MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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– Blatt 2 –
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PDG I
(Tutorium)

Tutorial 2

(Gauss-Green, Polar Coordinates)

In this tutorial we covered the following topics:

- The definition of a Lipschitz boundary for an open domain $\Omega \subset \mathbb{R}^n$ (also C^m boundaries, smooth boundaries)
- The Gauss-Green Theorem
- Polar coordinates

Question 1

(a) Deduce the *Divergence Theorem*: if $u \in C^1(\bar{\Omega}; \mathbb{R}^n)$ and $\partial\Omega$ is C^1, then

$$\int_\Omega \mathrm{div} u \, \mathrm{d}x = \int_{\partial\Omega} u \cdot \nu \, \mathrm{d}S$$

where $\nu \colon \partial\Omega \to \mathbb{R}^n$ is the unit normal to $\partial\Omega$.

(b) Deduce *Green's formulas*: if $u, v \in C^2(\bar{\Omega})$ and $\partial\Omega$ is C^1, then

(i) $\int_\Omega \Delta u \, \mathrm{d}x = \int_{\partial\Omega} \nabla u \cdot \nu \, \mathrm{d}S$

(ii) $\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = -\int_\Omega u \Delta v \, \mathrm{d}x + \int_{\partial\Omega} u(\nabla v \cdot \nu) \, \mathrm{d}S$

(iii) $\int_\Omega u\Delta v - v\Delta u \, \mathrm{d}x = \int_{\partial\Omega} u(\nabla v \cdot \nu) - v(\nabla u \cdot \nu) \, \mathrm{d}S$

Question 2

Let B be the unit ball in $\mathbb{R}^2$ and fix $0 < s < 1$. Consider the map

$$u(x) = |x|^{-s} .$$

(i) Show that $u \in L^p(B)$ for $1 \leq p < \frac{2}{s}$.

(ii) Compute the partial derivatives $\partial u / \partial x_1$, $\partial u / \partial x_2$ of u on $B \setminus \{(0,0)\}$.

(iii) Now suppose $1 \leq p < \frac{2}{s+1}$. Show that each partial derivative $\partial u / \partial x_i$ above satisfies

$$\int_B \left| \frac{\partial u}{\partial x_i} \right|^p \mathrm{d}x < \infty .$$