

# FROM SHARP TO UNSHARP

## EXPLORING THE FRONTIERS OF QUANTUM LOGIC

### LECTURE III

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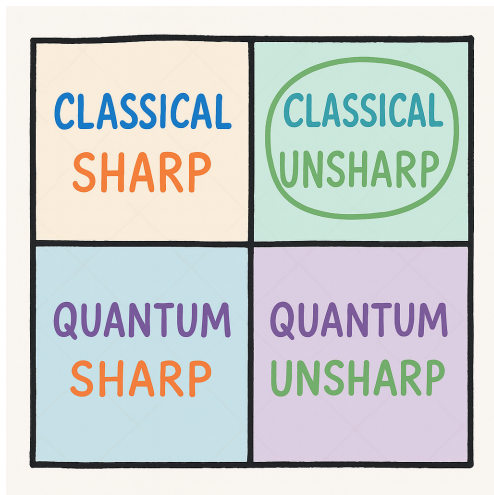


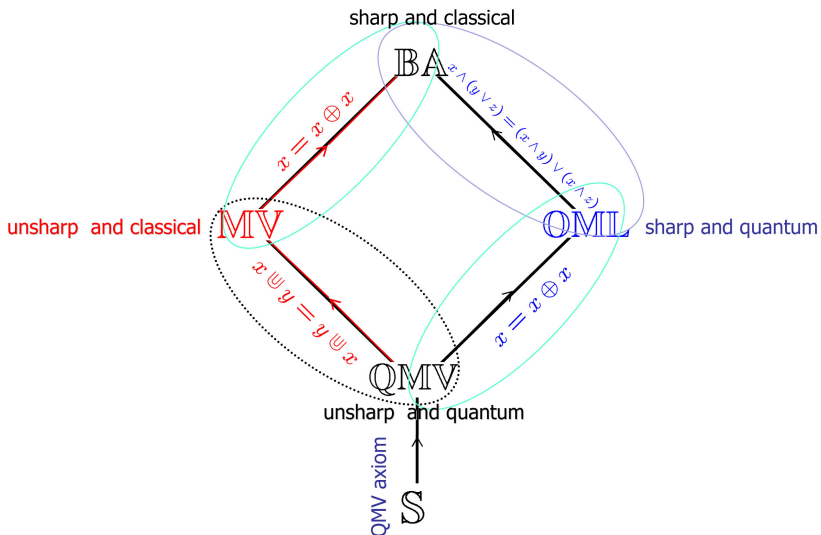
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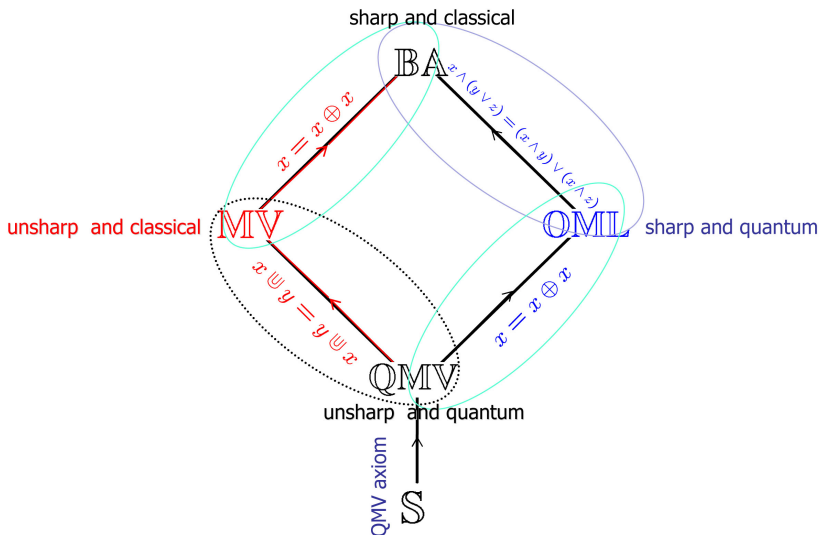








## THE SHARP AND THE UNSHARP UNIVERSES



## THE QUANTUM UNSHARP UNIVERSE AS A GENERALIZATION

## TWO CLASSICAL ROOTS

- Classical sharp (Boolean algebras)
  - Based on Law of Non-Contradiction:
  - $x \wedge x' = 0$ .
- Classical unsharp (MV-algebras)
  - Based on Łukasiewicz's Law:  $x \mathbin{\&}\! y = y \mathbin{\&}\! x$ .

- Both principles may fail:

- **Generalization:** Quantum MV-algebras (QMV-algebras) extend both Boolean and MV-algebras.



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## FROM PROJECTORS TO EFFECTS

# SHARP QUANTUM EVENTS

- Represented by **projectors**  $P$  on a Hilbert space:  
 $P^2 = P = P^*$ .
- Yes/No properties. Possible values:  $\{0, 1\}$ .
- Algebraic structure: Orthomodular Lattices (Quantum Logic).

- Represented by **effects**.
- Possible (eigen-)values of effects:  $[0, 1] \subseteq \mathbb{R}$ .
- Algebraic structure: **QMV-algebras**



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**Transition:** from *yes/no* projectors to *fuzzy* quantum properties.



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# SHARP QUANTUM EVENTS

- # UNSHARP QUANTUM EVENTS

- Transition:** from *yes/no* projectors to *fuzzy* quantum properties.





## UNSHARP QUANTUM MECHANICS: EFFECTS OF A HILBERT SPACE

- The spectrum  $\text{Spec}(E)$  is contained in  $[0, 1] \subset \mathbb{R}$ .
- There are some effects  $E$  such that

$$E^2 \neq E.$$

Thus,

$$\Pi(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}).$$

projectors      effects



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$$E = \begin{pmatrix} \frac{7}{16} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{16} \end{pmatrix}.$$

$$E^2 \neq E$$



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Fi



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- $E \leq F$  iff  $F - E$  is a **positive semidefinite** operator.



- $\mathcal{E}(\mathcal{H})$  can be equipped with an **involution** operation  $'$ :

$$E' = \mathbb{I} - E.$$









## KRIPKEAN INTERLUDE

Let us consider the following **accessibility relation**:

$$E \not\preceq F \text{ iff } E \not\leq F'.$$

The relation  $\not\preceq$  is **symmetric** but, in general, **not reflexive**. It may happen that  $F \perp F$ .

However,  $\not\preceq$  is **serial**:  $\forall E \exists F (E \not\preceq F)$ .

If  $\not\preceq$  is restricted to the set of all **of all projectors**,  $\not\preceq$  is **reflexive**.

Kripke Frame	Accessibility Rel.	Modal System	Modal Axiom
$\langle \mathcal{E}(\mathcal{H}), \not\preceq \rangle$	Symmetric and Serial	<b>D</b>	$\Box p \rightarrow \Diamond p$
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## THE ALGEBRAIC STRUCTURE(S?) OF EFFECTS

There are effects  $E$  that violate both the **non contradiction** and the **excluded-middle law**:

$$E \wedge E' \neq \mathbb{O} \text{ and } E \vee E' \neq \mathbb{I}$$



## THEOREM

- $F$  is a projector ( $E^2 = E$ ).
- $F \wedge F' = \mathbb{O}$

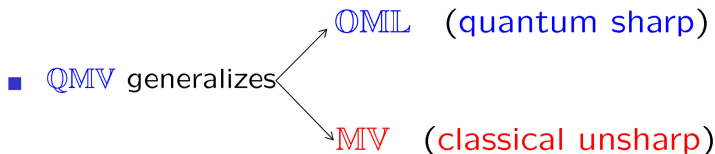


# WHY QMV

■  $\boxed{\text{BA} : \text{MV} = \text{OML} : \text{QMV}}$

↓
↓
↓
↓

Boolean algebras      MV algebras      orthomodular lattices      quantum MV algebras





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## S-ALGEBRAS

A **supplement algebra** (**S-algebra**) is a structure  $\mathcal{M} = (M, \oplus, ', 1, 0)$  of type  $\langle 2, 1, 0, 0 \rangle$  s.t.  $\forall x, y, z \in S$ :

- (S1)  $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ ;
- (S2)  $x \oplus y = y \oplus x$ ;
- (S3)  $(x')' = x$ ;
- (S4)  $x \oplus x' = 1$ ;
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## SOME DEFINITIONS

Let  $\mathcal{M} = (M, \oplus, ', 1, 0)$  be an S-algebra:

- $x \odot y := (x' \oplus y')'$ ;
- $x \mathbin{\textcolor{red}{\cap}} y := (x \oplus y') \odot y$   
(pseudo-inf; **generalized Sasaki projection**);
- $x \mathbin{\textcolor{red}{\cup}} y := (x \odot y') \oplus y$ ; (pseudo-sup)
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## QMV-ALGEBRAS

Recall:

MV := S + Łukasiewicz axiom

$$x \mathbin{\&}\! y = y \mathbin{\&}\! x.$$

QMV-algebras = S + QMV-axiom

$$x \oplus ((x' \mathbin{\&}\! y) \mathbin{\&}\! (z \mathbin{\&}\! x')) = (x \oplus y) \mathbin{\&}\! (x \oplus z)$$

QMV := the variety of all QMV-algebras

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## QMV-ALGEBRAS

# of elements	# of non-isomorphic QMV-algebras
1	1
2	1
3	1
4	4
5	5
6	16
7	21
8	88
9	127
10	817

---→ QMV-algebras



## THE STANDARD QMV-ALGEBRA(S)

$$\mathcal{E}(\mathcal{H}) := (E(\mathcal{H}), \oplus, ', \mathbb{I})$$

- $E(\mathcal{H})$  is the set of all effects of  $\mathcal{H}$ ;
- for any  $E, F \in E(\mathcal{H})$ :

$$E \oplus F := \begin{cases} E + F, & \text{if } E + F \in E(\mathcal{H}); \\ \mathbb{I}, & \text{otherwise,} \end{cases}$$

where  $+$  is the usual operator-sum.

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Let us define

$$E \preceq F \text{ iff } E \sqcap F = E.$$

$(E(\mathcal{H}), \preceq, ', \odot, \mathbb{I})$  is a (regular) *involutive bounded poset* that is *not* a lattice.





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## THE STANDARD QMV-ALGEBRA(S)

Let us consider the following effects (in the matrix-representation) on the Hilbert space  $\mathbb{R}^2$ :

$$E = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad F = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \quad G = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

It is not hard to see that  $G \preceq E, F$ .

Suppose, by contradiction, that  $L = E \wedge F$  exists in  $\mathcal{E}(\mathbb{R}^2)$ . An easy computation shows that  $L$  must be equal to  $G$ . Let

$$M = \begin{pmatrix} \frac{7}{16} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{16} \end{pmatrix}.$$

Then,  $M$  is an effect such that  $M \preceq E, F$ ; however,  $M \not\preceq L = G$ , which is a contradiction.



Let  $\mathcal{M} = (M, \oplus, ', 1, 0)$  be a QMV-algebra.

Recall:

- $x \odot y := (x' \oplus y')'$ ;
- $x \pitchfork y := (x \oplus y') \odot y$ ;
- $x \sqcup y := (x \odot y') \oplus y = (x' \pitchfork y')'$ .
- $x \preceq y$  if and only if  $x \pitchfork y = x$ .



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## SOME PROPERTIES OF QMV-ALGEBRAS

## THEOREM

- $\odot$  is associative and commutative;
- $x \odot x' = 0$ ;
- $x \odot 1 = x$  and  $x \odot 0 = 0$ ;
- $x \sqcap 1 = 1 \sqcap x = x$ ;
- $x \sqcap 0 = 0 \sqcap x = 0$ ;
- $x \sqcap x = x$ ; (idempotency);
- $(x \sqcap y)' = x' \sqcup y'$  and  $(x \sqcup y)' = x' \sqcap y'$ ;
- $x \leq y$  implies  $x = y \sqcap x$ ;





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- $x \sqcap 0 = 0 \sqcap x = 0$ ;
- $x \sqcap x = x$ ; (idempotency);
- $(x \sqcap y)' = x' \sqcup y'$  and  $(x \sqcup y)' = x' \sqcap y'$ ;
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## THEOREM

- $(x \sqcap y) \sqcap z = (x \sqcap y) \sqcap (y \sqcap z)$  (*weak associativity*)
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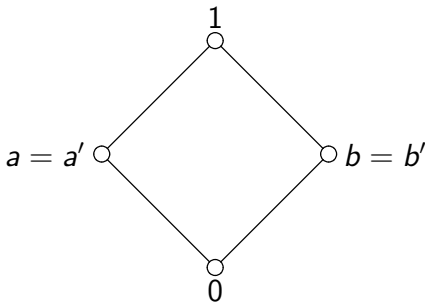
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## THE DIAMOND

The smallest genuine QMV-algebra is the “diamond” where, apart from the obvious conditions, the operation  $\oplus$  is defined as follows:  $a \oplus a = a \oplus b = b \oplus b = 1$ .





## QMV AS NON-COMMUTATIVE MV-ALGEBRAS

QMV-algebras can be thought of as **non-commutative** (w.r.t  $\bowtie$ ,  $\cup$ ) generalizations of MV-algebras since

$$\text{MV} = \text{QMV} + (x \bowtie y = y \bowtie x)$$



---



## ORTHOMODULAR LATTICES AS IDEMPOTENT QMV-ALGEBRAS

Orthomodular lattices can be thought of as **idempotent** (with respect to  $\oplus$ ) QMV-algebras .

## THEOREM

A QMV-algebra  $\mathcal{M}$  is an **orthomodular lattice** if and only if  $\forall x \in M: x \oplus x = x$ .

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## BA, MV, OML, QMV

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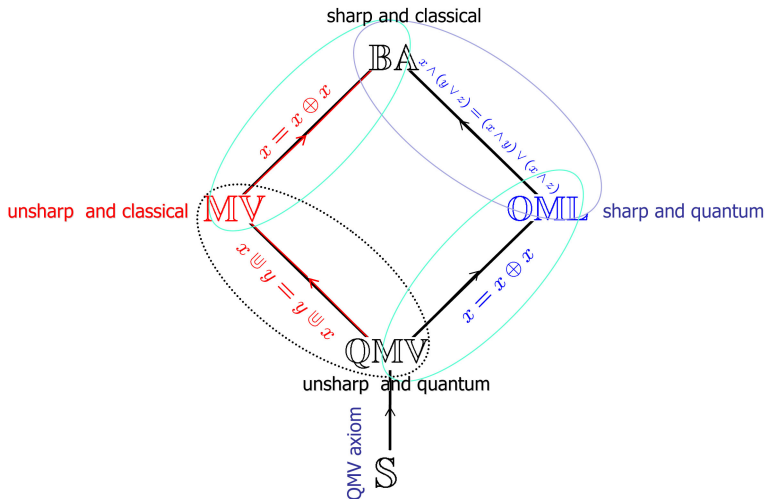
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## THE STANDARD MV-ALGEBRA

Recall:

$HSP(\mathcal{M}_{[0,1]})$  := the variety generated by the **standard** MV-algebra  $\mathcal{M}_{[0,1]}$ .

THEOREM (CHANG 1958; CHANG 1959)

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What about the **standard QMV-algebras**  $\mathcal{E}(\mathcal{H})$ ?

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$$\mathbf{EH} \subset \mathbf{QMV}$$

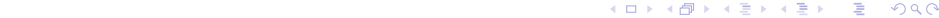
There is an equation that holds in the variety of all QMV-algebras of effects but fails in the variety of **all** QMV-algebras.







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*No... this is just the beginning!*

# THANK YOU

