Algorithmic Aspects in Financial Mathematics

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Part III: The Fundamental Theorem of Asset Pricing

Josef Berger and Gregor Svindland, *A separating hyperplane theorem, the fundamental theorem of asset pricing, and Markov's principle*, Annals of Pure and Applied Logic 167 (2016) 1161–1170

There are m assets. Their value at time 0 (present) is known. Their value at time 1 (future) is unknown. There are n cases and we know the values in each case.

This information is contained in a $\mathbb{R}^{m \times n}$-matrix A.

The value of the entry a_{ij} is the price development (price at time 1 minus price at time 0) of asset i in case j.

Set

$$P = \left\{ p \in \mathbb{R}^n \mid \textstyle\sum_{i=1}^n p_i = 1 \text{ and } 0 < p_i \text{ for all } i \right\}.$$

$p \in P$ is a *martingale measure* if $A \cdot p = 0$

Under a martinagle measure the average profit is zero, that is todays price of the assets is reasonable in the sense of being the expected value of the assets tomorrow.

For $x \in \mathbb{R}^n$ we define

$$x > 0 \quad :\Leftrightarrow \quad \forall i \, (x_i \geq 0) \ \wedge \ \exists i \, (x_i > 0) \, .$$

$\xi \in \mathbb{R}^m$ is an *arbitrage trading strategy* if $\xi \cdot A > 0$

The existence of an arbitrage trading stategy implies the possibility of risk-less profit.

The *fundamental theorem of asset pricing* says that the absence of an arbitrage trading strategy is equivalent to the existence of a martingale measure.

FTAP Fix a $\mathbb{R}^{m \times n}$-matrix A. Then

$$\neg \exists \xi \in \mathbb{R}^m \; (\xi \cdot A > 0) \;\Leftrightarrow\; \exists p \in P \; (A \cdot p = 0) \,.$$

"$\Leftarrow$" is clear (consider $\xi \cdot A \cdot p$)

Proposition 1

$$\text{FTAP} \Leftrightarrow \text{MP}$$

$\mathrm{MP} \Rightarrow \mathrm{FTAP}$

Fix a $\mathbb{R}^{m\times n}$-matrix A such that

$$\neg\exists\xi \in \mathbb{R}^m \; (\xi \cdot A > 0)\,.$$

Let Y be the linear subspace of $\mathbb{R}^n$ which is generated by the rows of A. Let C be the convex hull of the unit vectors of $\mathbb{R}^n$. By MP, we obtain

$$\forall c \in C, y \in Y \, (d(c, y) > 0)\,.$$

By the separation theorem, there exist $p \in \mathbb{R}^n$ and reals α, β such that

$$\forall y \in Y, c \in C \, (\langle p, c\rangle > \alpha > \beta > \langle p, y\rangle)\,.$$

This implies that $A \cdot p = 0$ and that all components of p are positive. We can assume further that $p_1 + \ldots + p_n = 1$.

FTAP $\Rightarrow$ MP

Fix a real number $a \neq 0$. Apply FTAP to

$$A = (|a|, -1).$$

The no-arbitrage condition is satisfied: the existence of $\xi \in \mathbb{R}$ with $(\xi \cdot |a|, -\xi) > 0$ would imply $a = 0$.

Now FTAP yields the existence of a $p \in P$ with

$$p_1 \cdot |a| = p_2.$$

This implies that $|a| > 0$.

We obtain the following constructive version of FTAP.

FTAP' Fix a $\mathbb{R}^{m\times n}$-matrix A. Then

$$\forall \xi \in \mathbb{R}^m\, y \in Y\; d(\xi{\cdot}A, y) > 0 \;\Rightarrow\; \exists p \in P\; (A \cdot p = 0)\,.$$

why?

Part IV: Brouwer's Fan Theorem

- $\{0,1\}^*$ the set of finite binary sequences
- $u, v, w \in \{0,1\}^*$
- $|u|$ the length of u
- $\overline{u}n$ the restriction of u to the first n elements
- $u * v$ the concatenation of u and v
- $i \in \{0,1\}$
- α, β infinite binary sequences

$\mathrm{B} \subseteq \{0,1\}^*$ is

- *detachable* if $\forall u\,(u \in \mathrm{B} \vee u \notin \mathrm{B})$
- a *bar* if $\forall \alpha\, \exists n\,(\overline{\alpha} n \in \mathrm{B})$
- a *uniform bar* if $\exists N\, \forall \alpha\, \exists n \leq N\,(\overline{\alpha} n \in \mathrm{B})$

FAN_Δ every detachable bar is a uniform bar

FAN every bar is a uniform bar

neither provable nor falsifiable in Bishop's constructive mathematics

Lemma (Julian, Richman 1984)

The following are equivalent:

- FAN_Δ
- $f : [0,1] \to \mathbb{R}^+$ u/c $\Rightarrow$ $\inf f > 0$

Lemma (B., Svindland 2016)

$f : [0,1] \to \mathbb{R}^+$ u/c + **convexity** $\Rightarrow$ $\inf f > 0$

Is there a corresponding extra condition on bars such that FAN becomes constructively valid?

$$u < v \;:\Leftrightarrow\; |u| = |v| \wedge \exists i < |u| \, (\overline{u}i = \overline{v}i \wedge u_i = 0 \wedge v_i = 1)$$

$$u \leq v \;:\Leftrightarrow\; u = v \vee u < v.$$

$\mathrm{A} \subseteq \{0,1\}^*$ is *co-convex* if $u \in \mathrm{A}$ implies that either

$$\{v \mid v \leq u\} \subseteq \mathrm{A}$$

or

$$\{v \mid u \leq v\} \subseteq \mathrm{A}\,.$$

Proposition. *Every co-convex bar is a uniform bar.*

Fix a co-convex bar B. We can assume that B is *closed under extension*:

$$u \in \mathrm{B} \Rightarrow u * 0 \in \mathrm{B} \wedge u * 1 \in \mathrm{B}$$

u is *secure* if

$$\exists n \forall w \in \{0,1\}^n \, (u * w \in \mathrm{B})$$

Claim 1. *For every u, either $u * 0$ is secure or $u * 1$ is secure.*

There exists a function

$$F : \{0, 1\}^* \to \{0, 1\}$$

such that

$$\forall u \; (u * F(u) \text{ is secure}) \, .$$

Define α by

$$\alpha_n = 1 - F(\overline{\alpha} n).$$

Claim 2. $\forall n \forall u \in \{0,1\}^n \,(u \neq \overline{\alpha} n \Rightarrow u \textit{ is secure})$

There exists n such that $\overline{\alpha} n$ is secure. Therefore, every u of length n is secure. Therefore, B is a uniform bar.

□

Proof of Claim 1. For

$$\beta := 1 * 0 * 0 * 0 * \dots$$

there exists a positive l with $\overline{\beta} l \in \mathrm{B}$. Set $m = l - 1$. By co-convexity of B, we either have

$$\left\{ v \mid v \leq \overline{\beta} l \right\} \subseteq \mathrm{B} \quad \text{or} \quad \left\{ v \mid \overline{\beta} l \leq v \right\} \subseteq \mathrm{B}.$$

In the first case,

$$0 * w \in \mathrm{B}$$

for every w of length m, which implies that 0 is secure. In the second case,

$$1 * w \in \mathrm{B}$$

for every w of length m, which implies that 1 is secure.

□

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