From Sharp to Unsharp
Exploring the Frontiers of Quantum Logic
Lecture III

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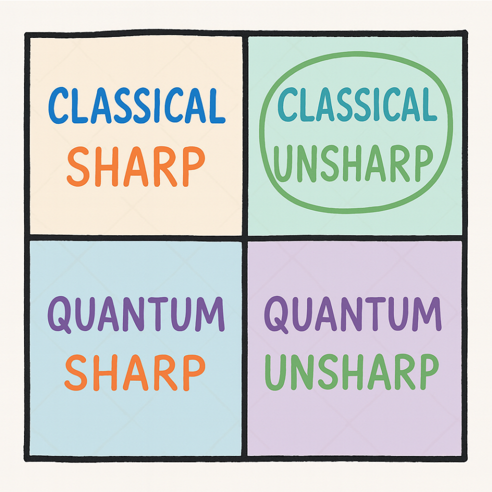
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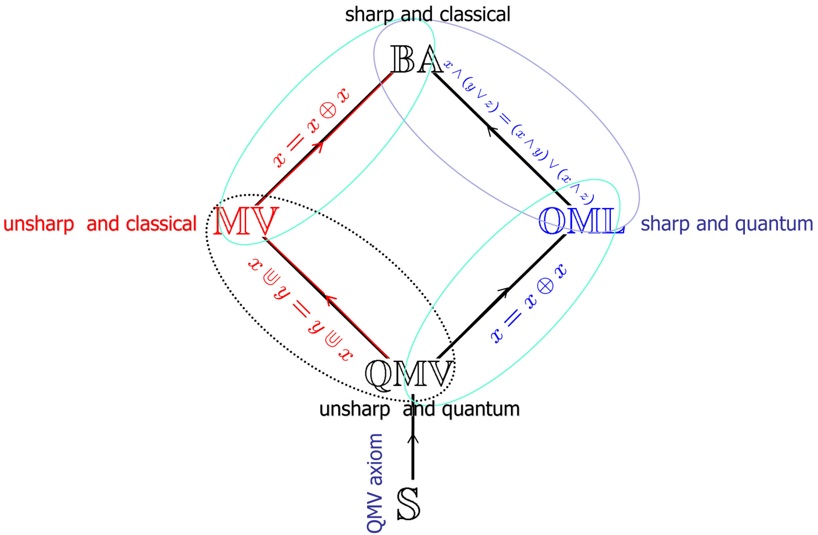
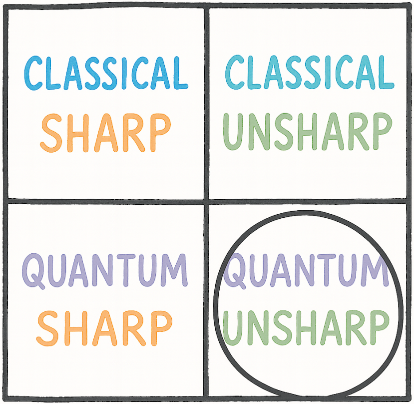
The classical unsharp universe

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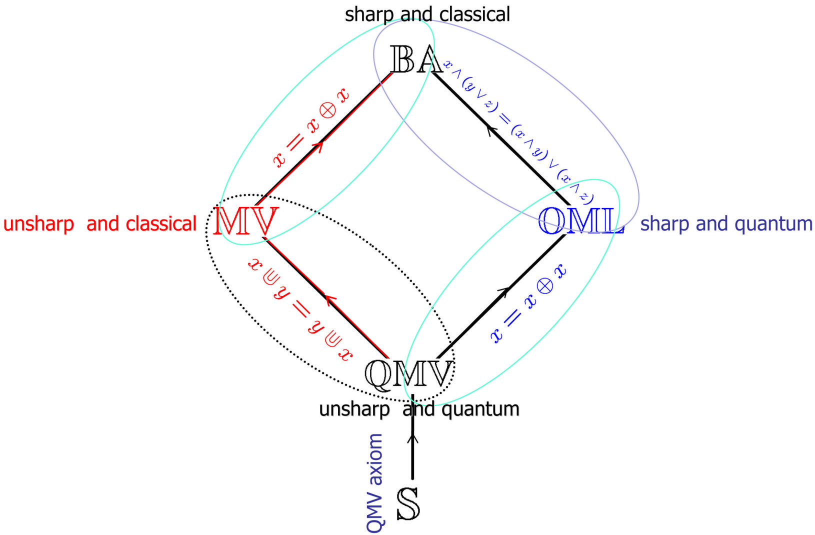
The Sharp and the Unsharp Universes



The Quantum Unsharp Universe as a Generalization

Two classical roots

- Classical sharp (Boolean algebras)
  - Based on Law of Non-Contradiction:
  - $x \wedge x' = 0$.
- Classical unsharp (MV-algebras)
  - Based on Łukasiewicz's Law: $x \Cap y = y \Cap x$.

Quantum unsharp universe

- Both principles may fail:

$$x \wedge x' = 0, \qquad x \Cap y \neq y \Cap x$$

- **Generalization:** Quantum MV-algebras (QMV-algebras) extend both Boolean and MV-algebras.

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From Projectors to Effects

Sharp quantum events

- Represented by projectors P on a Hilbert space: $P^2 = P = P^*$.
- Yes/No properties. Possible values: $\{0, 1\}$.
- Algebraic structure: Orthomodular Lattices (Quantum Logic).

Unsharp quantum events

- Represented by effects.
- Possible (eigen-)values of effects: $[0, 1] \subseteq \mathbb{R}$.
- Algebraic structure: QMV-algebras

Transition: from *yes/no* projectors to *fuzzy* quantum properties.

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Unsharp quantum mechanics (Ludwig, Kraus, Mittelstaedt, Busch)

The notion of **quantum event** is liberalized.

The set $\Pi(\mathcal{H})$ is replaced by the set of all effects of $\mathcal{H}$ (denoted by $\mathcal{E}(\mathcal{H})$), where an effect of $\mathcal{H}$ is a bounded linear operator E that satisfies the following condition:

$$\mathbb{O} \leq E \leq \mathbb{I},$$

where $R \leq S$ iff $S - E$ is a **positive semidefinite** operator.

Unsharp quantum mechanics: Effects of a Hilbert space

- The spectrum $Spec(E)$ is contained in $[0, 1] \subset \mathbb{R}$.
- There are some effects E such that

$$E^2 \neq E.$$

Thus,

$$\Pi(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}).$$

$$\swarrow \qquad \searrow$$

projectors effects

Unsharp quantum mechanics: Effects of a Hilbert space

Take

$$E = \begin{pmatrix} \frac{7}{16} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{16} \end{pmatrix}.$$

E is an effect.
The spectrum of E is $\{0.625, 0.135723\}$ and

$$E^2 \neq E$$

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- Can the set $\mathcal{E}(\mathcal{H})$ of all effects be equipped with an algebraic structure?

- $\mathcal{E}(\mathcal{H})$ can be partially ordered:

  $E \leq F$ iff $F - E$ is a **positive semidefinite** operator.

The algebraic strucure(s?) of effects

- $\mathcal{E}(\mathcal{H})$ can be equipped with an involution operation $'$:

$$E' = \mathbb{I} - E.$$

Theorem

$\langle \mathcal{E}(\mathcal{H}), \leq, ', \mathbb{O}, \mathbb{I} \rangle$ is a regular involutive bounde poset, i.e. an involutive bounded poset that satisfies the regularity condition:

$$\forall E, F \in \mathcal{E}(\mathcal{H}) : \text{if } E \leq E' \text{ and } F \leq F', \text{ then } E \leq F'.$$

However, $\langle \mathcal{E}(\mathcal{H}), \leq, ', \mathbb{O}, \mathbb{I} \rangle$ is **not** a lattice.

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Kripkean interlude

Let us consider the following accessibility relation:

$$E \not\perp F \text{ iff } E \not\leq F'.$$

The relation $\not\perp$ is symmetric but, in general, **not** reflexive. It may happen that $F \perp F$.
However, $\not\perp$ is serial: $\forall E \, \exists F \, (E \not\perp F)$.

If $\not\perp$ is restricted to the set of all of all projectors, $\not\perp$ is reflexive.

Kripke Frame	Accessibility Rel.	Modal System	Modal Axiom
$\langle \mathcal{E}(H), \not\perp \rangle$	Symmetric and Serial	**D**	$\Box p \rightarrow \Diamond p$
$\langle \Pi(\mathcal{H}), \not\perp \rangle$	Symmetric and Reflexive	**T**	$\Box p \rightarrow p$

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The algebraic strucure(s?) of effects

There are effects E that violate both the non contradiction and the excluded-middle law:

$$E \wedge E' \neq \mathbb{O} \text{ and } E \vee E' \neq \mathbb{I}$$

Projectors as sharp effects

Theorem

Let $\langle \mathcal{E}(\mathcal{H}), \leq, ', \mathbb{O}, \mathbb{I} \rangle$ be the involutive bounded poset of all effect of a Hilbert space $\mathcal{H}$. The following conditions are equivalent $\forall F \in \mathcal{E}(\mathcal{H})$:

- *F is a projector ($E^2 = E$).*
- $F \wedge F' = \mathbb{O}$

Why QMV

- $\mathbb{BA} : \mathbb{MV} = \mathbb{OML} : \mathbb{QMV}$

  Boolean algebras MV algebras orthomodular lattices quantum MV algebras

- $\mathbb{QMV}$ generalizes
  - $\mathbb{OML}$ (quantum sharp)
  - $\mathbb{MV}$ (classical unsharp)

S-algebras

A supplement algebra (S-algebra) is a structure $\mathcal{M} = (M, \oplus, ', 1, 0)$ of type $\langle 2, 1, 0, 0 \rangle$ s.t. $\forall x, y, z \in S$:

- (S1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (S2) $x \oplus y = y \oplus x$;
- (S3) $(x')' = x$;
- (S4) $x \oplus x' = 1$;
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$\mathbb{S}$:= the variety of all S-algebras.

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Some definitions

Let $\mathcal{M} = (M, \oplus, ', 1, 0)$ be an S-algebra:

- $x \odot y := (x' \oplus y')'$;
- $x \Cap y := (x \oplus y') \odot y$
  (pseudo-inf; generalized Sasaki projection);
- $x \Cup y := (x \odot y') \oplus y$; (pseudo-sup)
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QMV-ALGEBRAS

Recall:

$$\mathbb{MV} := \mathbb{S} \; + \; \text{Łukasiewicz axiom}$$

$$x \Cap y = y \Cap x.$$

QMV-algebras = S + QMV-axiom

$$x \oplus ((x' \Cap y) \Cap (z \Cap x')) = (x \oplus y) \Cap (x \oplus z)$$

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⇢ **QMV-algebras**

QMV-algebras

# of elements	# of non-isomorphic QMV-algebras
1	1
2	1
3	1
4	4
5	5
6	16
7	21
8	88
9	127
10	817

--→ **QMV-algebras**

THE STANDARD QMV-ALGEBRA(S)

$$\mathcal{E}(\mathcal{H}) := (E(\mathcal{H}), \oplus, {}', \mathbb{O}, \mathbb{I})$$

- $E(\mathcal{H})$ is the set of all effects of $\mathcal{H}$;
- for any $E, F \in E(\mathcal{H})$:

$$E \oplus F := \begin{cases} E + F, & \text{if } E + F \in E(\mathcal{H}); \\ \mathbb{I}, & \text{otherwise,} \end{cases}$$

  where $+$ is the usual operator-sum.
- $E' := \mathbb{I} - E$.

The standard QMV-algebra(s)

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The standard QMV-algebra(s)

Theorem

$\mathcal{E}(\mathcal{H}) := (E(\mathcal{H}), \oplus, ', \mathbb{O}, \mathbb{I})$ *is a QMV-algebra that is not an MV-algebra, since, in general,* $E \Cap F \neq F \Cap E$.

Let us define

$$E \preceq F \text{ iff } E \Cap F = E.$$

$(E(\mathcal{H}), \preceq, ', \mathbb{O}, \mathbb{I})$ is a (regular) involutive bounded poset that is not a lattice.

The standard QMV-algebra(s)

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The standard QMV-algebra(s)

Let us consider the following effects (in the matrix-representation) on the Hilbert space $\mathbb{R}^2$:

$$E = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad F = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \quad G = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

It is not hard to see that $G \preceq E, F$.
Suppose, by contradiction, that $L = E \wedge F$ exists in $\mathcal{E}(\mathbb{R}^2)$. An easy computation shows that L must be equal to G. Let

$$M = \begin{pmatrix} \frac{7}{16} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{16} \end{pmatrix}.$$

Then, M is an effect such that $M \preceq E, F$; however, $M \not\preceq L = G$, which is a contradiction.

Let $\mathcal{M} = (M, \oplus, ', 1, 0)$ be a QMV-algebra.

Recall:

- $x \odot y := (x' \oplus y')'$;
- $x \Cap y := (x \oplus y') \odot y$;
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Some properties of QMV-algebras

Theorem

- $\odot$ *is associative and commutative;*
- $x \odot x' = 0;$
- $x \odot 1 = x$ *and* $x \odot 0 = 0;$
- $x \Cap 1 = 1 \Cap x = x;$
- $x \Cap 0 = 0 \Cap x = 0;$
- $x \Cap x = x;$ *(idempotency);*
- $(x \Cap y)' = x' \Cup y'$ *and* $(x \Cup y)' = x' \Cap y';$
- $x \leq y$ *implies* $x = y \Cap x;$

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- $(x \Cap y) \Cap z = (x \Cap y) \Cap (y \Cap z)$ *(weak associativity)*
- $x \Cap (y \Cup x) = x$; *(weak absorption)*
- $x \Cup (y \Cap x) = x$. *(weak absorption)*
- $(M, \preceq, ', 0, 1)$ *is an involutive bounded poset:*
  - $(M, \preceq)$ *is a poset with maximum* (0) *and minimum* (1);
  - $x'' = x$;
  - $\forall x, y \in M$: *if* $x \preceq y$, *then* $y' \preceq x'$;

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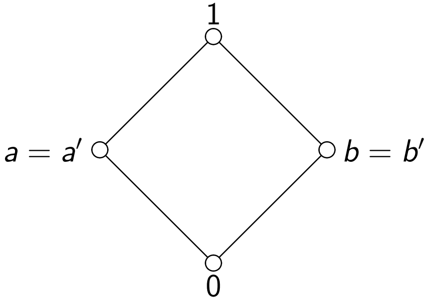
The diamond

The smallest genuine QMV-algebra is the "diamond" where, apart from the obvious conditions, the operation $\oplus$ is defined as follows: $a \oplus a = a \oplus b = b \oplus b = 1$.



QMVs as non-commutative MV-algebras

QMV-algebras can be thought of as non-commutative (w.r.t $\Cap$, $\Cup$) generalizations of MV-algebras since

$$\mathbb{MV} = \mathbb{QMV} \; + \; (x \Cap y = y \Cap x)$$

Orthomodular lattices as idempotent QMV-algebras

Orthomodular lattices can be thought of as idempotent (with respect to $\oplus$) QMV-algebras .

Theorem

A QMV-algebra $\mathcal{M}$ is an orthomodular lattice if and only if $\forall x \in M$: $x \oplus x = x$.

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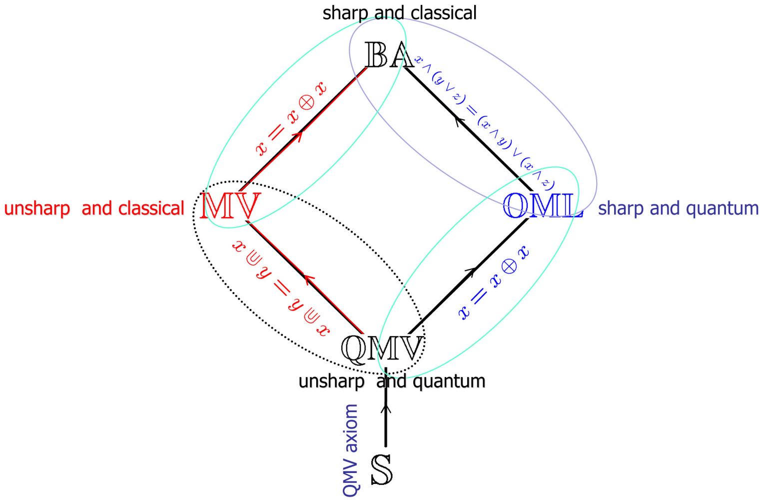
sharp and classical
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unsharp and classical
MV
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sharp and quantum
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S

The standard MV-algebra

Recall:

$HSP(\mathcal{M}_{[0,1]})$:= the variety generated by the standard MV-algebra $\mathcal{M}_{[0,1]}$.

Theorem (Chang 1958; Chang 1959)

$$\mathbb{MV} = HSP(\mathcal{M}_{[0,1]}).$$

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There is an equation that holds in the variety of all QMV-algebras of effects but fails in the variety of **all** QMV-algebras.

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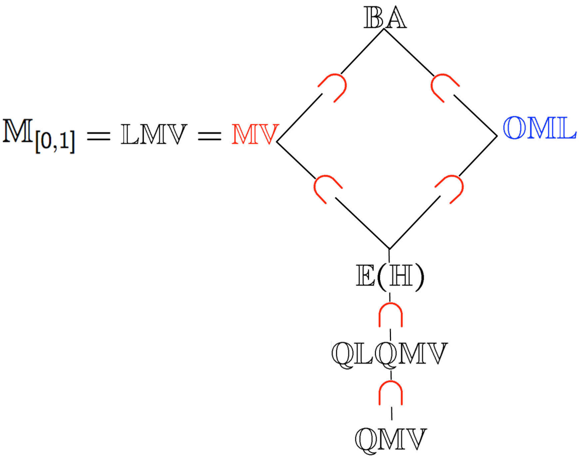
Summing up



Open problems

- Is the the variety $\mathbb{EH}$ (finitely) axiomatizable?

Łukasiewicz' Prophecy Revisited

Łukasiewicz (1920)

Three-valued logic has above all theoretical importance as an endeavour to construct a system of non-Aristotelian logic. Its practical importance will be seen only when the consequences of the indeterministic philosophy can be compared with empirical data.

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THE END?

No... this is just the beginning!

THANK YOU