Constructive (functional) analysis

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Contents (lectures 1 – 3)

- Intuitionistic logic
- Real numbers
- Metric spaces
- Normed and Banach spaces
- Hilbert spaces

Contents (lecture 1)

A history of constructivism

- History
    - Arithmetization of mathematics (Kronecker, 1887)
    - Three kinds of intuition (Poincaré, 1905)
    - French semi-intuitionism (Borel, 1914)
    - Intuitionism (Brouwer, 1914)
    - Predicativity (Weyl, 1918)
    - Finitism (Skolem, 1923; Hilbert-Bernays, 1934)
    - Constructive recursive mathematics (Markov, 1954)
    - Constructive mathematics (Bishop, 1967)
- Logic
    - Intuitionistic logic (Heyting, 1934; Kolmogorov, 1932)

Language

We use the standard language of (many-sorted) first-order predicate logic based on

- primitive logical operators $\land, \lor, \to, \bot, \forall, \exists$.

We introduce the abbreviations

- $\neg A \equiv A \to \bot$;
- $A \leftrightarrow B \equiv (A \to B) \land (B \to A)$.

The BHK interpretation

The Brouwer-Heyting-Kolmogorov (BHK) interpretation of the logical operators is the following.

- A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.
- A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B.
- A proof of $A \to B$ is a construction which transform any proof of A into a proof of B.
- Absurdity $\perp$ has no proof.
- A proof of $\forall x A(x)$ is a construction which transforms any t into a proof of $A(t)$.
- A proof of $\exists x A(x)$ is given by presenting a t and a proof of $A(t)$.

Natural Deduction System

We shall use $\mathcal{D}$, possibly with a subscript, for arbitrary deduction.

We write

$$\begin{matrix}\Gamma \\ \mathcal{D} \\ A\end{matrix}$$

to indicate that $\mathcal{D}$ is deduction with conclusion A and assumptions Γ.

Deduction (Basis)

For each formula A,

$$A$$

is a deduction with conclusion A and assumptions $\{A\}$.

Deduction (Induction step, →I)

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ B\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ B\end{array}}{A \rightarrow B}\ \rightarrow\mathrm{I}$$

is a deduction with conclusion $A \rightarrow B$ and assumptions $\Gamma \setminus \{A\}$.
We write

$$\dfrac{\begin{array}{c}[A]\\ \mathcal{D}\\ B\end{array}}{A \rightarrow B}\ \rightarrow\mathrm{I}$$

Deduction (Induction step, $\rightarrow\mathrm{E}$)

If

$$\begin{array}{c} \Gamma_1 \\ \mathcal{D}_1 \\ A \rightarrow B \end{array} \qquad\qquad \begin{array}{c} \Gamma_2 \\ \mathcal{D}_2 \\ A \end{array}$$

are deductions, then

$$\cfrac{\begin{array}{c} \Gamma_1 \\ \mathcal{D}_1 \\ A \rightarrow B \end{array} \qquad \begin{array}{c} \Gamma_2 \\ \mathcal{D}_2 \\ A \end{array}}{B} \; \rightarrow\mathrm{E}$$

is a deduction with conclusion B and assumptions $\Gamma_1 \cup \Gamma_2$.

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg A] \qquad \dfrac{\dfrac{[\neg\neg(A\to B)] \qquad \dfrac{\dfrac{[\neg B] \qquad \dfrac{[A\to B] \quad [A]}{B}\;{\to}\mathrm{E}}{\bot}\;{\to}\mathrm{E}}{\neg(A\to B)}\;{\to}\mathrm{I}}{\bot}\;{\to}\mathrm{E}}{\neg A}\;{\to}\mathrm{I}}{\bot}\;{\to}\mathrm{E}}{\neg\neg B}\;{\to}\mathrm{I}}{\neg\neg A\to\neg\neg B}\;{\to}\mathrm{I}}{\neg\neg(A\to B)\to(\neg\neg A\to\neg\neg B)}\;{\to}\mathrm{I}
$$

Minimal logic

$$\dfrac{\begin{matrix}[A]\\ \mathcal{D}\\ B\end{matrix}}{A \to B}\ \to\mathrm{I} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A \to B\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B}\ \to\mathrm{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ B\end{matrix}}{A \wedge B}\ \wedge\mathrm{I} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ A \wedge B\end{matrix}}{A}\ \wedge\mathrm{E}_r \qquad \dfrac{\begin{matrix}\mathcal{D}\\ A \wedge B\end{matrix}}{B}\ \wedge\mathrm{E}_l$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ A\end{matrix}}{A \vee B}\ \vee\mathrm{I}_r \qquad \dfrac{\begin{matrix}\mathcal{D}\\ B\end{matrix}}{A \vee B}\ \vee\mathrm{I}_l \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A \vee B\end{matrix} \quad \begin{matrix}[A]\\ \mathcal{D}_2\\ C\end{matrix} \quad \begin{matrix}[B]\\ \mathcal{D}_3\\ C\end{matrix}}{C}\ \vee\mathrm{E}$$

Minimal logic

$$\dfrac{\begin{matrix}\mathcal{D}\\ A\end{matrix}}{\forall y A[x/y]}\ \forall\mathrm{I} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \forall x A\end{matrix}}{A[x/t]}\ \forall\mathrm{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ A[x/t]\end{matrix}}{\exists x A}\ \exists\mathrm{I} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \exists y A[x/y]\end{matrix} \quad \begin{matrix}[A]\\ \mathcal{D}_2\\ C\end{matrix}}{C}\ \exists\mathrm{E}$$

- In $\forall\mathrm{E}$ and $\exists\mathrm{I}$, t must be free for x in A.
- In $\forall\mathrm{I}$, $\mathcal{D}$ must not contain assumptions containing x free, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$.
- In $\exists\mathrm{E}$, $\mathcal{D}_2$ must not contain assumptions containing x free except A, $x \notin \mathrm{FV}(C)$, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$.

Example

$$
\begin{prooftree}
\AxiomC{$[(A \to B) \wedge (A \to C)]$}
\RightLabel{$\wedge\mathrm{E}_r$}
\UnaryInfC{$A \to B$}
\AxiomC{$[A]$}
\RightLabel{$\to\mathrm{E}$}
\BinaryInfC{B}
\AxiomC{$[(A \to B) \wedge (A \to C)]$}
\RightLabel{$\wedge\mathrm{E}_l$}
\UnaryInfC{$A \to C$}
\AxiomC{$[A]$}
\RightLabel{$\to\mathrm{E}$}
\BinaryInfC{C}
\RightLabel{$\wedge\mathrm{I}$}
\BinaryInfC{$B \wedge C$}
\RightLabel{$\to\mathrm{I}$}
\UnaryInfC{$A \to B \wedge C$}
\RightLabel{$\to\mathrm{I}$}
\UnaryInfC{$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$}
\end{prooftree}
$$

Example

$$
\dfrac{\dfrac{[A \lor B] \qquad \dfrac{\dfrac{[(A \to C) \land (B \to C)]}{A \to C}\,\land\mathrm{E}_r \qquad [A]}{C}\,\to\mathrm{E} \qquad \dfrac{\dfrac{[(A \to C) \land (B \to C)]}{B \to C}\,\land\mathrm{E}_l \qquad [B]}{C}\,\to\mathrm{E}}{\dfrac{C}{A \lor B \to C}\,\to\mathrm{I}}\,\lor\mathrm{E}}{(A \to C) \land (B \to C) \to (A \lor B \to C)}\,\to\mathrm{I}
$$

Example

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[A \to \forall x B] \quad [A]}{\forall x B}\ \to\mathrm{E}}{B}\ \forall\mathrm{E}}{A \to B}\ \to\mathrm{I}}{\forall x (A \to B)}\ \forall\mathrm{I}}{(A \to \forall x B) \to \forall x (A \to B)}\ \to\mathrm{I}
$$

where $x \notin \mathrm{FV}(A)$.

Example

$$\dfrac{\dfrac{\dfrac{[\exists x(A \to B)] \qquad \dfrac{\dfrac{[A \to B] \quad [A]}{B}\ {\to}\mathrm{E}}{\exists x B}\ \exists\mathrm{I}}{\exists x B}\ \exists\mathrm{E}}{A \to \exists x B}\ {\to}\mathrm{I}}{\exists x(A \to B) \to (A \to \exists x B)}\ {\to}\mathrm{I}$$

where $x \notin \mathrm{FV}(A)$.

Intuitionistic logic

Intuitionistic logic is obtained from minimal logic by adding the intuitionistic absurdity rule (ex falso quodlibet).

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_i$$

is a deduction with conclusion A and assumptions Γ.

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg A \to \neg\neg B] \qquad \dfrac{\dfrac{[\neg(A \to B)] \qquad \dfrac{\dfrac{\dfrac{[\neg A] \quad [A]}{\bot}\,{\to}\mathrm{E}}{B}\,{\color{red}\bot_i}}{A \to B}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg\neg A}\,{\to}\mathrm{I}}{\neg\neg B}\,{\to}\mathrm{E} \qquad \dfrac{\dfrac{[\neg(A \to B)] \quad \dfrac{[B]}{A \to B}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg B}\,{\to}\mathrm{I}}{\bot}}{\neg\neg(A \to B)}\,{\to}\mathrm{I}}{(\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B)}\,{\to}\mathrm{I}$$

Example

$$\dfrac{\dfrac{\dfrac{[A \vee B] \qquad \dfrac{\dfrac{[\neg A] \quad [A]}{\bot}\ \rightarrow\mathrm{E}}{B}\ {\color{red}\bot_i} \qquad [B]}{B}\ \vee\mathrm{E}}{\neg A \rightarrow B}\ \rightarrow\mathrm{I}}{A \vee B \rightarrow (\neg A \rightarrow B)}\ \rightarrow\mathrm{I}$$

Classical logic

Classical logic is obtained from intuitionistic logic by strengthening the absurdity rule to the classical absurdity rule (reductio ad absurdum).

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_c$$

is a deduction with conclusion A and assumption $\Gamma \setminus \{\neg A\}$.

Example (classical logic)

The double negation elimination (DNE):

$$\dfrac{\dfrac{\dfrac{[\neg\neg A] \quad [\neg A]}{\bot}\ {\to}\mathrm{E}}{A}\ \bot_c}{\neg\neg A \to A}\ {\to}\mathrm{I}$$

Example (classical logic)

The principle of excluded middle (PEM):

$$\dfrac{\dfrac{[\neg(A \vee \neg A)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A \vee \neg A)] \qquad \dfrac{[A]}{A \vee \neg A}\ \vee\mathrm{I}_r}{\bot}\ \to\mathrm{E}}{\neg A}\ \to\mathrm{I}}{A \vee \neg A}\ \vee\mathrm{I}_l}{\bot}\ \to\mathrm{E}}{A \vee \neg A}\ \bot_c$$

Example (classical logic)

De Morgan's law (DML):

$$\dfrac{\dfrac{[\neg(\neg A \vee \neg B)] \qquad \dfrac{\dfrac{\dfrac{[\neg(\neg A \vee \neg B)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A \wedge B)] \qquad \dfrac{[A] \quad [B]}{A \wedge B}\,\wedge\mathrm{I}}{\bot}\,\rightarrow\mathrm{E}}{\neg A}\,\rightarrow\mathrm{I}}{\neg A \vee \neg B}\,\vee\mathrm{I}_r}{\bot}\,\rightarrow\mathrm{E}}{\neg B}\,\rightarrow\mathrm{I}}{\neg A \vee \neg B}\,\vee\mathrm{I}_l}{\dfrac{\bot}{\neg A \vee \neg B}\,\bot_c}\,\rightarrow\mathrm{E}}{\neg(A \wedge B) \rightarrow \neg A \vee \neg B}\,\rightarrow\mathrm{I}$$

RAA vs $\rightarrow\mathrm{I}$

$\bot_c$: deriving A by deducing absurdity ($\bot$) from $\neg A$.

$$\dfrac{\begin{array}{c}[\neg A]\\ \mathcal{D}\\ \bot\end{array}}{A}\ \bot_c$$

$\rightarrow\mathrm{I}$: deriving $\neg A$ by deducing absurdity ($\bot$) from A.

$$\dfrac{\begin{array}{c}[A]\\ \mathcal{D}\\ \bot\end{array}}{\neg A}\ \rightarrow\mathrm{I}$$

Notations

- $m, n, i, j, k, \ldots \in \mathbf{N}$
- $\alpha, \beta, \gamma, \delta, \ldots \in \mathbf{N}^{\mathbf{N}}$
  - $\mathbf{0} = \lambda n.0$
  - $\alpha \# \beta \Leftrightarrow \exists n(\alpha(n) \neq \beta(n))$

Omniscience principles

- The limited principle of omniscience (LPO, Σ^0_1-PEM):

$$\forall\alpha[\alpha \# \mathbf{0} \vee \neg\alpha \# \mathbf{0}]$$

- The weak limited principle of omniscience (WLPO, Π^0_1-PEM):

$$\forall\alpha[\neg\neg\alpha \# \mathbf{0} \vee \neg\alpha \# \mathbf{0}]$$

- The lesser limited principle of omniscience (LLPO, Σ^0_1-DML):

$$\forall\alpha\beta[\neg(\alpha \# \mathbf{0} \wedge \beta \# \mathbf{0}) \to \neg\alpha \# \mathbf{0} \vee \neg\beta \# \mathbf{0}]$$

Markov's principle

- Markov's principle (MP, Σ^0_1-DNE):

$$\forall\alpha[\neg\neg\alpha \# \mathbf{0} \to \alpha \# \mathbf{0}]$$

- Markov's principle for disjunction ($\mathrm{MP}^{\vee}$, Π^0_1-DML):

$$\forall\alpha\beta[\neg(\neg\alpha \# \mathbf{0} \wedge \neg\beta \# \mathbf{0}) \to \neg\neg\alpha \# \mathbf{0} \vee \neg\neg\beta \# \mathbf{0}]$$

- Weak Markov's principle (WMP):

$$\forall\alpha[\forall\beta(\neg\neg\beta \# \mathbf{0} \vee \neg\neg\beta \# \alpha) \to \alpha \# \mathbf{0}]$$

Remark

We may assume without loss of generality that α (and β) are ranging over

- binary sequences,
- nondecreasing sequences,
- sequences with at most one nonzero term, or
- sequences with $\alpha(0) = 0$.

Relationship among principles

$$\begin{tikzcd}
 & \mathrm{LPO} \arrow[dl] \arrow[dr] & \\
\mathrm{MP} \arrow[ddd] \arrow[dddrr] & & \mathrm{WLPO} \arrow[d] \\
 & & \mathrm{LLPO} \arrow[dd] \\
 & & \\
\mathrm{WMP} & & \mathrm{MP}^{\vee}
\end{tikzcd}$$

- $\mathrm{LPO} \Leftrightarrow \mathrm{WLPO} + \mathrm{MP}$
- $\mathrm{MP} \Leftrightarrow \mathrm{WMP} + \mathrm{MP}^{\vee}$

Remark

- MP (and hencce WMP and $\mathrm{MP}^{\vee}$) holds in constructive recuresive mathematics.
- WMP holds in intuitionism.

CZF and choice axioms

The materials in the lectures could be formalized in

the constructive Zermelo-Fraenkel set theory (**CZF**)

without the powerset axiom and the full separation axiom, together with the following choice axioms.

- The axiom of countable choice (AC_0):

$$\forall n \exists y \in Y A(n, y) \rightarrow \exists f \in Y^{\mathbf{N}} \forall n A(n, f(n))$$

- The axiom of dependent choice (DC):

$$\forall x \in X \exists y \in X A(x, y) \rightarrow \\ \forall x \in X \exists f \in X^{\mathbf{N}} [f(0) = x \wedge \forall n A(f(n), f(n+1))]$$

Number systems

- The set **Z** of integers is the set **N** × **N** with the equality

  $$(n, m) =_{\mathbf{Z}} (n', m') \Leftrightarrow n + m' = n' + m.$$

  The arithmetical relations and operations are defined on **Z** in a straightforwad way; natural numbers are embedded into **Z** by the mapping $n \mapsto (n, 0)$.

- The set **Q** of rationals is the set **Z** × **N** with the equality

  $$(a, m) =_{\mathbf{Q}} (b, n) \Leftrightarrow a \cdot (n + 1) =_{\mathbf{Z}} b \cdot (m + 1).$$

  The arithmetical relations and operations are defined on **Q** in a straightforwad way; integers are embedded into **Q** by the mapping $a \mapsto (a, 0)$.

Real numbers

Definition

A real number is a sequence $(p_n)_n$ of rationals such that

$$\forall mn \left(|p_m - p_n| < 2^{-m} + 2^{-n}\right).$$

We shall write **R** for the set of real numbers as usual.

Remark

Rationals are embedded into **R** by the mapping $p \mapsto p^* = \lambda n.p$.

Ordering relation

Definition

Let $<$ be the ordering relation between real numbers $x = (p_n)_n$ and $y = (q_n)_n$ defined by

$$x < y \Leftrightarrow \exists n \left(2^{-n+2} < q_n - p_n\right) .$$

Proposition

Let $x, y, z \in \mathbf{R}$. *Then*

- $\neg(x < y \wedge y < x)$,
- $x < y \rightarrow x < z \vee z < y$.

Ordering relation

Proof.

Let $x = (p_n)_n$, $y = (q_n)_n$ and $z = (r_n)_n$, and suppose that $x < y$. Then there exists n such that $2^{-n+2} < q_n - p_n$. Setting $N = n + 3$, either $(p_n + q_n)/2 < r_N$ or $r_N \leq (p_n + q_n)/2$. In the former case, we have

$$\begin{aligned} 2^{-N+2} &< 2^{-n+1} - (2^{-(n+3)} + 2^{-n}) < \frac{q_n - p_n}{2} - (p_N - p_n) \\ &= \frac{p_n + q_n}{2} - p_N < r_N - p_N, \end{aligned}$$

and hence $x < z$. In the latter case, we have

$$\begin{aligned} 2^{-N+2} &< -(2^{-(n+3)} + 2^{-n}) + 2^{-n+1} < (q_N - q_n) + \frac{q_n - p_n}{2} \\ &= q_N - \frac{p_n + q_n}{2} \leq q_N - r_N, \end{aligned}$$

and hence $z < y$. $\square$

Apartness and equality

Definition

We define the apartness $\#$, the equality $=$, and the ordering relation $\leq$ between real numbers x and y by

- $x \# y \Leftrightarrow (x < y \vee y < x)$,
- $x = y \Leftrightarrow \neg(x \# y)$,
- $x \leq y \Leftrightarrow \neg(y < x)$.

Lemma

Let $x, y, z \in \mathbf{R}$. *Then*

- $x \# y \leftrightarrow y \# x$,
- $x \# y \rightarrow x \# z \vee z \# y$.

Apartness and equality

Proposition

Let $x, y, z \in \mathbf{R}$. *Then*

- $x = x$,
- $x = y \rightarrow y = x$,
- $x = y \wedge y = z \rightarrow x = z$.

Proposition

Let $x, x', y, y' \in \mathbf{R}$. *Then*

- $x = x' \wedge y = y' \wedge x < y \rightarrow x' < y'$,
- $\neg\neg(x < y \vee x = y \vee y < x)$,
- $x < y \wedge y < z \rightarrow x < z$.

Apartness and equality

Corollary

Let $x, x', y, y', z \in \mathbf{R}$. *Then*

- $x = x' \wedge y = y' \wedge x \# y \rightarrow x' \# y'$,
- $x = x' \wedge y = y' \wedge x \leq y \rightarrow x' \leq y'$,
- $x \leq y \leftrightarrow \neg\neg(x < y \vee x = y)$,
- $\neg\neg(x \leq y \vee y \leq x)$,
- $x \leq y \wedge y \leq x \rightarrow x = y$,
- $x < y \wedge y \leq z \rightarrow x < z$,
- $x \leq y \wedge y < z \rightarrow x < z$,
- $x \leq y \wedge y \leq z \rightarrow x \leq z$.

Apartness and equality

Proposition

$\forall xy \in \mathbf{R}(x \# y \vee x = y) \Leftrightarrow \mathrm{LPO}$,

Proof.

($\Leftarrow$): Let $x = (p_n)_n$ and $y = (q_n)_n$, and define a binary sequence α by

$$\alpha(n) = 1 \Leftrightarrow 2^{-n+2} < |q_n - p_n|.$$

Then $\alpha \# \mathbf{0} \leftrightarrow x \# y$, and hence $x \# y \vee x = y$, by LPO.
($\Rightarrow$): Let α be a binary sequence α with at most one nonzero term, and define a sequence $(p_n)_n$ of rationals by

$$p_n = \sum_{k=0}^{n} \alpha(k) \cdot 2^{-k}.$$

Then $x = (p_n)_n \in \mathbf{R}$, and $x \# 0 \leftrightarrow \alpha \# \mathbf{0}$. Therefore $\alpha \# \mathbf{0} \vee \neg\alpha \# \mathbf{0}$, by $x \# 0 \vee x = 0$. $\square$

Apartness and equality

Proposition

- $\forall xy \in \mathbf{R}(\neg x = y \vee x = y) \Leftrightarrow \mathrm{WLPO}$,
- $\forall xy \in \mathbf{R}(x \leq y \vee y \leq x) \Leftrightarrow \mathrm{LLPO}$,
- $\forall xy \in \mathbf{R}(\neg x = y \to x \# y) \Leftrightarrow \mathrm{MP}$,
- $\forall xyz \in \mathbf{R}(\neg x = y \to \neg x = z \vee \neg z = y) \Leftrightarrow \mathrm{MP}^{\vee}$,
- $\forall xy \in \mathbf{R}(\forall z \in \mathbf{R}(\neg x = z \vee \neg z = y) \to x \# y) \Leftrightarrow \mathrm{WMP}$.

Arithmetical operations

The arithmetical operations are defined on $\mathbf{R}$ in a straightforwad way.

For $x = (p_n), y = (q_n) \in \mathbf{R}$, define

- $x + y = (p_{n+1} + q_{n+1})$;
- $-x = (-p_n)$;
- $|x| = (|p_n|)$;
- $\max\{x, y\} = (\max\{p_n, q_n\})$;
- $\vdots$

References

- Peter Aczel and Michael Rathjen, *CST Book draft*, 2010, http://www1.maths.leeds.ac.uk/~rathjen/book.pdf.
- Errett Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.
- Errett Bishop and Douglas Bridges, *Constructive Analysis*, Springer-Verlag, Berlin, 1985.
- Douglas Bridges and Fred Richman, *Varieties of Constructive Mathematics*, Cambridge Univ. Press, London, 1987.
- Douglas Bridges and Luminiţa Vîţă, *Techniques of Constructive Analysis*, Springer, New York, 2006.
- D. van Dalen, *Logic and Structure*, 5th ed., Springer, London, 2013.
- A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics, An Introduction*, Vol. I, North-Holland, Amsterdam, 1988.