PDG I
(Tutorium)

Tutorial 12

(Maximal Functions and Lebesgue's Differentiation Theorem)

In this tutorial we went through solutions to the "Holiday Problems".

Question 1 (Vitali's Covering Lemma)

Let $E \subset \mathbb{R}^n$ be the union of a finite number of balls $B(x_i, r_i)$, $i = 1, 2 \dots k$. Show that there exists a subset $I \subset \{1, \dots k\}$ such that the balls $B(x_i, r_i)$ with $i \in I$ are pairwise disjoint (that is, $B(x_i, r_i) \cap B(x_j, r_j) =$ whenever $i, j \in I$ and $i \neq j$), and

$$E \subset \bigcup_{i \in I} B(x_i, 3r_i)\,.$$

Question 2 (A "weak-type" inequality for the Hardy-Littlewood Maximal Function)

Let $u \in L^1(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, define the *Maximal Function* Mu as

$$(Mu)(x) := \sup_{r>0} \frac{1}{\mathscr{L}^n(B(x,r))} \int_{B(x,r)} |u(y)|\,\mathrm{d}y \quad \left(= \sup_{r>0} ⨍_{B(x,r)} |u(y)|\,\mathrm{d}y \right),$$

where $\mathscr{L}^n(\cdot)$ denotes the n-dimensional Lebesgue measure/volume (so $\mathscr{L}^n(B(x,r)) = r^n \alpha_n$). You may assume this is a measurable function. Let $t \in (0, \infty)$. Show that

$$\mathscr{L}^n\big(\{x \in \mathbb{R}^n : (Mu)(x) > t\}\big) \leq \frac{3^n}{t} \int_{\mathbb{R}^n} |u(y)|\,\mathrm{d}y\,. \tag{1}$$

To do this, use Vitali's Covering Lemma from Question 1. You may also use the fact that Lebesgue Measure is "inner regular" i.e. for any Lebesgue-measurable set E,

$$\mathscr{L}^n(E) = \sup\{\, \mathscr{L}^n(K) : K \subset E \text{ and } K \text{ compact}\}\,.$$

This means that it suffices to prove the estimate

$$\mathscr{L}^n(K) \leq \frac{3^n}{t} \int_{\mathbb{R}^n} |u(y)|\,\mathrm{d}y$$

for any compact set $K \subset \{x \in \mathbb{R}^n : (Mu)(x) > t\}$.

Question 3 (Lebesgue's Differentiation Theorem)

In this quetion we shall prove *Lebesgue's Differentiation Theorem*: i.e. if $u \in L^1(\mathbb{R}^n)$ then for almost all $x \in \mathbb{R}^n$,

$$\limsup_{r \searrow 0} ⨍_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}y = 0 \,. \tag{2}$$

Recall that if u is continuous, then this holds for all $x \in \mathbb{R}^n$ (in one dimension this is just the Fundamental Theorem of Calculus!) Our aim is to extend such a result to integrable functions.

(i) Let $u \in L^1(\mathbb{R}^n)$, and let $\varphi \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Fix $x \in \mathbb{R}^n$ and show that

$$\limsup_{r \searrow 0} ⨍_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}y \leq M(|u - \varphi|)(x) + |u(x) - \varphi(x)|$$

where $M(|u - \varphi|)$ is the Maximal Function of $(u - \varphi)$ at x, defined as in Question 2.

(ii) Let $\epsilon > 0$ and observe that the previous part implies that, for fixed x, if

$$\limsup_{r \searrow 0} ⨍_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}y > \epsilon \,,$$

then

$$M(|u - \varphi|)(x) > \tfrac{\epsilon}{2} \text{ or } |u(x) - \varphi(x)| > \tfrac{\epsilon}{2} \,.$$

Using the Maximal inequality (1) from Question 2 and the density of continuous functions in $L^1(\Omega)$ (in the $\| \cdot \|_1$ norm topology), deduce

$$\mathscr{L}^n \left(\left\{ x : \limsup_{r \searrow 0} ⨍_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}y > \epsilon \right\} \right) = 0 \,,$$

and then use this to establish (2).