

(iii) (Generalied Hölder) *Let* $n \in \mathbb{N}$ *(*$n \geq 2$*), and let* $p_1, \dots, p_n \in [1, \infty]$*, and let* $p \in [1, \infty]$ *satisfy* $\frac{1}{p} = \sum_{j=1}^n \frac{1}{p_j}$*. Then, for all* $f_j \in \mathcal{L}^{p_j}(X)$*,* $j = 1, \dots, n$*,*

$$\Big\| \prod_{j=1}^n f_j \Big\|_p \leq \prod_{j=1}^n \|f_j\|_{p_j}\,. \tag{28}$$

(iv) (Interpolation in $\mathcal{L}^p$-spaces)*. Let* $1 \leq p < r < q \leq \infty$*,* $f \in \mathcal{L}^p(X) \cap \mathcal{L}^q(X)$*. Let* $\theta \in (0, 1)$ *with* $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$*. Then* $f \in \mathcal{L}^r(X)$*, and*

$$\|f\|_r \leq \|f\|_p^\theta \, \|f\|_q^{1-\theta}\,. \tag{29}$$

Hence, for $f : X \to \mathbb{C}$ *measurable, the set*

$$\Gamma_f := \{ p \in [1, \infty] \,|\, f \in \mathcal{L}^p(X) \} \subset \mathbb{R} \tag{30}$$

is an interval.

(v) *Let* $p \in [1, \infty]$*,* $f \in \mathcal{L}^p(X) \cap \mathcal{L}^\infty(X)$*. Then* $f \in \cap_{q \geq p} \mathcal{L}^q(X)$*, and* $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$*.*

Theorem 4 (**The normed spaces** $L^p(X)$, $p \in [1,\infty]$)**.**
For $p \in [1,\infty]$*, the relation* $\sim_{a.e.}$ *defines an equivalence relation on* $\mathcal{L}^p(X)$*, and* $\|\cdot\|_p$ *defines a norm on the quotient vector space* $L^p(X)$*, which makes* $(L^p(X), \|\cdot\|_p)$ *a Banach space. For* $p = 2$*,* $L^2(X)$ *is a Hilbert space, with inner/scalar product* $\langle f, g\rangle := \int_X \overline{f(x)} g(x)\,\mathrm{d}\mu(x)$*.*

Remark 2. *By abuse of notation we will call* $f \in L^p(X)$ functions *when we should really be talking about equivalence classes (this abuse of notation/language is well established).*

Theorem 5 (**a.e. convergent subsequences**)**.**
Let $p \in [1,\infty]$*, and assume* $(f_j)_{j\in\mathbb{N}} \subset L^p(X), f \in L^p(X)$*, satisfy* $\lim_{j\to\infty} \|f_j - f\|_p = 0$*. Then there exists a subsequence* $(f_{j_k})_{k\in\mathbb{N}}$ *with* $\lim_{k\to\infty} f_{j_k}(x) = f(x)$ *a.e., that is, the subsequence* $(f_{j_k})_{k\in\mathbb{N}}$ *converges pointwise to* f *for* μ*-almost every* $x \in X$*.*

Theorem 6 (**Denseness of step functions in** $L^p(X)$)**.**
Let $p \in [1,\infty)$*, then the linear subspace of step functions,*

$$\begin{aligned} E :&= \operatorname{span}\{\mathbb{1}_A \mid A \in \mathcal{A}, \mu(A) < \infty\} \qquad (31)\\ &= \{g : X \to \mathbb{C} \mid g = \sum_{n=1}^{N} a_n \mathbb{1}_{A_n}, N \in \mathbb{N}, A_1, \ldots, A_N \in \mathcal{A}, \mu(A_j) < \infty, a_1, \ldots, a_N \in \mathbb{C}\} \end{aligned}$$

is dense in $L^p(X)$*: For all* $f \in L^p(X)$ *and all* $\varepsilon > 0$*, there exists* $g \in E$ *such that* $\|f - g\|_p < \varepsilon$*.*

Definition 11 (**Locally integrable functions**)**.** *Let* $(X, \mathcal{T})$ *be a topological space, and let* μ *be a measure on* $(X, \sigma(\mathcal{T}))$*. (Example:* $\mathbb{R}^d$ *with Lebesque(-Borel) measure.) For* $p \in [1,\infty]$*, we denote*

$$L^p_{\mathrm{loc}}(X) := \Big\{ f : X \to \mathbb{C} \,\Big|\, f \in \mathcal{M}(X, \mathbb{C}),\, f \in L^p(K) \text{ for all } K \subset X \text{ compact} \Big\}. \qquad (32)$$

Theorem 7 (**Monotone convergence / Beppo Levi**)**.** *Let* $(f_j)_{j\in\mathbb{N}}$*,* $f_j : X \to \mathbb{R}$*, be a sequence of measurable functions with*

$$0 \le f_1 \le f_2 \le f_3 \le \ldots. \qquad (33)$$

Then, with $f(x) := \lim_{j\to\infty} f_j(x)$*,*

$$\lim_{j\to\infty} \int_X f_j \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu\,. \qquad (34)$$

The possibility that both sides are $+\infty$ *is included.*

Theorem 8 (**(Lebesgue) Dominated convergence**)**.** *Let* $(f_j)_{j\in\mathbb{N}}$*,* $f_j : X \to \mathbb{R}$*, be a sequence of measurable functions. Assume there exists* $g \in L^1(X)$ *such that* $|f_j(x)| \le g(x)$ *for a.e.* $x \in X$ *and all* $j \in \mathbb{N}$*, and that* $f(x) := \lim_{j\to\infty} f_j(x)$ *exists a.e. on* X*.*
Then

$$\lim_{j\to\infty} \int_X f_j \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu\,. \qquad (35)$$

In this case both sides are finite.

Theorem 9 (**Fatou's Lemma**)**.** *Let* $(f_j)_{j\in\mathbb{N}}$*,* $f_j : X \to \mathbb{R}$*, be a sequence of measurable functions, with* $f_j(x) \ge 0$ *a.e. on* X *for all* $j \in \mathbb{N}$*. Then*

$$\int_X \Big(\liminf_{j\in\mathbb{N}} f_j\Big) \,\mathrm{d}\mu \le \liminf_{j\in\mathbb{N}} \Big(\int_X f_j \,\mathrm{d}\mu\Big)\,. \qquad (36)$$

Theorem 10 (Continuity and differentiability of parameter-dependent integrals). *Let (M,d) be a metric space, $(X,\mathcal{A},\mu)$ a measure space, and $f : M \times X \to \mathbb{R}$ a map satisfying*
(i) *The map $x \mapsto f(t,x)$ is integable for all $t \in M$.*
Let $F : M \to \mathbb{R}$ be given by $F(t) := \int_X f(t,x)\,\mathrm{d}\mu(x)$.

1. *Let $t_0 \in M$, and assume furthermore:*

   (ii) *The map $t \mapsto f(t,x)$ is continuous at t_0 for all $x \in X$.*

   (iii) *There exists integrable an function $g : X \to [0,\infty]$ such that $|f(t,x)| \leq g(x)$ for all $t \in M$ and $x \in X$.*

   Then F is continuous at t_0:

$$\begin{aligned}\lim_{t\to t_0} F(t) &= \lim_{t\to t_0}\Big(\int_X f(t,x)\,\mathrm{d}\mu(x)\Big) = \int_X \Big(\lim_{t\to t_0} f(t,x)\Big)\,\mathrm{d}\mu(x)\\ &= \int_X f(t_0,x)\,\mathrm{d}\mu(x) = F(t_0)\,. \end{aligned} \tag{37}$$

2. *Let $M = I \subset \mathbb{R}$ be an open interval, and assume* (i) *holds. Assume furthermore that*

   (ii') *The map $t \mapsto f(t,x)$ is differentiable on I for all $x \in X$.*

   (iii') *There exists an integrable function $g : X \to [0,\infty]$ such that $|\frac{\partial f}{\partial t}(t,x)| \leq g(x)$ for all $t \in M$ and $x \in X$.*

   Then F is differentiable on I, the map $x \mapsto \frac{\partial f}{\partial t}(t,x)$ is integrable for all $t \in I$, and

$$\frac{d}{dt}\Big(\int_X f(t,x)\,\mathrm{d}\mu(x)\Big) = F'(t) = \frac{dF}{dt}(t) = \int_X \frac{\partial f}{\partial t}(t,x)\,\mathrm{d}\mu(x)\,. \tag{38}$$

Definition 12 (Product-σ-algebra).
Let $(X_j,\mathcal{A}_j)$, $j = 1,\ldots,n$, be measurable spaces. The product-σ-algebra

$$\bigotimes_{j=1}^{n} \mathcal{A}_j := \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n := \sigma(p_1,\ldots,p_n) \tag{39}$$

(on $X := \bigtimes_{j=1}^n X_j$) is the smallest σ-algebra such that the projections *$p_j : X \to X_j$, $x = (x_1,\ldots,x_n) \mapsto x_j$, are all measurable.*

Example 4. $\mathcal{B}^d = \mathcal{B}^1 \otimes \ldots \otimes \mathcal{B}^1$ (d times). However, $\overline{\mathcal{B}^d} \supsetneq \overline{\mathcal{B}^1} \otimes \ldots \otimes \overline{\mathcal{B}^1}$.

Theorem 11 (Product measure).
Let $(X_j,\mathcal{A}_j,\mu_j)$, $j = 1,\ldots,n$, be σ-finite (!) measure spaces, and let $X := \bigtimes_{j=1}^n X_j$. Then there exists a unique measure $\mu := \otimes_{j=1}^n \mu_j := \mu_1 \otimes \ldots \otimes \mu_n$ (called the product measure (of $\mu_1,\ldots,\mu_n$)*) on $\mathcal{A} := \otimes_{j=1}^n \mathcal{A}_j$ such that*

$$\Big(\bigotimes_{j=1}^{n} \mu_j\Big)\Big(\bigtimes_{j=1}^{n} A_j\Big) = \prod_{j=1}^{n} \mu_j(A_j) \quad \textit{for all } A_j \in \mathcal{A}_j, j = 1,\ldots,n\,. \tag{40}$$

Furthermore, $(X,\mathcal{A},\mu)$ is σ-finite.

Theorem 12 (Fubini-Tonelli).
Let $(X_j, \mathcal{A}_j, \mu_j)$, $j = 1, 2$, be σ-finite (!) measure spaces, and let $(X, \mathcal{A}, \mu) := (X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$. Let $f : X \to \overline{\mathbb{R}}$ (or $\mathbb{C}$) be $\mathcal{A}$-measurable. Then is, for all $g \in \{\Re(f_+), \Re(f_-), \Im(f_+), \Im(f_-)\}$, the functions

$$X_1 \to [0, \infty] \ , \ x_1 \mapsto \int_{X_2} g(x_1, x_2) \, \mathrm{d}\mu_2(x_2) \ , \tag{41}$$

$$X_2 \to [0, \infty] \ , \ x_2 \mapsto \int_{X_1} g(x_1, x_2) \, \mathrm{d}\mu_1(x_1) \tag{42}$$

$\mathcal{A}_1$-measurable, respectively, $\mathcal{A}_2$-measurable. Furthermore,

1. (Tonelli) *If $f \geq 0$ a.e. (that is, $f(X \setminus N) \subset [0, \infty], \mu(N) = 0$), then*

$$\begin{aligned} \int_X f(x) \, \mathrm{d}\mu(x) &= \int_{X_1} \Big(\int_{X_2} f(x_1, x_2) \, \mathrm{d}\mu_2(x_2) \Big) \, \mathrm{d}\mu_1(x_1) \\ &= \int_{X_2} \Big(\int_{X_1} f(x_1, x_2) \, \mathrm{d}\mu_1(x_1) \Big) \, \mathrm{d}\mu_2(x_2) \, . \end{aligned} \tag{43}$$

   Note: It is possible that all three integrals are $+\infty$.

2. (Fubini) *If* one *of the three integrals*

$$\begin{aligned} \int_X |f(x)| \, \mathrm{d}\mu(x) \ , \ \int_{X_1} \Big(\int_{X_2} |f(x_1, x_2)| \, \mathrm{d}\mu_2(x_2) \Big) \, \mathrm{d}\mu_1(x_1) \ , \\ \int_{X_2} \Big(\int_{X_1} |f(x_1, x_2)| \, \mathrm{d}\mu_1(x_1) \Big) \, \mathrm{d}\mu_2(x_2) \end{aligned} \tag{44}$$

   is finite, *then they are* all *finite, and* (43) *holds.*

Theorem 13 (Layer Cake Principle). *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $\mathcal{B}_{\geq 0}$ be the Borel-algebra of $\mathbb{R}_{\geq 0}$. Let ν be a measure on $\mathcal{B}_{\geq 0}$ such that $\phi(t) := \nu([0, t))$ is finite for all $t > 0$, and let $f : X \to \mathbb{R}_{\geq 0}$ be $\mathcal{A}$-$\mathcal{B}_{\geq 0}$-measurable. Then*

$$\int_X \phi(f(x)) \, \mathrm{d}\mu(x) = \int_{\mathbb{R}_{\geq 0}} \mu(\{x \in X \mid f(x) > t\}) \, \mathrm{d}\nu(t) \, . \tag{45}$$

Recall that $\mu_f(t) = \mu(\{f > t\})$ is the distribution function of f relative to μ.
In particular, if $f \in \mathcal{L}^p(X)$, then (by choosing $\mathrm{d}\nu(t) = pt^{p-1} \mathrm{d}\lambda^1(t)$)

$$\int_X |f|^p \, \mathrm{d}\mu = p \int_{\mathbb{R}_{\geq 0}} t^{p-1} \mu(\{|f| > t\}) \, \mathrm{d}\lambda^1(t) \, , \tag{46}$$

and if $f \in \mathcal{L}^1(X)$ with $f \geq 0$, then (by choosing $p = 1$)

$$\int_X f \, \mathrm{d}\mu = \int_{\mathbb{R}_{\geq 0}} \mu(\{f > t\}) \, \mathrm{d}\lambda^1(t) \, . \tag{47}$$

Also (by choosing μ the Dirac measure δ_x at $x \in X$, and $p = 1$),

$$f(x) = \int_{\mathbb{R}_{\geq 0}} \mathbb{1}_{\{f > t\}}(x) \, \mathrm{d}\lambda^1(t) \quad \text{(Layer Cake Representation of } f) \, . \tag{48}$$

Theorem 14 (**Transformation formula for λ^d**).
Let $U \subset \mathbb{R}^d$ be open, and $\varphi : U \to \varphi(U) \subset \mathbb{R}^d$ a diffeomorphism. Then, for all $f \in \mathcal{L}^1(\varphi(U), \lambda^d)$,

$$\int_{\varphi(U)} f(y)\,\mathrm{d}\lambda^d(y) = \int_U f(\varphi(x))\Big|\det(D\varphi(x))\Big|\,\mathrm{d}\lambda^d(x)\,. \tag{49}$$

Lemma 1 (**Notation and certain concrete integrals in $\mathbb{R}^d$**).

1. *For $x \in \mathbb{R}^d$, $r > 0$, we denote $B_r^d(x) := B_r(x) := \{y \in \mathbb{R}^d \,|\, |x-y| < r\}$, and $\omega_d := \lambda^d(B_1(0)) = \lambda^d(\overline{B_1(0)})$. One has $\omega_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$, with $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}\,dt$, $z > 0$, the* Gamma-function.

2. *One has*

$$\int_{B_1(0)} |x|^\alpha \,\mathrm{d}\lambda^d(x) < \infty \;\Leftrightarrow\; \alpha > -d\,, \tag{50}$$

$$\int_{\mathbb{R}^d\setminus B_1(0)} |x|^\alpha \,\mathrm{d}\lambda^d(x) < \infty \;\Leftrightarrow\; \alpha < -d\,, \tag{51}$$

$$\int_{\mathbb{R}^d} \frac{1}{(1+|x|)^\alpha}\,\mathrm{d}\lambda^d(x) < \infty \;\Leftrightarrow\; \alpha > -d\,. \tag{52}$$

Definition 13 (**Spaces of differentiable functions on $\mathbb{R}^d$**). *Denote, for $k \in \mathbb{N}$,*

$$C^0(\mathbb{R}^d) := C(\mathbb{R}^d) := C(\mathbb{R}^d,\mathbb{C}) := \Big\{f : \mathbb{R}^d \to \mathbb{C} \,\Big|\, f \text{ is continuous } \Big\}, \tag{53}$$

$$C^k(\mathbb{R}^d) := \Big\{f : \mathbb{R}^d \to \mathbb{C} \,\Big|\, f \text{ is } k \text{ times continuous differentiable } \Big\}, \tag{54}$$

$$C^\infty(\mathbb{R}^d) := \bigcap_{k\in\mathbb{N}} C^k(\mathbb{R}^d)\,, \tag{55}$$

and define, for $f \in C(\mathbb{R}^d)$, the support of f *by* $\operatorname{supp}(f) := \overline{\{x \in \mathbb{R}^d \,|\, f(x) \neq 0\}}$. *Denote, for $k \in \mathbb{N} \cup \{\infty\}$,*

$$C_c^k(\mathbb{R}^d) := \Big\{f \in C^k(\mathbb{R}^d) \,\Big|\, \operatorname{supp}(f) \subset \mathbb{R}^d \text{ is compact } \Big\}. \tag{56}$$

Theorem 15 (**Denseness of $C_c^k(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$**).

1. *The set $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ with respect to $\|\cdot\|_p$ for $1 \leqslant p < \infty$.*

   More precisely: For all $f \in L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$, and all $\varepsilon > 0$ there exists $\phi \in C_c(\mathbb{R}^d)$ with $\|\phi - f\|_p < \varepsilon$. Note: The result fails in $L^\infty(\mathbb{R}^d)$.

2. *The set $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ with respect to $\|\cdot\|_p$ for $1 \leqslant p < \infty$.*

   Again, the result fails in $L^\infty(\mathbb{R}^d)$.

3. *As a consequence, $C_c^k(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ with respect to $\|\cdot\|_p$ for $1 \leqslant p < \infty$, and all $k \in \mathbb{N} \cup \{\infty\}$.*

   Again, the result fails in $L^\infty(\mathbb{R}^d)$.

Remark 3 (**Notation in $\mathbb{R}^d$**). *We will most often write $\int_{\mathbb{R}^d} f(x)\,dx$ or $\int f(x)\,\mathrm{d}x$ or simply $\int f\mathrm{d}x$ instead of $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}\lambda^d(x)$ from now on. Also, we will often use the notation $|A| := \lambda^d(A)$ for the Lebesgue(-Borel) measure of a (measurable) set $A \subset \mathbb{R}^d$. This way, for the distribution function of $f : \mathbb{R}^d \to \mathbb{R}$ (relative to Lebesgue measure λ^d) we have*

$$(\lambda^d)_f(t) = \lambda^d(\{f > t\}) = \lambda^d(\{x \in \mathbb{R}^d \,|\, f(x) > t\}) = |\{f > t\}|\,. \tag{57}$$