Constructive (functional) analysis

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Proof and Computation, Fischbachau, 3 – 8 October, 2016

Contents (lecture 3)

- Normed and Banach spaces
  - The Hahn-Banach theorem
  - The uniform boundedness theorem
  - The open mapping theorem
- Hilbert spaces
  - Adjoit operators
  - Compact operators

Normed spaces

Definition

A normed space is a linear space E equipped with a norm $\|\cdot\| : E \to \mathbf{R}$ such that

- $\|x\| = 0 \leftrightarrow x = 0$,
- $\|ax\| = |a|\|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$,

for each $x, y \in E$ and $a \in \mathbf{R}$.

Note that a normed space E is a metric space with the metric

$$d(x, y) = \|x - y\|.$$

Definition

A Banach space is a normed space which is complete with respect to the metric.

Examples

For $1 \leq p < \infty$, let

$$l_p = \{(x_n) \in \mathbf{R}^{\mathbf{N}} \mid \sum_{n=0}^{\infty} |x_n|^p < \infty\}$$

and define a norm by

$$\|(x_n)\| = (\sum_{n=0}^{\infty} |x_n|^p)^{1/p}.$$

Then l_p is a (separable) Banach space.

Examples

Classically the normed space

$$l_\infty = \{(x_n) \in \mathbf{R}^{\mathbf{N}} \mid (x_n) \text{ is bounded}\}$$

with the norm

$$\|(x_n)\| = \sup_n |x_n|$$

is an inseparable Banach space.

However, constructively, it is not a normed space.

Linear mappings

Definition

A mapping T between linear spaces E and F is linear if

- $T(ax) = aTx$,
- $T(x + y) = Tx + Ty$

for each $x, y \in E$ and $a \in \mathbf{R}$.
A linear functional f on a linear space E is a linear mapping from E into $\mathbf{R}$.

Definition

The kernel $\ker(T)$ of a linear mapping T between linear spaces E and F is defined by

$$\ker(T) = \{x \in E \mid Tx = 0\}.$$

Bounded linear mappings

Definition

A linear mapping T between normed spaces E and F is **bounded** if

$$T(B_E) = \{\, Tx \mid x \in B_E \,\}$$

is bounded, where $B_E = \{x \in E \mid \|x\| \leq 1\}$.

Proposition

Let T be a linear mapping between normed spaces E and F. Then the following are equivalent.

- *T is continuous,*
- *T is uniformly continuous,*
- *T is bounded.*

Normable linear mappings

Definition

A linear mapping T between normed spaces E and F is normable if

$$\|T\| = \sup\{\|Tx\| \mid x \in B_E\}$$

exists.

Theorem (classical)

The set of bounded linear functionals on a normed space is a Banach space.

Normable linear functionals

Proposition

If every bounded linear functional on l_2 is normable, then LPO *holds.*

Proof.

Let α be a binary sequence with at most one nonzero term, and define a linear functional f on l_2 by

$$f((x_n)) = \sum_{k=0}^{\infty} \alpha(k) x_k .$$

Then f is bounded. If f is normable, then either $0 < \|f\|$ or $\|f\| < 1$; in the former case, we have $\alpha \# \mathbf{0}$; in the latter case, we have $\neg\alpha \# \mathbf{0}$. □

Normable linear functionals

Proposition

If the set $(l_1)^*$ of normable linear functionals on l_1 is linear, then LPO *holds.*

Proof.

Let α be a binary sequence with $\alpha(0) = 0$, and define linear functionals on l_1 by

$$f((x_n)) = \sum_{k=0}^{\infty} x_k, \quad g((x_n)) = \sum_{k=0}^{\infty} (\alpha(k) - 1)x_k.$$

Then f and g are normable with $\|f\| = \|g\| = 1$. If $f + g$ is normable, then either $0 < \|f + g\|$ or $\|f + g\| < 1$; in the former case, we have $\alpha \# \mathbf{0}$; in the latter case, we have $\neg\alpha \# \mathbf{0}$. □

Normable linear functionals

Let E^* be the set of normable linear fuctionals on a normed space E.

Open Problem

Under what condition does E^* become a linear space?

Note that $(l_p)^*$ is a linear space for $1 < p < \infty$, and H^* is a linear space for a Hilbert space H.

Normable linear functionals

Proposition

A nonzero bounded linear functional f on a normed space E is normable if and only if its kernel

$$\ker(f) = \{x \in E \mid f(x) = 0\}$$

is located.

Classical Hahn-Banach theorem

Theorem

Let M be a subspace of a normed space E, and let f be a bounded linear functional on M. Then there exists a bounded linear functional g on E such that $g(x) = f(x)$ for each $x \in M$ and $\|g\| = \|f\|$.

Corollary

Let x be a nonzero element of a normed space E. Then there exists a bounded linear functional f on E such that $f(x) = \|x\|$ and $\|f\| = 1$.

Classical Hahn-Banach theorem

Proposition

The classical Hahn-Banach theorem implies LLPO.

Proof.

Let $(1, a)$ be a nonzero element of the normed space $\mathbf{R}^2$ with a norm $\|(x, y)\| = |x| + |y|$. Then there exists a bounded linear functional f such that $f(1, a) = 1 + |a|$ and $\|f\| = 1$. Since $|f(1, 0)| \leq 1$ and $|f(0, 1)| \leq 1$, we have

$$1 + |a| = f(1, a) = f(1, 0) + af(0, 1) \leq f(1, 0) + |a|,$$

and therefore $f(1, 0) = 1$ and $af(0, 1) = |a|$. Either $-1 < f(0, 1)$ or $f(0, 1) < 1$. In the former case, we have $0 \leq a$; in the latter case, we have $a \leq 0$. □

Constructive Hahn-Banach theorem

Theorem (Bishop 1967)

Let M be a subspace of a separable normed space E, and let f be a nonzero normable linear functional on M. Then for each $\epsilon > 0$ there exists a normable linear functional g on E such that $g(x) = f(x)$ for each $x \in M$ and $\|g\| \leq \|f\| + \epsilon$.

Corollary

Let x be a nonzero element of a separable normed space E. Then for each $\epsilon > 0$ there exists a normable linear functional f on E such that $f(x) = \|x\|$ and $\|f\| \leq 1 + \epsilon$.

Gâteaux differentiable norm

Definition

The norm of a normed space E is Gâteaux differentiable at $x \in E$ with the derivative $f : E \to \mathbf{R}$ if for each $y \in E$ with $\|y\| = 1$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall t \in \mathbf{R}(|t| < \delta \to |\|x + ty\| - \|x\| - tf(y)| < \epsilon|t|).$$

Note that the derivative f is linear.

Definition

The norm of a normed space E is Gâteaux differentiable if it is Gâteaux differentiable at each $x \in E$ with $\|x\| = 1$.

Remark

The norm of l_p for $1 < p < \infty$ and the norm of a Hilbert space are Gâteaux differentiable at each $x \in E$ with $x \# 0$.

A constructive corollary

Proposition (I 1989)

Let x be a nonzero element of a normed linear space E whose norm is Gâteaux differentiable at x. Then there exists a unique normable linear functional f on E such that $f(x) = \|x\|$ and $\|f\| = 1$.

Proof.

Take the derivative f of the norm at x. □

Uniformly convex spaces

Definition

A normed space E is uniformly convex if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x - y\| \geq \epsilon \rightarrow \|(x + y)/2\| \leq 1 - \delta$$

for each $x, y \in E$ with $\|x\| = \|y\| = 1$.

Proposition (Bishop-Bridges 1985)

Let f be a nonzero normable linear functional on a uniformly convex Banach space E. Then there exists $x \in E$ such that $f(x) = \|f\|$ and $\|x\| = 1$.

Remark

l_p for $1 < p < \infty$ and a Hilbert space are uniformly convex.

Constructive Hahn-Banach theorem

Theorem (I 1989)

Let M be a subspace of a uniformly convex Banach space E with a Gâteaux differentiable norm, and let f be a normable linear functional on M. Then there exists a unique normable linear functional g on H such that $g(x) = f(x)$ for each $x \in M$ and $\|g\| = \|f\|$.

Proof.

We may assume without loss of generality that $\|f\| = 1$. Let $\overline{M}$ be the closure of M. Then there exists a normable extension $\overline{f}$ of f on $\overline{M}$. Since $\overline{M}$ is a uniformly convex Banach, there exists $x \in \overline{M}$ such that $\overline{f}(x) = \|x\| = 1$. Take the derivative g of the norm at x. □

Classical uniform boundedness theorem

Theorem

Let $(T_m)_m$ be a sequence of bounded linear mappings from a Banach space E into a normed space F such that the set

$$\{T_m x \mid m \in \mathbf{N}\}$$

is bounded in F for each $x \in E$. Then $(T_m)_m$ is equicontinuous, that is, $\{T_m x \mid m \in \mathbf{N}, x \in B_E\}$ is bounded.

Corollary

Let $(T_m)_m$ be a sequence of bounded linear mappings from a Banach space E into a normed space F such that the limit

$$Tx = \lim_{m \to \infty} T_m x$$

exists for each $x \in E$. Then (being obviously linear) T is bounded.

Classical uniform boundedness theorem

Proposition (I 2012)

The classical uniform boundedness theorem implies BD-N.

Proof.

Let $S = \{s_n \mid n \in \mathbf{N}\}$ be a pseudobounded countable subset of $\mathbf{N}$, and define a sequence $(T_m)_m$ of bounded linear mappings from l_2 itself by

$$T_m(x_n) = (s_0 x_0, \ldots, s_m x_m, 0 \ldots).$$

Then, since S is pseudobounded, we can show that the limit

$$Tx = \lim_{m \to \infty} T_m x = (s_n x_n)$$

exists for each $x \in l_2$. If T is bounded, then we see that S is bounded. $\square$

A constructive uniform boundedness theorem

Theorem (Bishop 1967)

Let $(T_m)_m$ be a sequence of bounded linear mappings from a Banach space E into a normed space F. If (x_m) is a sequnece of B_E such that $\{T_m x_m \mid m \in \mathbf{N}\}$ is unbounded, then there exists $x \in E$ such that

$$\{T_m x \mid m \in \mathbf{N}\}$$

is unbounded.

A constructive uniform boundedness theorem

Proposition (I 2012)

Assume BD-N. *If* $(T_m)_m$ *is a sequence of bounded linear mappings from a separable Banach space* E *into a normed space* F *such that the set*

$$\{T_m x \mid m \in \mathbf{N}\}$$

is bounded for each $x \in E$, *then* $(T_m)_m$ *is equicontinuous.*

Classical open mapping theorem

Definition

A linear mapping T between normed spaces E and F is open if $T(B_E)$ has an inhabited interior.

Theorem (Open mapping theorem)

Let T be a bounded linear mapping between Banach spaces. Then T is an open mapping.

Corollary (Closed graph theorem)

Let T be a linear mapping between Banach spaces. Then T is bounded if and only if its graph is closed.

Corollary (Banach's inverse mapping theorem)

Let T be a bounded one-to-one linear mapping from a Banach space onto a Banach space. Then its inverse T^{-1} is bounded.

Classical open mapping theorem

Proposition

Classical Banach's inverse mapping theorem implies BD-N.

Proof.

Let $S = \{s_n \mid n \in \mathbf{N}\}$ be a pseudobounded countable subset of $\mathbf{N}$, and define a bounded linear mapping T from l_2 itself by

$$T(x_n) = (x_n/2^{s_n}).$$

Then T is one-to-one, and, since S is pseudobounded, we can show that T is onto. If T^{-1} is bounded, then S is bounded. □

A constructive open mapping theorem

Theorem (I 1994)

Let T be a sequentially continuous one-to-one linear mapping from a separable Banach space onto a Banach space. Then its inverse T^{-1} is sequentially continuous.

Corollary (I 1994)

Let T be a sequentially continuous linear mapping from a separable Banach space onto a Banach space such that $\ker(T)$ is located. Then T is sequentially open.

Corollary (I 1994)

Let T be a linear mapping between Banach spaces such that its graph is separable. Then T is sequentially continuous if and only if its graph is closed.

A constructive open mapping theorem

Theorem

Assume $\mathrm{BD\text{-}N}$. *If* T *is a bounded one-to-one linear mapping from a separable Banach space onto a Banach space, then its inverse* T^{-1} *is bounded.*

Hilbert spaces

Definition

An inner product space is a linear space E equipped with an inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathbf{R}$ such that

- $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \leftrightarrow x = 0$,
- $\langle x, y \rangle = \langle y, x \rangle$,
- $\langle ax, y \rangle = a\langle x, y \rangle$,
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

for each $x, y, z \in E$ and $a \in \mathbf{R}$.

Note that an inner product space E is a normed space with the norm

$$\|x\| = \langle x, x \rangle^{1/2}.$$

Definition

A Hilbert space is an inner product space which is a Banach space.

Example

Let

$$l_2 = \{(x_n) \in \mathbf{R}^{\mathbf{N}} \mid \sum_{n=0}^{\infty} |x_n|^2 < \infty\}$$

and define an inner product by

$$\langle (x_n), (y_n) \rangle = \sum_{n=0}^{\infty} x_n y_n.$$

Then l_2 is a Hilbert space.

The Riesz theorem

Proposition (Bishop-Bridges 1985)

Let f be a bounded linear functional on a Hilbert space H. Then f is normable if and only if there exists (unique) $x_0 \in H$ such that

$$f(x) = \langle x, x_0 \rangle$$

for each $x \in H$.

Adjoint operators

Definition

An operator A on a Hilbert space H is a bounded linear mapping from H into itself.

Definition

An operator A^* on a Hilbert space H is an adjoint of an operator A on H if

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

for each $x, y \in H$.

Remark

Classically, every operator has an adjoint.

Adjoint operators

Proposition

If every operator on l_2 has an adjoint, then LPO *holds.*

Proof.

Let α be a binary sequence with at most one nonzero term, and define a linear mapping C from l_2 into itself by

$$C(x_n) = (\textstyle\sum_{k=0}^{\infty} \alpha(k) x_k / \sqrt{2^{n+1}}).$$

Then C is an operator. Note that

$$\langle C(x_n), y \rangle = \sum_{k=0}^{\infty} \alpha(k) x_k$$

for $y = (1/\sqrt{2^{n+1}})$. If C has an adjoint, then, the linear functional $f : (x_n) \mapsto \langle C(x_n), y \rangle$ is normable, by the Riesz theorem, and therefore either $0 < \|f\|$ or $\|f\| < 1$; in the former case, we have $\alpha \# \mathbf{0}$; in the latter case, we have $\neg\alpha \# \mathbf{0}$. □

Weakly compact operators

Definition

An operator A on a Hilbert space H is weakly compact if

$$\{\langle Ax, y\rangle \mid x \in B_H\}$$

is totally bounded for each $y \in H$.

Proposition (I 1991)

An operator A has an adjoint if and only if A is weakly compact.

Proof.

By the Riesz theorem, A has an adjoint if and only if the linear functional $x \mapsto \langle Ax, y\rangle$ is normable for each $y \in H$ if and only if $\{\langle Ax, y\rangle \mid x \in H, \|x\| \leq 1\}$ is totally bounded for each $y \in H$. $\square$

Compact Operators

Definition

An operator A on a Hilbert space H is compact if $A(B_H)$ is totally bounded.

Remark

- Every compact operator is weakly compact.
- Every compact operator is normable.

Note that the identity operator $I : x \mapsto x$ on l^2 is not compact, but weakly compact.

Compact operators

Theorem (classical)

Let A and B be compact operators on a Hilbert space H, let C be an operator on H, and let $a \in \mathbf{R}$. Then

- *aA, $A + B$ and A^* are compact,*
- *CA and AC are compact.*

Compact operators

Proposition

If AC is compact for each compact operator A and bounded operator C on l_2, then LPO holds.

Proof.

Let α be a binary sequence with at most one nonzero term, and define linear mappings A and C from l_2 into itself by

$$A(x_n) = (x_n/\sqrt{2^{n+1}}),$$
$$C(x_n) = (\textstyle\sum_{k=0}^{\infty} \alpha(k)x_k/\sqrt{2^{n+1}}).$$

Then A is compact and C is bounded, and

$$\|AC(x_n)\|^2 = |\textstyle\sum_{k=0}^{\infty} \alpha(k)x_k|^2/3.$$

Therefore either $0 < \|AC\|$ or $\|AC\| < 1/3$; in the former case, we have $\alpha \# \mathbf{0}$; in the latter case, we have $\neg\alpha \# \mathbf{0}$. □

Compact operators

Theorem (I 1991)

Let A and B be compact operators on a Hilbert space H, let C be an operator on H, and let $a \in \mathbf{R}$. Then

- *aA and $A + B$ are compact,*
- *A^* exists and is compact,*
- *CA is compact,*
- *if C is weakly compact, then AC is compact.*

Future challenges

- Developing a constructive theory of distributions.
  We have shown that the completeness of the space $\mathcal{D}(\mathbf{R})$ of test functions is equivalent to $\mathrm{BD\text{-}N}$.
- Developing a constructive reverese (functional) analysis.
  Which nonconstructive principle is equivalent to the Baire theorem for complete metric spaces?
- Developing a constructive theory of uniform spaces and topological spaces.
  We have given constructions of a completion of a uniform space and a quotient topology in **CZF**.

References

- Josef Berger, Hajime Ishihara, Erik Palmgren and Peter Schuster, *A predicative completion of a uniform space*, Ann. Pure Appl. Logic **163** (2012), 975–980.
- Errett Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.
- Errett Bishop and Douglas Bridges, *Constructive Analysis*, Springer-Verlag, Berlin, 1985.
- Douglas Bridges and Fred Richman, *Varieties of Constructive Mathematics*, Cambridge Univ. Press, London, 1987.
- Douglas Bridges and Luminiţa Vîţă, *Techniques of Constructive Analysis*, Springer, New York, 2006.
- Hajime Ishihara, *On the constructive Hahn-Banach theorem*, Bull. London Math. Soc. **21** (1989), 79–81.

References

- Hajime Ishihara, *An omniscience principle, the König lemma and the Hahn-Banach theorem*, Z. Math. Logik Grundlagen Math. **36** (1990), 237–240.
- Hajime Ishihara, *A constructive version of Banach's inverse mapping theorem*, New Zealand J. Math. **23** (1994), 71–75.
- Hajime Ishihara, *Constructive reverse mathematics: compactness properties*, in: L. Crosilla and P. Schuster eds., From Sets and Types to Topology and Analysis, Oxford Univ. Press, 2005, 245–267.
- Hajime Ishihara, *The uniform boundedness theorem and a boundedness principle*, Ann. Pure Appl. Logic **163** (2012), 1057–1061.

References

- Hajime Ishihara and Erik Palmgren, *Quotient topologies in constructive set theory and type theory*, Ann. Pure Appl. Logic **141** (2006), 257–265.
- Hajime Ishihara and Peter Schuster, *A continuity principle, a version of Baire's theorem and a boundedness principle*, J. Symbolic Logic **73** (2008), 1354–1350.
- Hajime Ishihara and Satoru Yoshida, *A constructive look at the completeness of $\mathcal{D}(\mathbf{R})$*, J. Symbolic Logic **67** (2002), 1511–1519.
- Walter Rudin, *Functional Analysis*, Second Edition, McGraw-Hill, New York, 1991.