Algorithmic Aspects in Financial Mathematics

Part I: Convex Functions

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$t \in [0,1]$ is a *minimum point* of $f : [0,1] \to \mathbb{R}$ if

$$\forall s \in [0,1]\,(f(t) \leq f(s))\,.$$

Lemma 1

Every continuous function $f : [0,1] \to \mathbb{R}$ has a minimum point.

Proof.

Let (t_n) be a sequence such that

$$f(t_n) \to \inf\{f(t) \mid t \in [0,1]\}\,.$$

This sequence has a convergent subsequence, its limit is a minimum point. □

good points:

- strong statement
- easy proof

bad point:

- no algorithm

In *Constructive Mathematics*, we avoid to use the law of excluded middle. When we prove the existence of an object, we really can construct it.

How to learn Constructive Mathematics?

- avoid indirect proofs of undecidable statements
- be careful with infima, comparing reals, compactness
- the intuition for constructive reasoning comes automatically and quickly

Constructive Reverse Mathematics investigates how constructive a theorem is.

Why combining Logic and Financial Mathematics?

- Finance depends on Stochastics, Stochastics is highly non-constructive
- people in Finance like algorithms (how should I invest?)
- Optimization Theory is a promising area for Constructive (Reverse) Mathematics
- project *CORE Constructive Operations Research*
- LMUexcellent start-up grant

for $f : [0, 1] \to \mathbb{R}$ the implications

uniform continuity

$$\Downarrow$$

pointwise continuity

$$\Downarrow$$

sequentially continuity

are strict

A sequence (I_n) of subintervals $I_n = [a_n, b_n] \subseteq [0,1]$ is *well behaved* if

- $a_n, b_n \in \mathbb{Q}$
- $I_0 = [0,1]$
- $0 < b_1 - a_1 < 1$
- $I_{n+1} \subseteq I_n$
- $b_n - a_n = (b_1 - a_1)^n$.

Lemma 2

Suppose that (I_n) is well behaved.

- *There exists a unique $t \in \cap I_n$.*
- *For all rational $s \in [0,1]$ with $|s - t| > 0$ there exist n with $s \in I_n \setminus I_{n+1}$.*

$f : [0,1] \to \mathbb{R}$ is *strictly quasi-convex* if

$$r < s < t \quad \Rightarrow \quad f(s) < \max\left(f(r), f(t)\right).$$

This is equivalent to

$$r < s < t \quad \Rightarrow \quad f(s) < f(r) \quad \vee \quad f(s) < f(t).$$

Lemma 3

Suppose that $f : [0,1] \to \mathbb{R}$ is sequentially continuous and strictly quasi-convex. Then f has a minimum point.

Proof. If $f(\frac{1}{2}) < f(\frac{1}{3})$, set

$$[a_1, b_1] = \left[\tfrac{1}{3}, 1\right]$$

and if $f(\frac{1}{2}) < f(\frac{2}{3})$, set

$$[a_1, b_1] = \left[0, \tfrac{2}{3}\right].$$

In both cases, we have

$$s \in [0,1] \setminus [a_1, b_1] \;\; \Rightarrow \;\; f(\tfrac{1}{2}) < f(s).$$

Iterating this, we obtain a well behaved sequence of intervals $I_n = [a_n, b_n]$ with midpoints t_n such that

- $t_n \in I_{n+1}$
- $s \in I_n \setminus I_{n+1} \Rightarrow f(t_n) < f(s)$.

Let t be in $\cap I_n$. Note that

$$s \in I_n \cap \mathbb{Q} \wedge f(s) < f(t) \rightarrow \exists m \geq n \left(f(t_m) < f(s)\right).$$

We can conclude

- $f(t) \leq f(t_n)$
- t is a minimum point of f.

□

We assume that Lemma 3 can be easily generalised as follows.

Lemma 4

Suppose that $C \subseteq \mathbb{R}^n$ is compact and $f : C \to \mathbb{R}$ is sequentially continuous and strictly quasi-convex. Then f has a minimum point.

A function $f : [0,1] \to \mathbb{R}$ is *convex* if

$$f(\lambda \cdot s + (1-\lambda) \cdot t) \leq \lambda \cdot f(s) + (1-\lambda) \cdot f(t).$$

for all $s, t, \lambda \in [0,1]$.

Lemma 5

Suppose that $f : [0,1] \to \mathbb{R}$ is pointwise continuous and convex. Then there exists $\delta > 0$ such that

$$f(t) \geq \delta$$

for all $t \in [0,1]$.

Proof.

There exist a subinterval $[a, b]$ of $[0, 1]$ with $b - a = \frac{2}{3}$ and $\delta > 0$ such that

$$f(s) < \delta \Rightarrow s \in [a, b] .$$

Iterating this, we obtain a well behaved sequence (I_n) and a sequence (δ_n) with

$$f(s) < \delta_n \Rightarrow s \in I_n.$$

Let t be in $\cap I_n$. Let n be large enough such that

$$s \in I_n \to f(s) \geq \frac{f(t)}{2}$$

and set $\delta = \min\left(\frac{f(t)}{2}, \delta_n\right)$. □

We do not assume that Lemma 5 can be easily generalised as follows.

Lemma 6

Suppose that $C \subseteq \mathbb{R}^n$ is compact and $f : C \to \mathbb{R}$ is pointwise continuous and convex. Then there exists $\delta > 0$ such that

$$f(t) \geq \delta$$

for all $t \in C$.