MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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WS 2014/15
– Blatt 2 –
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PDG I
(Zentralübung)

Problem Sheet 2

(Spaces of continuously differentiable functions, the Transport equation)

Recall that given an open subset Ω of $\mathbb{R}^n$ and a continuous function $u\colon \Omega \to \mathbb{R}$, we define the *support* of u as

$$\operatorname{supp}(u) := \overline{\{x \in \Omega : u(x) \neq 0\}}\,.$$

We then define

$$C_c(\Omega) := \{u \in C(\Omega) : \operatorname{supp}(u) \text{ is compact }\}\,,$$

and, for $k \in \mathbb{N} \cup \{\infty\}$,

$$C_c^k(\Omega) := C^k(\Omega) \cap C_c(\Omega)\,.$$

Question 1

Let $\Omega = \mathbb{R}^n \times (0,\infty)$, $g \in L^\infty(\mathbb{R}^n)$ and $b \in \mathbb{R}^n$. Show that the function $u\colon \bar{\Omega} \to \mathbb{R}$, $u(x,t) := g(x - bt)$ is a weak solution of the Initial Value Problem

$$\begin{cases} u_t + b \cdot D_x u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}\,. \end{cases} \tag{1}$$

Here we say that u is a *weak solution* of (1) precisely when for all $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \big(\varphi_t(x,t) + b \cdot D_x\varphi(x,t)\big)u(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0$$

and $u(x,0) = g(x)$ for almost all $x \in \mathbb{R}^n$.

Hint: Use the change-of-variables $(x,t) \mapsto (y,t) := (x - bt, t)$.

Question 2

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We define

$$C^k(\bar{\Omega}) := \{u \in C^k(\Omega) : D^\alpha u \text{ has a continuous extension on } \bar{\Omega}\ \ \forall \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k\}\,.$$

Show that the following statements are equivalent:

(a) $u \in C^k(\bar{\Omega})$

(b) $u \in C^k(\Omega)$ and $D^\alpha u$ is uniformly continuous on Ω for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$.

Question 3

Let $u \in L^1(\mathbb{R}^n)$, $\varphi \in C_c^1(\mathbb{R}^n)$. Show that the convolution

$$\varphi * u(x) = \int_{\mathbb{R}^n} \varphi(x-y)u(y)\,\mathrm{d}y$$

is in $C^1(\mathbb{R}^n)$, with

$$\frac{\partial}{\partial x_i}(\varphi * u)(x) = \int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial x_i}(x-y)u(y)\,\mathrm{d}y\,. \tag{1}$$

Hint: For $h > 0$, $x \in \mathbb{R}^n$, $1 \leq i \leq n$, consider the difference quotient

$$\frac{(\varphi * u)(x + he_i) - (\varphi * u)(x)}{h}\,,$$

and use the (continuous) Dominated Convergence Theorem to establish the identity (1). Argue similarly to show that the derivative is continuous.

Deadline for handing in: 0800 Wednesday 29 October

Please put solutions in Box 17, 1st floor (near the library)

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`