

CHAPTER 3

A theory of computable functionals

After getting clear about the domains we intend to reason about, the partial continuous functionals and in particular the computable ones, we now set up a theory to prove their properties. The main concepts are those of inductively and coinductively defined predicates.

3.1. Inductive predicates

When we want to make propositions about computable functionals and their domains of partial continuous functionals, it is perfectly natural to take, as initial propositions, ones formed inductively or coinductively. For simplicity we postpone the treatment of coinductive definitions and until then deal with inductive definitions only.

Consider for example the concept of a well-founded (or “total”) binary tree. It is inductively defined by the clauses

$$0 \in T_{\mathbf{D}}, \quad \forall_{t,s}(t, s \in T_{\mathbf{D}} \rightarrow Cts \in T_{\mathbf{D}}),$$

and understood to be the least predicate satisfying these clauses. We could also obtain this predicate by means of its “characteristic function”, i.e., a boolean-valued function f defined by the computation rules

$$f(0) = \mathbf{tt}, \quad f(Cts) = (f(t) \text{ and } f(s)).$$

But this is not possible any more if we want to refer to infinitely many predecessors. An example is the *accessible part* of a binary relation $\prec$, which is inductively defined by

$$\forall_x(\forall_{y \prec x}(y \in \text{Acc}_{\prec} \rightarrow x \in \text{Acc}_{\prec})).$$

Here we need a universal quantifier in the premise.

Generally, an inductively defined predicate I is given by k clauses, which are of the form

$$K_i := \forall_{\vec{x}_i}((A_{i\nu}(I))_{\nu < n_i} \rightarrow I\vec{r}_i) \quad (i < k).$$

Here $A_{i\nu}(X)$ is a “formula” containing possibly a “predicate variable” X , which is substituted by I in the clause above. For the clause to make sense we need to require that X appears at most “strictly positive” in $A(X)$.

Our formulas will be defined by the operations of implication $A \rightarrow B$ and universal quantification $\forall_x A$ from inductively defined predicates $\mu_X \vec{K}$, where the K_i are “clauses”. Every predicate has an *arity*, which is a possibly empty list of types.

DEFINITION (Predicates and formulas). By simultaneous induction we define *predicate forms*

$$P, Q ::= X \mid \{ \vec{x} \mid A \} \mid \mu_X (\forall_{\vec{x}_i} ((A_{i\nu})_{\nu < n_i} \rightarrow X \vec{r}_i))_{i < k}$$

and *formula forms*

$$A, B ::= P \vec{r} \mid A \rightarrow B \mid \forall_x A$$

with X a predicate variable, $k \geq 1$ and $\vec{x}_i$ all free variables in $(A_{i\nu})_{\nu < n_i} \rightarrow X \vec{r}_i$ (it is not necessary to allow object parameters in inductively defined predicates, since they can be taken as extra arguments). Let C denote both predicate and formula forms, and $PV(C)$ denote the set of (free) predicate variables in C . We define $SP(Y, C)$ “ Y occurs at most strictly positive in C ” by induction on C .

$$\begin{array}{c} SP(Y, X) \quad \frac{SP(Y, A)}{SP(Y, \{ \vec{x} \mid A \})} \quad \frac{SP(Y, A_{i\nu}) \text{ for all } i < k, \nu < n_i}{SP(Y, \mu_X (\forall_{\vec{x}_i} ((A_{i\nu})_{\nu < n_i} \rightarrow X \vec{r}_i))_{i < k})} \\ \frac{SP(Y, P)}{SP(Y, P \vec{r})} \quad \frac{Y \notin PV(A) \quad SP(Y, B)}{SP(Y, A \rightarrow B)} \quad \frac{SP(Y, A)}{SP(Y, \forall_x A)} \end{array}$$

Now we can define $P(P)$ “ P is a predicate” and $F(A)$ “ A is a formula”, again by simultaneous induction.

$$\begin{array}{c} P(X) \quad \frac{F(A)}{P(\{ \vec{x} \mid A \})} \\ \frac{F(A_{i\nu}) \text{ and } SP(X, A_{i\nu}) \text{ for all } i < k, \nu < n_i \quad P(\vec{Q})}{P(I(\vec{\rho}, \vec{Q}))} \\ \frac{P(P)}{P(P \vec{r})} \quad \frac{F(A) \quad F(B)}{F(A \rightarrow B)} \quad \frac{F(A)}{F(\forall_x A)} \end{array}$$

with

$$I(\vec{\alpha}, \vec{Y}) := \mu_X (\forall_{\vec{x}_i} ((A_{i\nu})_{\nu < n_i} \rightarrow X \vec{r}_i))_{i < k}$$

where $\vec{Y}$ are all predicate variables except X strictly positive in some $A_{i\nu}$. We call I an *inductively defined predicate* or shortly *inductive predicate*.

Here $\vec{A} \rightarrow B$ means $A_0 \rightarrow \cdots \rightarrow A_{n-1} \rightarrow B$, associated to the right. The terms $\vec{r}$ are those introduced in Section 2.2.1, i.e., typed terms built from variables and constants by abstraction and application, and (importantly) those with a common reduct are identified. In $\forall_{\vec{x}} ((A_{\nu}(X))_{\nu < n} \rightarrow X \vec{r})$ we call $A_{\nu}(X)$ a *parameter* premise if X does not occur in it, and a *recursive*

premise otherwise. A recursive premise $A_\nu(X)$ is *nested* if it has an occurrence of X in a strictly positive parameter position of another (previously defined) inductive predicate, and *unnested* otherwise. An inductive predicate I is called *nested* if it has a clause with at least one nested recursive premise, and *unnested* otherwise. For simplicity we only deal with unnested inductive predicates here.

A predicate of the form $\{\vec{x} \mid C\}$ is called a *comprehension term*. We identify $\{\vec{x} \mid C(\vec{x})\}\vec{r}$ with $C(\vec{r})$. For a predicate C of arity $(\rho, \vec{\sigma})$ we write Cr for $\{\vec{y} \mid Cr\vec{y}\}$. An inductive predicate is *finitary* if its clauses have recursive premises of the form $X\vec{s}$ only.

REMARK (Substitution for predicate parameters). Let $C(\vec{X})$ be a predicate or formula and $\vec{P}$ be predicates of the same arities as $\vec{X}$. By induction on C one can see easily that $C(\vec{P})$ is a predicate or formula again.

EXAMPLES. The *even numbers* are inductively defined by

$$\text{Even} := \mu_X(0 \in X, \forall_n(n \in X \rightarrow S(Sn) \in X)).$$

The *transitive closure* of a binary relation $\prec$ is inductively defined by

$$\text{TC}_\prec := \mu_X(\forall_{x,y}(x \prec y \rightarrow Xxy), \forall_{x,y,z}(x \prec y \rightarrow Xyz \rightarrow Xxz)).$$

An important example of an inductive predicate is *Leibniz equality*, defined simply by

$$\text{EqD} := \mu_X(\forall_x Xxx).$$

Existence, intersection and union can be defined inductively by

$$\begin{aligned} \text{Ex}_Y &:= \mu_X(\forall_x(x \in Y \rightarrow X)), \\ \text{Cap}_{Y,Z} &:= \mu_X(\forall_{\vec{x}}(Y\vec{x} \rightarrow Z\vec{x} \rightarrow X\vec{x})), \\ \text{Cup}_{Y,Z} &:= \mu_X(\forall_{\vec{x}}(Y\vec{x} \rightarrow X\vec{x}), \forall_{\vec{x}}(Z\vec{x} \rightarrow X\vec{x})). \end{aligned}$$

We will use the abbreviations

$$\begin{aligned} (x \equiv y) &:= \text{EqD}(x, y), & P \cap Q &:= \text{Cap}_{P,Q}, \\ \exists_x A &:= \text{Ex}_{\{x \mid A\}}, & P \cup Q &:= \text{Cup}_{P,Q}. \end{aligned}$$

We define a theory of computable functionals, called TCF. Formulas are those in F defined above, involving typed variables. Derivations use the rules of minimal logic for $\rightarrow$ and $\forall$, and the following axioms. For each inductive predicate, there are “closure” or introduction axioms, together with a “least-fixed-point” or elimination axiom. In more detail, consider an inductive predicate

$$I := \mu_X(\forall_{\vec{x}_i}((A_{i\nu}(X))_{\nu < n_i} \rightarrow X\vec{r}_i))_{i < k}.$$

For every $i < k$ we have a *clause* (or *introduction axiom*)

$$(10) \quad I_i^+ : \forall \vec{x}_i ((A_{i\nu}(I))_{\nu < n_i} \rightarrow I\vec{r}_i).$$

Moreover, we have an *elimination axiom*

$$(11) \quad I^- : \forall \vec{x} (I\vec{x} \rightarrow (\forall \vec{x}_i ((A_{i\nu}(I \cap X))_{\nu < n_i} \rightarrow X\vec{r}_i))_{i < k} \rightarrow X\vec{x})$$

($I \cap X$ was inductively defined above). Here X can be thought of as a “competitor” predicate. We take all substitution instances of I_i^+ , I^- (w.r.t. substitutions for type and predicate variables) as axioms.

The reason for writing the elimination axiom (11) in the present way is that it fits more conveniently with the logical elimination rules. It expresses that every “competitor” X satisfying the same clauses contains I . Notice that we have used a “strengthened” form of the “step formula”, namely $\forall \vec{x}_i (A_{i\nu}(I \cap X))_{\nu < n_i} \rightarrow X\vec{r}_i$ rather than $\forall \vec{x}_i (A_{i\nu}(X))_{\nu < n_i} \rightarrow X\vec{r}_i$. In applications of the least-fixed-point axiom this simplifies the proof of the “step”, since we have an additional I -hypothesis available.

EXAMPLES. (i) For Even the introduction axioms are

$$0 \in \text{Even}, \quad \forall n (n \in \text{Even} \rightarrow S(Sn) \in \text{Even})$$

and the elimination axiom is

$$\forall n (n \in \text{Even} \rightarrow 0 \in X \rightarrow \forall n (n \in \text{Even} \rightarrow n \in X \rightarrow S(Sn) \in X) \rightarrow n \in X).$$

(ii) For the transitive closure $\text{TC}_<$ the introduction axioms are

$$\begin{aligned} \forall x, y (x < y \rightarrow \text{TC}_<(x, y)), \\ \forall x, y, z (x < y \rightarrow \text{TC}_<(y, z) \rightarrow \text{TC}_<(x, z)) \end{aligned}$$

and the elimination axiom is

$$\begin{aligned} \forall x, y (\text{TC}_<(x, y) \rightarrow \forall x, y (x < y \rightarrow Xxy) \rightarrow \\ \forall x, y, z (x < y \rightarrow \text{TC}_<(y, z) \rightarrow Xyz \rightarrow Xxz) \rightarrow \\ Xxy). \end{aligned}$$

(iii) Let $<$ be a binary relation. Its *accessible part* is inductively defined by

$$\text{Acc}_< := \mu_X (\forall x (\forall y < x (y \in X) \rightarrow x \in X)).$$

The introduction axiom is

$$\forall x (\forall y < x (y \in \text{Acc}_<) \rightarrow x \in \text{Acc}_<),$$

where $\forall y < x A$ stands for $\forall y (y < x \rightarrow A)$. The elimination axiom is

$$\forall x (x \in \text{Acc}_< \rightarrow \forall x (\forall y < x (y \in \text{Acc}_<) \rightarrow \forall y < x (y \in X) \rightarrow x \in X) \rightarrow x \in X).$$

(iv) Existence, intersection and union were defined inductively by

$$\text{Ex}_Y := \mu_X (\forall x (x \in Y \rightarrow X)),$$

$$\begin{aligned}\text{Cap}_{Y,Z} &:= \mu_X(\forall_{\vec{x}}(Y\vec{x} \rightarrow Z\vec{x} \rightarrow X\vec{x})), \\ \text{Cup}_{Y,Z} &:= \mu_X(\forall_{\vec{x}}(Y\vec{x} \rightarrow X\vec{x}), \forall_{\vec{x}}(Z\vec{x} \rightarrow X\vec{x})).\end{aligned}$$

together with the abbreviations

$$\begin{aligned}\exists_x A &:= \text{Ex}_{\{x|A\}}, \\ P \cap Q &:= \text{Cap}_{P,Q}, \\ P \cup Q &:= \text{Cup}_{P,Q}.\end{aligned}$$

For nullary predicates $P = \{ \mid A \}$ and $Q = \{ \mid B \}$ we write $A \wedge B$ for $P \cap Q$ and $A \vee B$ for $P \cup Q$. Then – as in Chapter 1 – the introduction axioms are

$$\begin{aligned}\forall_x(A \rightarrow \exists_x A), \\ A \rightarrow B \rightarrow A \wedge B, \\ A \rightarrow A \vee B, \quad B \rightarrow A \vee B\end{aligned}$$

and the elimination axioms

$$\begin{aligned}\exists_x A \rightarrow \forall_x(A \rightarrow B) \rightarrow B \quad (x \notin \text{FV}(B)), \\ A \wedge B \rightarrow (A \rightarrow B \rightarrow C) \rightarrow C, \\ A \vee B \rightarrow (A \rightarrow C) \rightarrow (B \rightarrow C) \rightarrow C.\end{aligned}$$

(v) For Leibniz equality the introduction axiom is

$$\forall_x(x^\rho \equiv x^\rho)$$

and the elimination axiom

$$\forall_{x,y}(x \equiv y \rightarrow \forall_x Xxx \rightarrow Xxy),$$

where $x \equiv y$ abbreviates $\text{EqD}(\rho)(x^\rho, y^\rho)$.

We conclude this section with some comments on Leibniz equality.

LEMMA (Compatibility of EqD). $\forall_{x,y}(x \equiv y \rightarrow A(x) \rightarrow A(y))$.

PROOF. Exercise. □

Using compatibility of EqD one easily proves symmetry and transitivity. Define *falsity* by $\mathbf{F} := (\mathbf{ff} \equiv \mathbf{tt})$.

THEOREM (Ex-falso-quodlibet). *For every formula A we can derive $\mathbf{F} \rightarrow A$ from assumptions $\text{Ef}_Y: \forall_{\vec{x}}(\mathbf{F} \rightarrow Y\vec{x})$ for predicate variables Y strictly positive in A , and $\text{Ef}_I: \forall_{\vec{x}}(\mathbf{F} \rightarrow I\vec{x})$ for inductive predicates I without a nullary clause.*

PROOF. We first show $\text{EfEqD}: \mathbf{F} \rightarrow x^\rho \equiv y^\rho$. To see this, we first obtain $\mathcal{R}_{\mathbf{B}}^\rho \mathbf{ff}xy \equiv \mathcal{R}_{\mathbf{B}}^\rho \mathbf{ff}xy$ from the introduction axiom. Then from $\mathbf{ff} \equiv \mathbf{tt}$ we get $\mathcal{R}_{\mathbf{B}}^\rho \mathbf{tt}xy \equiv \mathcal{R}_{\mathbf{B}}^\rho \mathbf{ff}xy$ by compatibility. Now $\mathcal{R}_{\mathbf{B}}^\rho \mathbf{tt}xy$ converts to x and $\mathcal{R}_{\mathbf{B}}^\rho \mathbf{ff}xy$ converts to y . Hence $x^\rho \equiv y^\rho$, since we identify terms with a common reduct.

The claim can now be proved by induction on $A \in \mathbf{F}$. *Case $I\vec{s}$.* If I has no nullary clause take Ef_I . Otherwise let K_i be the nullary clause, with final conclusion $I\vec{t}$. By induction hypothesis from $\mathbf{F}$ we can derive all parameter premises. Hence $I\vec{t}$. From $\mathbf{F}$ we also obtain $s_i \equiv t_i$, by the remark above. Hence $I\vec{s}$ by compatibility. The cases $Y\vec{s}$, $A \rightarrow B$ and $\forall_x A$ are obvious. $\square$

A crucial use of the equality predicate EqD is that it allows us to lift a boolean term $r^{\mathbf{B}}$ to a formula, using $\text{atom}(r^{\mathbf{B}}) := (r^{\mathbf{B}} \equiv \mathbf{tt})$. This opens up a convenient way to deal with equality on finitary algebras. The computation rules ensure that, for instance, the boolean term $Sr =_{\mathbf{N}} Ss$, or more precisely $=_{\mathbf{N}}(Sr, Ss)$, is identified with $r =_{\mathbf{N}} s$. We can now turn this boolean term into the formula $(Sr =_{\mathbf{N}} Ss) \equiv \mathbf{tt}$, which again is abbreviated by $Sr =_{\mathbf{N}} Ss$, but this time with the understanding that it is a formula. Then (importantly) the two formulas $Sr =_{\mathbf{N}} Ss$ and $r =_{\mathbf{N}} s$ are identified because the latter is a reduct of the first. Consequently there is no need to prove the implication $Sr =_{\mathbf{N}} Ss \rightarrow r =_{\mathbf{N}} s$ explicitly.

3.2. Pointwise equality, totality and extensionality

We define totality T_ρ simultaneously with pointwise equality $\dot{=}_\rho$. The definition follows Gandy (1956) and is by induction on the type ρ . Because of the general case in the definition of totality we also write E_ρ for T_ρ and call it an extensionality predicate.

For an arrow type $\rho \rightarrow \sigma$ we define

$$\begin{aligned} (f \in E_{\rho \rightarrow \sigma}) &= \forall_x (x \in E_\rho \rightarrow fx \in E_\sigma) \wedge \forall_{x,y} (x \dot{=}_\rho y \rightarrow fx \dot{=}_\sigma fy), \\ (f \dot{=}_{\rho \rightarrow \sigma} g) &= (f, g \in E_{\rho \rightarrow \sigma} \wedge \forall_x (x \in E_\rho \rightarrow fx \dot{=}_\sigma gx)). \end{aligned}$$

In case of an algebra ι we only consider finitary algebras. For instance, $E_{\mathbf{N}}$ is inductively defined by the clauses

$$\begin{aligned} 0 &\in E_{\mathbf{N}}, \\ \forall_n (n \in E_{\mathbf{N}} \rightarrow Sn \in E_{\mathbf{N}}) \end{aligned}$$

and pointwise equality $\dot{=}_{\mathbf{N}}$ is defined by

$$\begin{aligned} 0 &\dot{=}_{\mathbf{N}} 0, \\ \forall_{n,m} (n \dot{=}_{\mathbf{N}} m \rightarrow Sn \dot{=}_{\mathbf{N}} Sm). \end{aligned}$$

REMARK. For finitary algebras – for instance $\mathbf{N}$ – we can prove

$$\begin{aligned} n \dot{=}_{\mathbf{N}} m &\rightarrow n \equiv m, \\ n, m \in E_{\mathbf{N}} &\rightarrow n = m \rightarrow n \equiv m, \\ n \in E_{\mathbf{N}} &\rightarrow n = n. \end{aligned}$$

Here $n = m$ means $= (n, m) \equiv \mathbf{tt}$, where $=: \mathbf{N} \rightarrow \mathbf{N} \rightarrow \mathbf{B}$ is the binary boolean-valued function defined in Section 2.2.1. Hence $\dot{=}$, $\equiv$ and $=$ are equivalent on $E_{\mathbf{N}}$. Usage of $=$ has the advantage that proofs become shorter, since we identify terms with a common reduct.

LEMMA. $x \dot{=} y \rightarrow x, y \in E_{\rho}$.

PROOF. By induction on ρ . *Case ι .* Use the elimination axiom for $\dot{=}_{\iota}$. *Case $\rho \rightarrow \sigma$.* By definition. $\square$

LEMMA (Reflexivity of $\dot{=}$ on E). $x \in E_{\rho} \rightarrow x \dot{=} x$.

PROOF. By induction on ρ . *Case ι .* Use the elimination axiom for E_{ι} . *Case $\rho \rightarrow \sigma$.* Let $f \in E_{\rho \rightarrow \sigma}$, i.e.,

$$\forall x(x \in E_{\rho} \rightarrow fx \in E_{\sigma}) \wedge \forall_{x,y}(x \dot{=}_{\rho} y \rightarrow fx \dot{=}_{\sigma} fy).$$

We need to show $f \dot{=}_{\rho \rightarrow \sigma} f$. $f \in E_{\rho \rightarrow \sigma}$ is already given; it remains to show $\forall x(x \in E_{\rho} \rightarrow fx \dot{=}_{\sigma} fx)$. Let $x \in E_{\rho}$. By induction hypothesis on ρ we have $x \dot{=} x$. But then $f \in E_{\rho \rightarrow \sigma}$ implies the claim $fx \dot{=}_{\sigma} fx$. $\square$

Using these two lemmas (which are based on the definitions above) we can prove the following properties, which can simplify some arguments.

LEMMA. (a) *For finitary ρ from $\forall x(x \in E_{\rho} \rightarrow fx \in E_{\sigma})$ we can infer $\forall_{x,y}(x \dot{=}_{\rho} y \rightarrow fx \dot{=}_{\sigma} fy)$.*
 (b) *The converse holds generally:*

$$\forall_{x,y}(x \dot{=}_{\rho} y \rightarrow fx \dot{=}_{\sigma} fy) \rightarrow \forall x(x \in E_{\rho} \rightarrow fx \in E_{\sigma}).$$

PROOF. (a). Assume $\forall x(x \in E_{\rho} \rightarrow fx \in E_{\sigma})$ and $x \dot{=}_{\rho} y$. The goal is $fx \dot{=}_{\sigma} fy$. From $x \dot{=}_{\rho} y$ we obtain $x \equiv y$ by the remark above, and also $x \in E_{\rho}$ (by the first lemma), hence $fx \in E_{\sigma}$ (by assumption), hence $fx \dot{=}_{\sigma} fx$ (by reflexivity of $\dot{=}$ on E_{σ}), hence $fx \dot{=}_{\sigma} fy$ by compatibility.

(b). Assume $\forall_{x,y}(x \dot{=}_{\rho} y \rightarrow fx \dot{=}_{\sigma} fy)$ and $x \in E_{\rho}$. The goal is $fx \in E_{\sigma}$. Then $x \dot{=}_{\rho} x$ (by reflexivity of $\dot{=}$ on E_{ρ}), hence $fx \dot{=}_{\sigma} fx$ (by assumption), hence $fx \in E_{\sigma}$ (by the first lemma). $\square$

We now prove symmetry and transitivity of $\dot{=}$.

LEMMA (Symmetry of $\dot{=}$). $x \dot{=}_{\rho} y \rightarrow y \dot{=}_{\rho} x$.

PROOF. By induction on ρ . *Case ι .* Use the elimination axiom for $\dot{=}_{\iota}$. *Case $\rho \rightarrow \sigma$.* The claim follows from the fact that $f \dot{=}_{\rho \rightarrow \sigma} g$ is symmetric in f and g . $\square$

LEMMA (Transitivity of $\dot{=}$). $x \dot{=}_{\rho} y \rightarrow y \dot{=}_{\rho} z \rightarrow x \dot{=}_{\rho} z$.

PROOF. By induction on ρ . *Case ι .* Since $\dot{=}_\iota$ implies E_ι it suffices to prove

$$\forall x(x \in E_\iota \rightarrow \forall y(y \in E_\iota \rightarrow x \dot{=}_\iota y \rightarrow \forall z(z \in E_\iota \rightarrow y \dot{=}_\iota z \rightarrow x \dot{=}_\iota z))).$$

This can be done by (nested) inductions on E_ι , using the fact that constructors have disjoint ranges and are injective, both w.r.t. $\equiv$ and $\dot{=}$.

Case $\rho \rightarrow \sigma$. The claim follows from the definition of $\dot{=}_{\rho \rightarrow \sigma}$ and the induction hypothesis for σ . $\square$

LEMMA (Closure of E under application).

$$f \in E_{\rho \rightarrow \sigma} \rightarrow x \in E_\rho \rightarrow fx \in E_\sigma.$$

PROOF. From $x \in E_\rho$ we obtain $x \dot{=}_\rho x$ by reflexivity. Then $f \in E_{\rho \rightarrow \sigma}$ gives $fx \dot{=}_\sigma fx$, whence $fx \in E_\sigma$. $\square$

LEMMA (Compatibility of application with $\dot{=}$).

$$f \dot{=}_{\rho \rightarrow \sigma} g \leftrightarrow \forall x, y(x \dot{=}_\rho y \rightarrow fx \dot{=}_\sigma gy).$$

PROOF. $\rightarrow$. Assume $f \dot{=}_{\rho \rightarrow \sigma} g$. Then by definition $f, g \in E_{\rho \rightarrow \sigma}$ and

$$\forall x(x \in E_\rho \rightarrow fx \dot{=}_\sigma gx).$$

Assume further $x \dot{=}_\rho y$. We must show $fx \dot{=}_\sigma gy$. From $x \dot{=}_\rho y$ we get $x \in E_\rho$, whence $fx \dot{=}_\sigma gx$. From $g \in E_{\rho \rightarrow \sigma}$ and $x \dot{=}_\rho y$ we obtain $gx \dot{=}_\sigma gy$. Now transitivity of $\dot{=}$ gives the claim.

$\leftarrow$. Assume $\forall x, y(x \dot{=}_\rho y \rightarrow fx \dot{=}_\sigma gy)$. The goal is $f \dot{=}_{\rho \rightarrow \sigma} g$. We first show $f \in E_{\rho \rightarrow \sigma}$; for g the argument is similar. By a lemma above it suffices to prove $\forall x, y(x \dot{=}_\rho y \rightarrow fx \dot{=}_\sigma fy)$. Let x, y with $x \dot{=}_\rho y$ be given. Then $y \in E_\rho$, and $y \dot{=}_\rho y$ by reflexivity. The assumption applied to $x \dot{=}_\rho y$ gives $fx \dot{=}_\sigma gy$, and applied to $y \dot{=}_\rho y$ gives $fy \dot{=}_\sigma gy$. Transitivity and symmetry of $\dot{=}$ now implies the claim $fx \dot{=}_\sigma fy$. We finally show $\forall x(x \in E_\rho \rightarrow fx \dot{=}_\sigma gx)$. Let $x \in E_\rho$ be given. Then $x \dot{=}_\rho x$ by reflexivity and hence $fx \dot{=}_\sigma gx$ by our assumption. $\square$

3.3. Coinductive definitions

We now extend TCF by allowing “coinductive” definitions as well as inductive ones. For instance, in the algebra $\mathbf{N}$ we can coinductively define *cototality* by the “closure” axiom

$$n \in {}^{\text{co}}T_{\mathbf{N}} \rightarrow n \equiv 0 \vee \exists m(m \in {}^{\text{co}}T_{\mathbf{N}} \wedge n \equiv Sm).$$

Generally, a coinductive predicate J is given by exactly one clause, which is of the form

$$\forall \vec{x}(J\vec{x} \rightarrow \bigvee_{i < k} \exists \vec{x}_i \bigwedge_{\nu < n_i} A_{i\nu}(J)).$$

However, here we do not need this generality, and restrict ourselves to a special situation: every inductive predicate I gives rise to an important example of a coinductive predicate, its *dual* or *companion* ${}^{\text{co}}I$. Let I be inductively defined by the clauses

$$\forall \vec{x}_i ((A_{i\nu}(I))_{\nu < n_i} \rightarrow I\vec{r}_i) \quad (i < k).$$

The conjunction of these k clauses is equivalent to

$$\forall \vec{x} (\bigwedge_{i < k} \exists \vec{x}_i (\bigwedge_{\nu < n_i} A_{i\nu}(I) \wedge \vec{x} \equiv \vec{r}_i) \rightarrow I\vec{x}).$$

Now the dual ${}^{\text{co}}I$ of I is coinductively defined by its *closure axiom* ${}^{\text{co}}I^-$:

$$\forall \vec{x} ({}^{\text{co}}I\vec{x} \rightarrow \bigwedge_{i < k} \exists \vec{x}_i (\bigwedge_{\nu < n_i} A_{i\nu}({}^{\text{co}}I) \wedge \vec{x} \equiv \vec{r}_i)).$$

Its *greatest-fixed-point axiom* ${}^{\text{co}}I^+$ is

$$\forall \vec{x} (X\vec{x} \rightarrow \forall \vec{x} (X\vec{x} \rightarrow \bigwedge_{i < k} \exists \vec{x}_i (\bigwedge_{\nu < n_i} A_{i\nu}({}^{\text{co}}I \vee X) \wedge \vec{x} \equiv \vec{r}_i)) \rightarrow {}^{\text{co}}I\vec{x}).$$

Here we may also use another binary relation instead of Leibniz equality $\equiv$.

More precisely, we extend the definition of formulas and predicates in Section 3.1 to also include the dual of an inductive predicate I , defined by

$${}^{\text{co}}I(\vec{\alpha}, \vec{Y}) := \nu_X (\forall \vec{x}_i ((A_{i\nu})_{\nu < n_i} \rightarrow X\vec{r}_i))_{i < k}.$$

The proof of the ex-falso-quodlibet theorem in Section 3.1 can be extended to also cover ${}^{\text{co}}I$, even in cases where no nullary clause is present. To see this, use the greatest-fixed-point axiom for ${}^{\text{co}}I$ with $X := \{\vec{x} \mid \mathbf{F}\}$. Then any $\exists \vec{x}_i (\bigwedge_{\nu < n_i} A_{i\nu}({}^{\text{co}}I \vee X) \wedge \vec{x} \equiv \vec{r}_i)$ is provable, since all $A_{i\nu}({}^{\text{co}}I \vee \mathbf{F})$ can be proved from $\mathbf{F}$ by induction hypothesis and we have $\text{EfEqD} : \mathbf{F} \rightarrow x^\rho \equiv y^\rho$.

EXAMPLES. (i) For Even with the introduction axioms

$$0 \in \text{Even}, \quad \forall n (n \in \text{Even} \rightarrow S(Sn) \in \text{Even})$$

its dual ${}^{\text{co}}\text{Even}$ is defined by the closure axiom ${}^{\text{co}}\text{Even}^-$:

$$\forall n (n \in {}^{\text{co}}\text{Even} \rightarrow n \equiv 0 \vee \exists m (m \in {}^{\text{co}}\text{Even} \wedge n \equiv S(Sm))).$$

Its greatest-fixed-point axiom ${}^{\text{co}}\text{Even}^+$ is

$$\begin{aligned} \forall n (n \in X \rightarrow \\ \forall n (n \in X \rightarrow n \equiv 0 \vee \exists m ((m \in {}^{\text{co}}\text{Even} \vee m \in X) \wedge n \equiv S(Sm))) \rightarrow \\ n \in {}^{\text{co}}\text{Even}). \end{aligned}$$

It expresses that every “competitor” X satisfying the same closure axiom is a subset of ${}^{\text{co}}\text{Even}$.

(ii) An important example of a coinductive predicate occurs when we represent real numbers by “streams” of signed digits (cf. Section 2.2.3). We inductively define a predicate I by the single clause

$$(12) \quad \forall_{d,x',x} (d \in \text{Sd} \rightarrow x' \in I \rightarrow x = \frac{x' + d}{2} \rightarrow x \in I).$$

Here x ranges over real numbers and d over integers. Sd is a (formally inductive) predicate expressing that the integer d is a signed digit, i.e., $|d| \leq 1$.

We have chosen (12) rather than the simpler

$$(13) \quad \forall_{d,x} (d \in \text{Sd} \rightarrow x \in I \rightarrow \frac{x + d}{2} \in I),$$

since we want I to be compatible with the defined equality $=$ on real numbers

$$(14) \quad \forall_{x,y} (x = y \rightarrow x \in I \rightarrow y \in I),$$

which easily follows from (12) (with reflexivity, symmetry and transitivity of $=$). Using (14) we then obtain (13) from (12).

Dually to I we coinductively define a predicate ${}^{\text{co}}I$ by the closure axiom

$$(15) \quad \forall_x (x \in {}^{\text{co}}I \rightarrow \exists_{d,x',y} (d \in \text{Sd} \wedge x' \in {}^{\text{co}}I \wedge y = \frac{x' + d}{2} \wedge x = y)).$$

Similar to what was done for I above we can simplify (15) to ${}^{\text{co}}I^-$:

$$\forall_x (x \in {}^{\text{co}}I \rightarrow \exists_{d,x'} (d \in \text{Sd} \wedge x' \in {}^{\text{co}}I \wedge x = \frac{x' + d}{2})).$$

More formally, both I and ${}^{\text{co}}I$ are defined as fixed points of an operator

$$\Phi(X) := \{ x \mid \exists_{d,x'} (d \in \text{Sd} \wedge x' \in X \wedge x = \frac{x' + d}{2}) \}.$$

Then

$$\begin{aligned} I &:= \mu_X \Phi(X) && \text{least fixed point} \\ {}^{\text{co}}I &:= \nu_X \Phi(X) && \text{greatest fixed point} \end{aligned}$$

satisfy the (strengthened) axioms

$$\begin{aligned} \Phi(I \cap X) \subseteq X &\rightarrow I \subseteq X && \text{induction} \\ X \subseteq \Phi({}^{\text{co}}I \cup X) &\rightarrow X \subseteq {}^{\text{co}}I && \text{coinduction} \end{aligned}$$

(they are called “strengthened” because their hypotheses are weaker than the fixed point property $\Phi(X) = X$). More explicitly, the greatest-fixed-point (or coinduction) axiom is ${}^{\text{co}}I^+$:

$$\begin{aligned} &\forall_x (x \in X \rightarrow \\ &\forall_x (x \in X \rightarrow \exists_{d,x'} (d \in \text{Sd} \wedge (x' \in {}^{\text{co}}I \vee x' \in X) \wedge x = \frac{x' + d}{2})) \rightarrow x \in {}^{\text{co}}I). \end{aligned}$$

3.4. Bisimilarity, cototality and coextensionality

We define cototality ${}^{\text{co}}T_\rho$ simultaneously with the dual ${}^{\text{co}}\dot{=}_\rho$ of pointwise equality $\dot{=}_\rho$. The definition again follows Gandy (1956) and is by induction on the type ρ . Because of the general case in the definition of cototality we also write ${}^{\text{co}}E_\rho$ for ${}^{\text{co}}T_\rho$ and call it a coextensionality predicate. The relation ${}^{\text{co}}\dot{=}_\rho$ is called bisimilarity; we write it as $\approx_\rho$.

For an arrow type $\rho \rightarrow \sigma$ we define

$$\begin{aligned} (f \in {}^{\text{co}}E_{\rho \rightarrow \sigma}) &= \forall_x (x \in {}^{\text{co}}E_\rho \rightarrow fx \in {}^{\text{co}}E_\sigma) \wedge \forall_{x,y} (x \approx_\rho y \rightarrow fx \approx_\sigma fy), \\ (f \approx_{\rho \rightarrow \sigma} g) &= (f, g \in {}^{\text{co}}E_{\rho \rightarrow \sigma} \wedge \forall_x (x \in {}^{\text{co}}E_\rho \rightarrow fx \approx_\sigma gx)). \end{aligned}$$

In case of an algebra ι we only consider finitary algebras. For instance, ${}^{\text{co}}E_{\mathbf{N}}$ is defined by the closure axiom ${}^{\text{co}}E_{\mathbf{N}}^-$:

$$\forall_n (n \in {}^{\text{co}}E_{\mathbf{N}} \rightarrow n \equiv 0 \vee \exists_m (m \in {}^{\text{co}}E_{\mathbf{N}} \wedge n \equiv Sm))$$

and $\approx_{\mathbf{N}}$ by the closure axiom $\approx_{\mathbf{N}}^-$:

$$\forall_{n,m} (n \approx_{\mathbf{N}} m \rightarrow (n \equiv 0 \wedge m \equiv 0) \vee \exists_{n_1, m_1} (n_1 \approx_{\mathbf{N}} m_1 \wedge n \equiv Sn_1 \wedge m \equiv Sm_1)).$$

LEMMA. $x \approx y \rightarrow x, y \in {}^{\text{co}}E_\rho$.

PROOF. By induction on ρ . *Case ι .* As an example we give the proof for $\mathbf{N}$. We use the greatest-fixed-point axiom ${}^{\text{co}}E_{\mathbf{N}}^+$:

$$\begin{aligned} \forall_n (n \in X \rightarrow \forall_n (n \in X \rightarrow n \equiv 0 \vee \exists_m ((m \in {}^{\text{co}}E_{\mathbf{N}} \vee m \in X) \wedge n \equiv Sm))) \rightarrow \\ n \in {}^{\text{co}}E_{\mathbf{N}}). \end{aligned}$$

with the competitor predicate $X := \{n \mid \exists_m (n \approx m)\}$. It suffices to prove the step formula

$$\forall_n (\exists_m (n \approx m) \rightarrow n \equiv 0 \vee \exists_l ((l \in {}^{\text{co}}E_{\mathbf{N}} \vee \exists_{m_1} (l \approx m_1)) \wedge n \equiv Sl)).$$

Assume $n \approx m$. To prove the disjunction we use the closure axiom $\approx_{\mathbf{N}}^-$:

$$\forall_{n,m} (n \approx m \rightarrow (n \equiv 0 \wedge m \equiv 0) \vee \exists_{n_1, m_1} (n_1 \approx m_1 \wedge n \equiv Sn_1 \wedge m \equiv Sm_1)).$$

In the first case we already know $n \equiv 0$. In the second case we have n_1, m_1 with $n_1 \approx m_1$, $n \equiv Sn_1$ and $m \equiv Sm_1$. Take l to be n_1 . *Case $\rho \rightarrow \sigma$.* By definition. $\square$

LEMMA (Reflexivity of $\approx$ on ${}^{\text{co}}E$). $x \in {}^{\text{co}}E_\rho \rightarrow x \approx x$.

PROOF. By induction on ρ . *Case ι .* As an example we give the proof for $\mathbf{N}$. We use the greatest-fixed-point axiom $\approx_{\mathbf{N}}^+$:

$$\begin{aligned} \forall_{n,m} (Xnm \rightarrow \forall_{n,m} (Xnm \rightarrow (n \equiv 0 \wedge m \equiv 0) \vee \\ \exists_{n_1, m_1} ((n_1 \approx m_1 \vee Xn_1m_1) \wedge n \equiv Sn_1 \wedge m \equiv Sm_1))) \rightarrow \\ n \approx m) \end{aligned}$$

with competitor predicate $X := \{n, m \mid \exists l(l \in {}^{\text{co}}E \wedge n \equiv l \wedge m \equiv l)\}$. It suffices to prove the step formula, for then we have

$$\exists l(l \in {}^{\text{co}}E \wedge n \equiv l \wedge n \equiv l) \rightarrow n \approx n$$

and hence the claim. For the step formula assume $l \in {}^{\text{co}}E \wedge n \equiv l \wedge m \equiv l$. We have to prove

$$(n \equiv 0 \wedge m \equiv 0) \vee$$

$$\exists_{n_1, m_1}((n_1 \approx m_1 \vee \exists l(l \in {}^{\text{co}}E \wedge n_1 \equiv l \wedge m_1 \equiv l)) \wedge n \equiv Sn_1 \wedge m \equiv Sm_1).$$

To prove the disjunction we use the closure axiom ${}^{\text{co}}E_{\mathbf{N}}^-$ for both n and m

$$n \in {}^{\text{co}}E_{\mathbf{N}} \rightarrow n \equiv 0 \vee \exists l(l \in {}^{\text{co}}E_{\mathbf{N}} \wedge n \equiv Sl),$$

$$m \in {}^{\text{co}}E_{\mathbf{N}} \rightarrow m \equiv 0 \vee \exists l(l \in {}^{\text{co}}E_{\mathbf{N}} \wedge m \equiv Sl).$$

Since constructors have disjoint ranges from the two disjunctions either the two left hand sides or else the two right hand sides hold. In the first case we are done. In the second case we have $n_1 \in {}^{\text{co}}E$ with $n \equiv Sn_1$ and also $m_1 \in {}^{\text{co}}E$ with $m \equiv Sm_1$. Go for the right hand side of the goal disjunction with these n_1, m_1 . Pick l to be m_1 . It remains to prove $n_1 \equiv m_1$. But this follows from $Sn_1 \equiv Sm_1$ by the injectivity of the constructor S w.r.t. $\equiv$.

Case $\rho \rightarrow \sigma$. Let $f \in {}^{\text{co}}E_{\rho \rightarrow \sigma}$:

$$\forall x(x \in {}^{\text{co}}E_{\rho} \rightarrow fx \in {}^{\text{co}}E_{\sigma}) \wedge \forall_{x,y}(x \approx_{\rho} y \rightarrow fx \approx_{\sigma} fy).$$

We must show $f \approx_{\rho \rightarrow \sigma} f$. $f \in {}^{\text{co}}E_{\rho \rightarrow \sigma}$ is already given; it remains to show $\forall x(x \in {}^{\text{co}}E_{\rho} \rightarrow fx \approx_{\sigma} fx)$. Let $x \in {}^{\text{co}}E_{\rho}$. We must show $fx \approx_{\sigma} fx$. By IH_{ρ} we have $x \approx_{\rho} x$. But then $f \in {}^{\text{co}}E_{\rho \rightarrow \sigma}$ implies the claim. $\square$

LEMMA (Symmetry of $\approx$). $x \approx_{\rho} y \rightarrow y \approx_{\rho} x$.

PROOF. By induction on ρ . *Case ι .* As an example one may take the proof for $\mathbf{N}$. It uses the closure axiom $\approx_{\mathbf{N}}^-$; details are left as an exercise.

Case $\rho \rightarrow \sigma$. The claim follows from the definition of $\approx_{\rho \rightarrow \sigma}$ with the induction hypothesis on σ . $\square$

LEMMA (Transitivity of $\approx$). $x \approx_{\rho} y \rightarrow y \approx_{\rho} z \rightarrow x \approx_{\rho} z$.

PROOF. By induction on ρ . *Case ι .* As an example we give the proof for $\mathbf{N}$. We use the greatest-fixed-point axiom $\approx_{\mathbf{N}}^+$:

$$\begin{aligned} & \forall_{n,l}(Xnl \rightarrow \forall_{n,l}(Xnl \rightarrow (n \equiv 0 \wedge l \equiv 0)) \vee \\ & \quad \exists_{n_1, l_1}((n_1 \approx l_1 \vee Xn_1l_1) \wedge n \equiv Sn_1 \wedge l \equiv Sl_1)) \rightarrow \\ & \quad n \approx l) \end{aligned}$$

with competitor predicate $X := \{n, l \mid \exists m(n \approx m \wedge m \approx l)\}$. It suffices to prove the step formula, for then we have

$$\exists_m(n \approx m \wedge m \approx l) \rightarrow n \approx l$$

and hence the claim. For the step formula assume $n \approx m \wedge m \approx l$. We must prove

$$(n \equiv 0 \wedge l \equiv 0) \vee \exists_{n_1, l_1} ((n_1 \approx l_1 \vee \exists_{m_1} (n_1 \approx m_1 \wedge m_1 \approx l_1)) \wedge n \equiv Sn_1 \wedge l \equiv Sl_1).$$

To prove the disjunction use the closure axiom for both $n \approx m$ and $m \approx l$:

$$\begin{aligned} & \forall_{n, m} (n \approx m \rightarrow (n \equiv 0 \wedge m \equiv 0) \vee \exists_{n_1, m_1} (n_1 \approx m_1 \wedge n \equiv Sn_1 \wedge m \equiv Sm_1)), \\ & \forall_{m, l} (m \approx l \rightarrow (m \equiv 0 \wedge l \equiv 0) \vee \exists_{m_1, l_1} (m_1 \approx l_1 \wedge m \equiv Sm_1 \wedge l \equiv Sl_1)). \end{aligned}$$

Since constructors have disjoint ranges from the two disjunctions either the two left hand sides or else the two right hand sides hold. In the first case we are done. In the second case we have n_1, m_1 with $n_1 \approx m_1$, $n \equiv Sn_1$ and $m \equiv Sm_1$ and also m_1, l_1 with $m_1 \approx l_1$, $m \equiv Sm_1$ and $l \equiv Sl_1$. Go for the right hand side of the goal disjunction with n_1, l_1 and the present m_1 .

Case $\rho \rightarrow \sigma$. The claim follows from the definition of $\approx_{\rho \rightarrow \sigma}$ with the induction hypothesis on σ . $\square$

LEMMA (Closure of ${}^{\text{co}}E$ under application).

$$f \in {}^{\text{co}}E_{\rho \rightarrow \sigma} \rightarrow x \in {}^{\text{co}}E_{\rho} \rightarrow fx \in {}^{\text{co}}E_{\sigma}.$$

PROOF. From $x \in {}^{\text{co}}E_{\rho}$ we have $x \approx_{\rho} x$ by reflexivity. Then $f \in {}^{\text{co}}E_{\rho \rightarrow \sigma}$ gives $fx \approx_{\sigma} fx$, whence $fx \in {}^{\text{co}}E_{\sigma}$. $\square$

LEMMA (Compatibility of application with $\approx$).

$$f \approx_{\rho \rightarrow \sigma} g \leftrightarrow \forall_{x, y} (x \approx_{\rho} y \rightarrow fx \approx_{\sigma} gy).$$

PROOF. $\rightarrow$. Assume $f \approx_{\rho \rightarrow \sigma} g$. Then by definition $f, g \in {}^{\text{co}}E_{\rho \rightarrow \sigma}$ and

$$\forall_x (x \in {}^{\text{co}}E_{\rho} \rightarrow fx \approx_{\sigma} gx).$$

Assume further $x \approx_{\rho} y$. We must show $fx \approx_{\sigma} gy$. From $x \approx_{\rho} y$ we get $x \in {}^{\text{co}}E_{\rho}$, whence $fx \approx_{\sigma} gx$. From $g \in {}^{\text{co}}E_{\rho \rightarrow \sigma}$ and $x \approx_{\rho} y$ we obtain $gx \approx_{\sigma} gy$. Now transitivity of $\approx$ gives the claim.

$\leftarrow$. Assume $\forall_{x, y} (x \approx_{\rho} y \rightarrow fx \approx_{\sigma} gy)$. The goal is $f \approx_{\rho \rightarrow \sigma} g$. We first show $f \in {}^{\text{co}}E_{\rho \rightarrow \sigma}$; for g the argument is similar. For $\forall_x (x \in {}^{\text{co}}E_{\rho} \rightarrow fx \in {}^{\text{co}}E_{\sigma})$: let $x \in {}^{\text{co}}E_{\rho}$. Then $x \approx x$ (by reflexivity of $\approx$ on ${}^{\text{co}}E_{\rho}$), hence $fx \approx_{\sigma} gx$ (by assumption), hence $fx \in {}^{\text{co}}E_{\sigma}$ (by a lemma). For $\forall_{x, y} (x \approx_{\rho} y \rightarrow fx \approx_{\sigma} fy)$. Let x, y with $x \approx_{\rho} y$ be given. Then $y \in {}^{\text{co}}E_{\rho}$, and $y \approx_{\rho} y$ by reflexivity. The assumption applied to $x \approx_{\rho} y$ gives $fx \approx_{\sigma} gy$, and applied to $y \approx_{\rho} y$ gives $fy \approx_{\sigma} gy$. Transitivity and symmetry of $\approx$ now implies the claim $fx \approx_{\sigma} fy$. We finally show $\forall_x (x \in {}^{\text{co}}E_{\rho} \rightarrow fx \approx_{\sigma} gx)$. Let $x \in {}^{\text{co}}E_{\rho}$ be given. Then $x \approx_{\rho} x$ by reflexivity and hence $fx \approx_{\sigma} gx$ by our assumption. $\square$

REMARK. We can easily show that

$$n \in E_{\mathbf{N}} \rightarrow n \in {}^{\text{co}}E_{\mathbf{N}}.$$

This is left as an exercise.