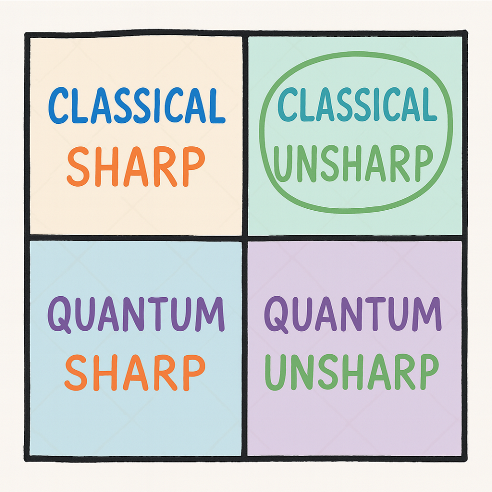
From Sharp to Unsharp
Exploring the Frontiers of Quantum Logic
Lecture II

Roberto Giuntini[1,2]

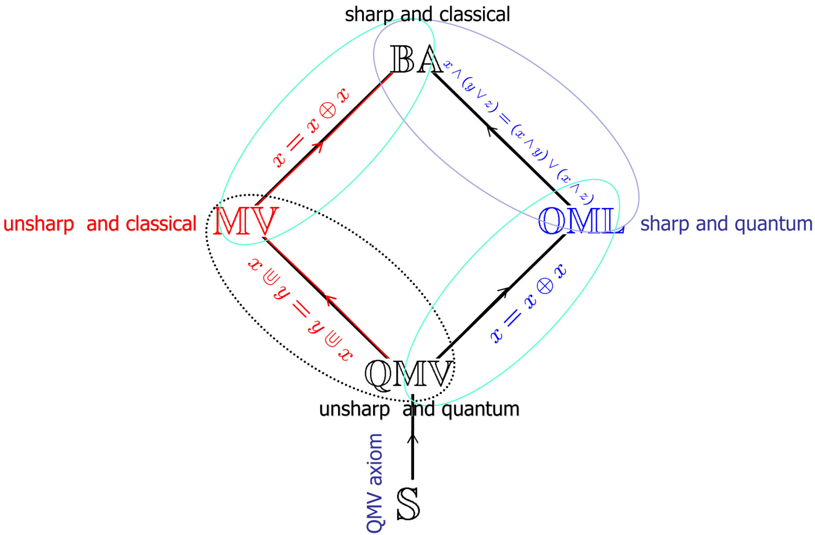
[1]University of Cagliari
ALOPHIS GROUP - Applied Logic, Philosophy and History of Science

[2]Institute for Advanced Study, Technische Universität München

SPONSORED BY THE
Federal Ministry of Education and Research
Bayerisches Staatsministerium für Wissenschaft und Kunst


CLASSICAL
SHARP
CLASSICAL
UNSHARP
QUANTUM
SHARP
QUANTUM
UNSHARP

The classical unsharp universe



Philosophical Prelude

- In 1920, Łukasiewicz published his two-page article *On three-valued logic*.
- Motivation: escape the determinism implied by bivalence.
    - If every sentence is either true or false, then the future is already determined.
    - But our intuition about contingency suggests otherwise.

Łukasiewicz (1920)

Three-valued logic has above all theoretical importance as an endeavour to construct a system of non-Aristotelian logic. Its practical importance will be seen only when the consequences of the indeterministic philosophy can be compared with empirical data.

Philosophical Prelude

- In 1920, Łukasiewicz published his two-page article *On three-valued logic*.
- Motivation: escape the determinism implied by bivalence.
  - If every sentence is either true or false, then the future is already determined.
  - But our intuition about contingency suggests otherwise.

Łukasiewicz (1920)

Three-valued logic has above all theoretical importance as an endeavour to construct a system of non-Aristotelian logic. Its practical importance will be seen only when the consequences of the indeterministic philosophy can be compared with empirical data.

Philosophical Prelude

- In 1920, Łukasiewicz published his two-page article *On three-valued logic*.
- Motivation: escape the determinism implied by bivalence.
    - If every sentence is either true or false, then the future is already determined.
    - But our intuition about contingency suggests otherwise.

Łukasiewicz (1920)

Three-valued logic has above all theoretical importance as an endeavour to construct a system of non-Aristotelian logic. Its practical importance will be seen only when the consequences of the indeterministic philosophy can be compared with empirical data.

Philosophical Prelude

- In 1920, Łukasiewicz published his two-page article *On three-valued logic*.
- Motivation: escape the determinism implied by bivalence.
    - If every sentence is either true or false, then the future is already determined.
    - But our intuition about contingency suggests otherwise.

Łukasiewicz (1920)

Three-valued logic has above all theoretical importance as an endeavour to construct a system of non-Aristotelian logic. Its practical importance will be seen only when the consequences of the indeterministic philosophy can be compared with empirical data.

Philosophical Prelude

- In 1920, Łukasiewicz published his two-page article *On three-valued logic*.
- Motivation: escape the determinism implied by bivalence.
  - If every sentence is either true or false, then the future is already determined.
  - But our intuition about contingency suggests otherwise.

Łukasiewicz (1920)

Three-valued logic has above all theoretical importance as an endeavour to construct a system of non-Aristotelian logic. Its practical importance will be seen only when the consequences of the indeterministic philosophy can be compared with empirical data.

From Prophecy to Reality

- Łukasiewicz foresaw only a *theoretical* role for many-valued logics.
- Yet, two surprising developments occurred:
  1. **Fuzzy logics** (natural heirs of Łukasiewicz' logics) entered mathematics and technology.
  2. **Quantum theory** gave empirical meaning to indeterminism: no-go theorems and experimental tests (Bell, Aspect, ...).
- Today: fuzzy and quantum approaches converge in the *unsharp* view Quantum MV-algebras (QMV-algebras) belong to this frontier.

FROM PROPHECY TO REALITY

- Łukasiewicz foresaw only a *theoretical* role for many-valued logics.
- Yet, two surprising developments occurred:
  1. **Fuzzy logics** (natural heirs of Łukasiewicz' logics) entered mathematics and technology.
  2. **Quantum theory** gave empirical meaning to indeterminism: no-go theorems and experimental tests (Bell, Aspect, . . .).
- Today: fuzzy and quantum approaches converge in the *unsharp* view Quantum MV-algebras (QMV-algebras) belong to this frontier.

From Prophecy to Reality

- Łukasiewicz foresaw only a *theoretical* role for many-valued logics.
- Yet, two surprising developments occurred:
  1. **Fuzzy logics** (natural heirs of Łukasiewicz' logics) entered mathematics and technology.
  2. **Quantum theory** gave empirical meaning to indeterminism: no-go theorems and experimental tests (Bell, Aspect, ...).
- Today: fuzzy and quantum approaches converge in the *unsharp* view Quantum MV-algebras (QMV-algebras) belong to this frontier.

From Prophecy to Reality

- Łukasiewicz foresaw only a *theoretical* role for many-valued logics.
- Yet, two surprising developments occurred:
  1. **Fuzzy logics** (natural heirs of Łukasiewicz' logics) entered mathematics and technology.
  2. **Quantum theory** gave empirical meaning to indeterminism: no-go theorems and experimental tests (Bell, Aspect, . . .).
- Today: fuzzy and quantum approaches converge in the *unsharp* view Quantum MV-algebras (QMV-algebras) belong to this frontier.

From Prophecy to Reality

- Łukasiewicz foresaw only a *theoretical* role for many-valued logics.
- Yet, two surprising developments occurred:
  1. **Fuzzy logics** (natural heirs of Łukasiewicz' logics) entered mathematics and technology.
  2. **Quantum theory** gave empirical meaning to indeterminism: no-go theorems and experimental tests (Bell, Aspect, . . .).
- Today: fuzzy and quantum approaches converge in the *unsharp* view Quantum MV-algebras (QMV-algebras) belong to this frontier.

From Prophecy to Reality

- Łukasiewicz foresaw only a *theoretical* role for many-valued logics.
- Yet, two surprising developments occurred:
  1. **Fuzzy logics** (natural heirs of Łukasiewicz' logics) entered mathematics and technology.
  2. **Quantum theory** gave empirical meaning to indeterminism: no-go theorems and experimental tests (Bell, Aspect, . . .).
- Today: fuzzy and quantum approaches converge in the *unsharp* view Quantum MV-algebras (QMV-algebras) belong to this frontier.

From Prophecy to Reality

- Łukasiewicz foresaw only a *theoretical* role for many-valued logics.
- Yet, two surprising developments occurred:
  1. **Fuzzy logics** (natural heirs of Łukasiewicz' logics) entered mathematics and technology.
  2. **Quantum theory** gave empirical meaning to indeterminism: no-go theorems and experimental tests (Bell, Aspect, . . .).
- Today: fuzzy and quantum approaches converge in the *unsharp* view Quantum MV-algebras (QMV-algebras) belong to this frontier.

From Bivalence to Gradual Truth

- **Boolean algebras:** truth is sharp, two-valued: $\{0, 1\}$.
- Rooted in the excluded-middle principle and the non-contradiction principle.
- But many real phenomena resist sharp boundaries:
- **Multi-valued algebras** (MV-algebras): extend Boole's paradigm to a **continuum of truth-values**, capturing vagueness, approximation.
- **MV-algebras:** (Chang, 1958, 1958) fuzzy (non-idempotent) generalization of Boolean algebras.
- Capture Łukasiewicz infinite-valued logic.
- Provide **algebraic semantics** for graded truth-values.

From Bivalence to Gradual Truth

- **Boolean algebras:** truth is sharp, two-valued: $\{0, 1\}$.
- Rooted in the excluded-middle principle and the non-contradiction principle.
- But many real phenomena resist sharp boundaries:
- **Multi-valued algebras** (MV-algebras): extend Boole's paradigm to a **continuum of truth-values**, capturing vagueness, approximation.
- **MV-algebras:** (Chang, 1958, 1958) fuzzy (non-idempotent) generalization of Boolean algebras.
- Capture Łukasiewicz infinite-valued logic.
- Provide **algebraic semantics** for graded truth-values.

From Bivalence to Gradual Truth

- **Boolean algebras:** truth is sharp, two-valued: $\{0, 1\}$.
- Rooted in the excluded-middle principle and the non-contradiction principle.
- But many real phenomena resist sharp boundaries:
- **Multi-valued algebras** (MV-algebras): extend Boole's paradigm to a **continuum of truth-values**, capturing vagueness, approximation.
- **MV-algebras:** (Chang, 1958, 1958) fuzzy (non-idempotent) generalization of Boolean algebras.
- Capture Łukasiewicz infinite-valued logic.
- Provide **algebraic semantics** for graded truth-values.

From Bivalence to Gradual Truth

- **Boolean algebras:** truth is sharp, two-valued: $\{0, 1\}$.
- Rooted in the excluded-middle principle and the non-contradiction principle.
- But many real phenomena resist sharp boundaries:
- **Multi-valued algebras** (MV-algebras): extend Boole's paradigm to a **continuum of truth-values**, capturing vagueness, approximation.
- **MV-algebras:** (Chang, 1958, 1958) fuzzy (non-idempotent) generalization of Boolean algebras.
- Capture Łukasiewicz infinite-valued logic.
- Provide **algebraic semantics** for graded truth-values.

From Bivalence to Gradual Truth

- **Boolean algebras:** truth is sharp, two-valued: $\{0, 1\}$.
- Rooted in the excluded-middle principle and the non-contradiction principle.
- But many real phenomena resist sharp boundaries:
- **Multi-valued algebras** (MV-algebras): extend Boole's paradigm to a **continuum of truth-values**, capturing vagueness, approximation.
- **MV-algebras:** (Chang, 1958, 1958) fuzzy (non-idempotent) generalization of Boolean algebras.
- Capture Łukasiewicz infinite-valued logic.
- Provide **algebraic semantics** for graded truth-values.

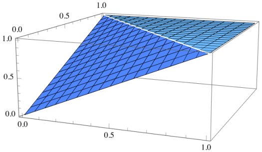
The standard model of the classical unsharp universe

Let us consider the structure

$$\mathcal{M}_{[0,1]} := ([0,1], \oplus, ', 0, 1),$$

where

- $[0,1] \subset \mathbb{R}$;
- $x \oplus y := \min(\{x + y, 1\})$ (truncated sum)



- $x' = 1 - x$

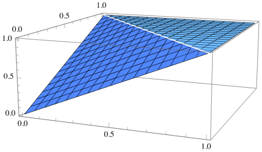
The standard model of the classical unsharp universe

Let us consider the structure

$$\mathcal{M}_{[0,1]} := ([0,1], \oplus, ', 0, 1),$$

where

- $[0,1] \subset \mathbb{R}$;
- $x \oplus y := \min(\{x + y, 1\})$ (truncated sum)



- $x' = 1 - x$

ì

MV-algebras

$\mathcal{M}_{[0,1]}$ is an MV-algebra.

TOWARDS MV-ALGEBRAS: S-ALGEBRAS

A supplement algebra (S-algebra) is a structure $\mathcal{M} = (M, \oplus, ', 1, 0)$ of type $\langle 2, 1, 0, 0 \rangle$ s.t. $\forall x, y, z \in S$:

- (S1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (S2) $x \oplus y = y \oplus x$;
- (S3) $x'' = x$;
- (S4) $x \oplus x' = 1$;
- (S5) $x \oplus 0 = x$;
- (S6) $x \oplus 1 = 1$.

$\mathbb{S}$:= the variety of all S-algebras.

Towards MV-algebras: S-algebras

A supplement algebra (S-algebra) is a structure $\mathcal{M} = (M, \oplus, ', 1, 0)$ of type $\langle 2, 1, 0, 0 \rangle$ s.t. $\forall x, y, z \in S$:

- (S1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (S2) $x \oplus y = y \oplus x$;
- (S3) $x'' = x$;
- (S4) $x \oplus x' = 1$;
- (S5) $x \oplus 0 = x$;
- (S6) $x \oplus 1 = 1$.

$\mathbb{S}$:= the variety of all S-algebras.

TOWARDS MV-ALGEBRAS: S-ALGEBRAS

A supplement algebra (S-algebra) is a structure $\mathcal{M} = (M, \oplus, {}', 1, 0)$ of type $\langle 2, 1, 0, 0\rangle$ s.t. $\forall x, y, z \in S$:

- (S1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (S2) $x \oplus y = y \oplus x$;
- (S3) $x'' = x$;
- (S4) $x \oplus x' = 1$;
- (S5) $x \oplus 0 = x$;
- (S6) $x \oplus 1 = 1$.

$\mathbb{S}$:= the variety of all S-algebras.

Towards MV-algebras: S-algebras

A supplement algebra (S-algebra) is a structure $\mathcal{M} = (M, \oplus, ', 1, 0)$ of type $\langle 2, 1, 0, 0\rangle$ s.t. $\forall x, y, z \in S$:

- (S1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (S2) $x \oplus y = y \oplus x$;
- (S3) $x'' = x$;
- (S4) $x \oplus x' = 1$;
- (S5) $x \oplus 0 = x$;
- (S6) $x \oplus 1 = 1$.

$\mathbb{S}$:= the variety of all S-algebras.

Towards MV-algebras: S-algebras

A supplement algebra (S-algebra) is a structure $\mathcal{M} = (M, \oplus, ', 1, 0)$ of type $\langle 2, 1, 0, 0 \rangle$ s.t. $\forall x, y, z \in S$:

- (S1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (S2) $x \oplus y = y \oplus x$;
- (S3) $x'' = x$;
- (S4) $x \oplus x' = 1$;
- (S5) $x \oplus 0 = x$;
- (S6) $x \oplus 1 = 1$.

$\mathbb{S}$:= the variety of all S-algebras.

Towards MV-algebras: S-algebras

A supplement algebra (S-algebra) is a structure $\mathcal{M} = (M, \oplus, ', 1, 0)$ of type $\langle 2, 1, 0, 0 \rangle$ s.t. $\forall x, y, z \in S$:

- (S1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (S2) $x \oplus y = y \oplus x$;
- (S3) $x'' = x$;
- (S4) $x \oplus x' = 1$;
- (S5) $x \oplus 0 = x$;
- (S6) $x \oplus 1 = 1$.

$\mathbb{S}$:= the variety of all S-algebras.

Towards MV-algebras: S-algebras

A supplement algebra (S-algebra) is a structure $\mathcal{M} = (M, \oplus, ', 1, 0)$ of type $\langle 2, 1, 0, 0 \rangle$ s.t. $\forall x, y, z \in S$:

- (S1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (S2) $x \oplus y = y \oplus x$;
- (S3) $x'' = x$;
- (S4) $x \oplus x' = 1$;
- (S5) $x \oplus 0 = x$;
- (S6) $x \oplus 1 = 1$.

$\mathbb{S}$:= the variety of all S-algebras.

Towards MV-algebras: S-algebras

A supplement algebra (S-algebra) is a structure $\mathcal{M} = (M, \oplus, ', 1, 0)$ of type $\langle 2, 1, 0, 0 \rangle$ s.t. $\forall x, y, z \in S$:

- (S1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (S2) $x \oplus y = y \oplus x$;
- (S3) $x'' = x$;
- (S4) $x \oplus x' = 1$;
- (S5) $x \oplus 0 = x$;
- (S6) $x \oplus 1 = 1$.

$\mathbb{S}$:= the variety of all S-algebras.

Some definitions

Let $\mathcal{M} = (M, \oplus, ', 1, 0)$ be an S-algebra:

- $x \odot y := (x' \oplus y')'$;
- $x \Cap y := (x \oplus y') \odot y$ (pseudo-inf; generalized Sasaki projection);
- $x \Cup y := (x \odot y') \oplus y$; (pseudo-sup)
- $x \leq y$ iff $x \Cap y = x$.

It turns that the relation $\leq$ is a **partial order**.

Some definitions

Let $\mathcal{M} = (M, \oplus, ', 1, 0)$ be an S-algebra:

- $x \odot y := (x' \oplus y')'$;
- $x \Cap y := (x \oplus y') \odot y$
  (pseudo-inf; generalized Sasaki projection);
- $x \Cup y := (x \odot y') \oplus y$; (pseudo-sup)
- $x \leq y$ iff $x \Cap y = x$.

It turns that the relation $\leq$ is a **partial order**.

Some definitions

Let $\mathcal{M} = (M, \oplus, ', 1, 0)$ be an S-algebra:

- $x \odot y := (x' \oplus y')'$;
- $x \Cap y := (x \oplus y') \odot y$
  (pseudo-inf; generalized Sasaki projection);
- $x \Cup y := (x \odot y') \oplus y$; (pseudo-sup)
- $x \leq y$ iff $x \Cap y = x$.

It turns that the relation $\leq$ is a **partial order**.

Some definitions

Let $\mathcal{M} = (M, \oplus, ', 1, 0)$ be an S-algebra:

- $x \odot y := (x' \oplus y')'$;
- $x \Cap y := (x \oplus y') \odot y$
  (pseudo-inf; generalized Sasaki projection);
- $x \Cup y := (x \odot y') \oplus y$; (pseudo-sup)
- $x \leq y$ iff $x \Cap y = x$.

It turns that the relation $\leq$ is a **partial order**.

Some definitions

Let $\mathcal{M} = (M, \oplus, ', 1, 0)$ be an S-algebra:

- $x \odot y := (x' \oplus y')'$;
- $x \Cap y := (x \oplus y') \odot y$
  (pseudo-inf; generalized Sasaki projection);
- $x \Cup y := (x \odot y') \oplus y$; (pseudo-sup)
- $x \leq y$ iff $x \Cap y = x$.

It turns that the relation $\leq$ is a **partial order**.

S-algebras

# of elements	# of non-isomorphic S-algebras
1	1
2	1
3	1
4	5
5	14
6	158
7	1276

MV-algebras

An MV-algebra is an S-algebra $\mathcal{M} = (M\,, \oplus\,, {}'\,, 1, 0)$ such that $\forall x, y \in M$:

$$x \Cap y = y \Cap x \qquad \text{(Łukasiewicz axiom)}$$

Theorem

The standard model $\mathcal{M}_{[0,1]} := ([0,1]\,, \oplus\,,'\,, 0, 1)$ *is an MV-algebra.*

In $\mathcal{M}_{[0,1]}$, we have:

$$x \Cap y = \min(\{x, y\})$$

MV-algebras

An MV-algebra is an S-algebra $\mathcal{M} = (M, \oplus, ', 1, 0)$ such that $\forall x, y \in M$:

$$x \Cap y = y \Cap x \qquad \text{(Łukasiewicz axiom)}$$

Theorem

The standard model $\mathcal{M}_{[0,1]} := ([0,1], \oplus, ', 0, 1)$ is an MV-algebra.

In $\mathcal{M}_{[0,1]}$, we have:

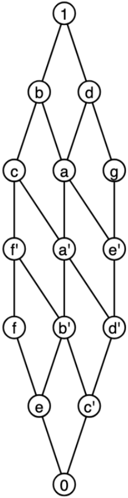
$$x \Cap y = \min(\{x, y\})$$

MV-algebras

# of elements	# of non-isomorphic MV-algebras
1	1
2	1
3	1
4	2
5	1
6	2
7	1
8	3
9	2
10	2
11	1
12	4
13	1
14	2
15	2

⇢ **MV-algebras**

15-element MV-algebra



Some properties of MV-algebras

Theorem

Let $\mathcal{M} = (M, \oplus, ', 1, 0)$ be an MV-algebra. The structure $(M, \Cap, \Cup, ', 1, 0)$ is a De Morgan lattice. In other words: $(M, \Cap, \Cup, 1, 0)$ is a distributive involutive bounded lattice.

It turns out that:

$$x \leq y \ \ (x \Cap y = x) \ \text{ iff } \ x \to y := x' \oplus y = 1.$$

Boolean algebras as sharp MV-algebras

Let $\mathcal{M} = (M, \oplus, ', 1, 0)$ be an MV-algebra. In general:

$$x \Cap x' \neq 0 \text{ and } x \Cup x' \neq 1.$$

Let

$Sh(\mathcal{M}) := \{x \in M \mid x \Cap x' = 0\}$ (the set of all sharp elements of M)

Theorem

The structure $(Sh(\mathcal{M}), \Cap, \Cup, ', 1, 0)$ is a Boolean sub-algebra of $\mathcal{M}$ s.t. $\oplus = \Cup$ and $\odot = \Cap$.

Thus,

$$\mathbb{BA} = \mathbb{MV} \cup \{x \Cup x' = 1\}.$$

Boolean algebras as sharp MV-algebras

Let $\mathcal{M} = (M\,, \oplus\,,'\ 1\,, 0)$ be an MV-algebra. In general:

$$x \Cap x' \neq 0 \text{ and } x \Cup x' \neq 1.$$

Let

$Sh(\mathcal{M}) := \{x \in M \mid x \Cap x' = 0\}$ (the set of all sharp elements of M)

Theorem

The structure $(\mathcal{S}h(\mathcal{M})\,, \Cap\,, \Cup\,, '\,, 1, 0)$ is a Boolean sub-algebra of $\mathcal{M}$ s.t. $\oplus = \Cup$ and $\odot = \Cap$.

Thus,

$$\mathbb{BA} = \mathbb{MV} \cup \{x \Cup x' = 1\}.$$

Boolean algebras as idempotent MV-algebras

Let

$$\mathcal{I}(\mathcal{M}) := \{x \in M \mid x \oplus x = x\} \quad (\text{idempotents of } \mathcal{M}).$$

Theorem

The structure $(\mathcal{I}(\mathcal{M}), \Cap, \Cup, ', 1, 0)$ *is a Boolean sub-algebra of* $\mathcal{M}$ *s.t.* $\oplus = \Cup$ *and* $\odot = \Cap$.

Thus,

$$\mathbb{BA} = \mathbb{MV} \cup \{x \oplus x = x\}.$$

$$\mathcal{I}(\mathcal{M}) = Sh(\mathcal{M}).$$

In the case of the standard MV- algebra: $\mathcal{I}(\mathcal{M}_{[0,1]}) = \{0, 1\}$.

Boolean algebras as idempotent MV-algebras

Let

$$\mathcal{I}(\mathcal{M}) := \{x \in M \mid x \oplus x = x\} \quad \text{(idempotents of } \mathcal{M}\text{)}.$$

Theorem

The structure $(\mathcal{I}(\mathcal{M}), \Cap, \Cup, ', 1, 0)$ is a Boolean sub-algebra of $\mathcal{M}$ s.t. $\oplus = \Cup$ and $\odot = \Cap$.

Thus,

$$\mathbb{BA} = \mathbb{MV} \cup \{x \oplus x = x\}.$$

$$\mathcal{I}(\mathcal{M}) = Sh(\mathcal{M}).$$

In the case of the standard MV- algebra: $\mathcal{I}(\mathcal{M}_{[0,1]}) = \{0, 1\}$.

Boolean algebras as idempotent MV-algebras

Let

$$\mathcal{I}(\mathcal{M}) := \{x \in M \mid x \oplus x = x\} \quad (\text{idempotents of} \mathcal{M}).$$

Theorem

The structure $(\mathcal{I}(\mathcal{M}), \Cap, \Cup, ', 1, 0)$ *is a Boolean sub-algebra of* $\mathcal{M}$ *s.t.* $\oplus = \Cup$ *and* $\odot = \Cap$.

Thus,

$$\mathbb{BA} = \mathbb{MV} \cup \{x \oplus x = x\}.$$

$$\mathcal{I}(\mathcal{M}) = Sh(\mathcal{M}).$$

In the case of the standard MV- algebra: $\mathcal{I}(\mathcal{M}_{[0,1]}) = \{0, 1\}$.

Boolean algebras as idempotent MV-algebras

Let

$$\mathcal{I}(\mathcal{M}) := \{x \in M \mid x \oplus x = x\} \quad (\text{idempotents of} \mathcal{M}).$$

Theorem

The structure $(\mathcal{I}(\mathcal{M}), \Cap, \Cup, ', 1, 0)$ *is a Boolean sub-algebra of* $\mathcal{M}$ *s.t.* $\oplus = \Cup$ *and* $\odot = \Cap$.

Thus,

$$\mathbb{BA} = \mathbb{MV} \cup \{x \oplus x = x\}.$$

$$\mathcal{I}(\mathcal{M}) = Sh(\mathcal{M}).$$

In the case of the standard MV- algebra: $\mathcal{I}(\mathcal{M}_{[0,1]}) = \{0, 1\}$.

Theorem (Chang 1958, 1959)

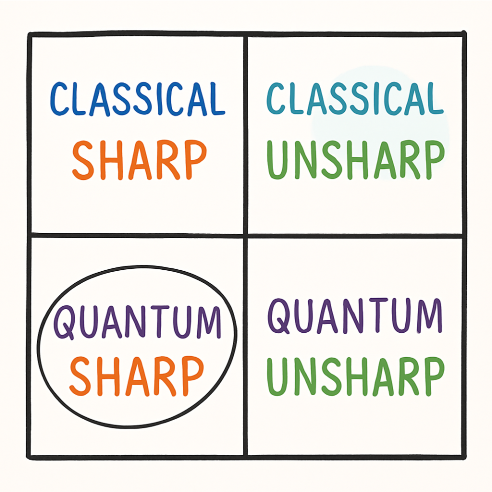
Let $\alpha \approx \beta$ an MV-equation.
$\alpha \approx \beta$ holds in the variety $\mathbb{MV}$ iff $\alpha \approx \beta$ holds in the standard model $\mathcal{M}_{[0,1]}$.

Infinite many-valued (Łukasiewicz) logic is characterized by $\mathcal{M}_{[0,1]}$.

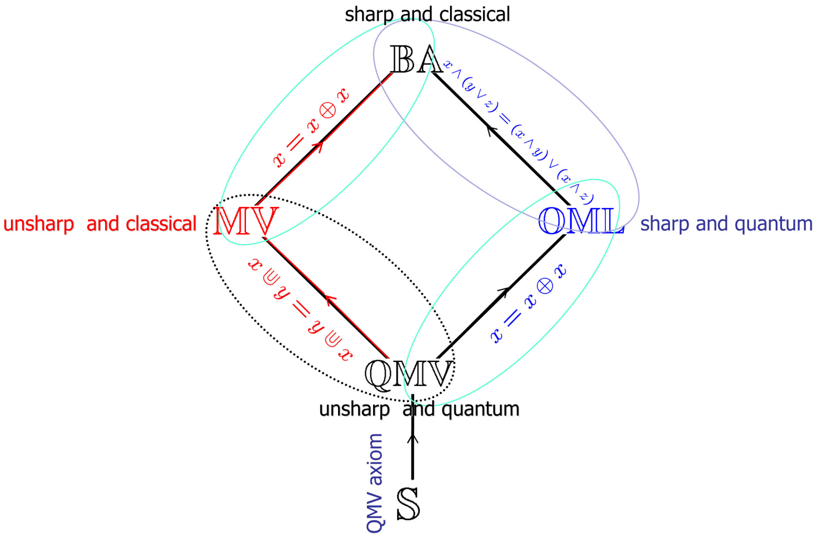
Theorem (Chang 1958, 1959)

Let $\alpha \approx \beta$ an MV-equation.
$\alpha \approx \beta$ holds in the variety $\mathbb{MV}$ iff $\alpha \approx \beta$ holds in the standard model $\mathcal{M}_{[0,1]}$.

Infinite many-valued (Łukasiewicz) logic is characterized by $\mathcal{M}_{[0,1]}$.


CLASSICAL
SHARP
CLASSICAL
UNSHARP
QUANTUM
SHARP
QUANTUM
UNSHARP

The quantum sharp universe



The quantum sharp universe

- The problem of quantum logic can be formulated as the question of whether the classical duality also holds when **Boolean algebras** (or **MV-algebras**) are replaced by weaker algebraic structures naturally arising from the mathematical formalism of quantum mechanics (QM).
- In their seminal 1936 paper, Birkhoff and von Neumann were the first to suggest "logicizing" quantum properties in terms of a non-Boolean lattice.

G. Birkhoff J. von Neumann (1936). "The logic of quantum mechanics", *Annals of Mathematics 37* (4): 823–843.

THE QUANTUM SHARP UNIVERSE

- The problem of quantum logic can be formulated as the question of whether the classical duality also holds when **Boolean algebras** (or **MV-algebras**) are replaced by weaker algebraic structures naturally arising from the mathematical formalism of quantum mechanics (QM).
- In their seminal 1936 paper, Birkhoff and von Neumann were the first to suggest "logicizing" quantum properties in terms of a non-Boolean lattice.

G. Birkhoff J. von Neumann (1936). "The logic of quantum mechanics", *Annals of Mathematics 37* (4): 823–843.

The quantum sharp universe

- There are, however, several kinds of *non-Boolean lattices*.
- **Question:** Which of them should be "logicized" as the most suitable candidate for the logic of quantum mechanics?

(SHARP) QUANTUM MECHANICS

PHYSICAL OBJECT		MATHEMATICAL INTERPRETATION
(Physical) **system**	⟹	**Hilbert space** $\mathcal{H}$
(Physical) **state**	⟹	**Density operator** of $\mathcal{H}$
(Physical) **property**	⟹	**Projection operator** of $\mathcal{H}$

J. Von Neumann, 1932; G. Birkhoff, J. Von Neumann, 1936.

The quantum sharp universe

- Hilbert space $\mathcal{H}$ plays in quantum mechanics the same role that the phase space plays in classical particle mechanics.

The quantum sharp universe: pure states

- Pure states are represented by unit vectors $|\psi\rangle$ of $\mathcal{H}$.

  Pure states correspond to maximal information about the physical system.

- Any unit vector $|\psi\rangle$ determines a linear operator $P_{|\psi\rangle}$ (the projector associated to the 1-dimensional subspace spanned by $|\psi\rangle$).

The quantum sharp universe: quantum states

Pure states are particular examples of density operators, i.e., positive, self-adjoint, trace-class linear operator of $\mathcal{H}$ of trace 1.

Both pure states and mixed states (convex combinations of pure states) are mathematically interpreted as density operators of $\mathcal{H}$.

$\mathcal{S}(\mathcal{H})$:= the set of all density operators of $\mathcal{H}$.

The quantum sharp universe: quantum states

Pure states are particular examples of density operators, i.e., positive, self-adjoint, trace-class linear operator of $\mathcal{H}$ of trace 1.

Both pure states and mixed states (convex combinations of pure states) are mathematically interpreted as density operators of $\mathcal{H}$.

$\mathcal{S}(\mathcal{H})$:= the set of all density operators of $\mathcal{H}$.

The quantum sharp universe: quantum states

Let $|\psi\rangle := (0.625953, -0.337518 + 0.703039i) \in \mathbb{C}^2$ be pure state. The projection $P_{|\psi\rangle}$ associated to $|\psi\rangle$ is

$$P_{|\psi\rangle} = \begin{pmatrix} 0.3918 & -0.2113 - 0.4401\,i \\ -0.2113 + 0.4401\,i & 0.6082 \end{pmatrix}$$

Clearly: $(P_{|\psi\rangle})^* = P_{|\psi\rangle} = P^2_{|\psi\rangle}$.

The quantum sharp universe: quantum states

Let $|\psi\rangle := (0.625953, -0.337518 + 0.703039i) \in \mathbb{C}^2$ be pure state. The projection $P_{|\psi\rangle}$ associated to $|\psi\rangle$ is

$$P_{|\psi\rangle} = \begin{pmatrix} 0.3918 & -0.2113 - 0.4401\, i \\ -0.2113 + 0.4401\, i & 0.6082 \end{pmatrix}$$

Clearly: $(P_{|\psi\rangle})^* = P_{|\psi\rangle} = P^2_{|\psi\rangle}$.

The quantum sharp universe

- In classical mechanics: properties of a physical system are mathematically represented by subsets of the phase space $\mathfrak{R}^n$.

$$\Downarrow$$

$$(\mathcal{P}(\mathfrak{R}^n), \cap, \cup, {}^c, \emptyset, \mathfrak{R}^n).$$

The Quantum Sharp Universe

Why are the *mere* subsets of $\mathcal{H}$ not adequate as mathematical representatives of quantum properties, unlike in the phase-space case?

The reason lies in the superposition principle, one of the fundamental dividing lines between the quantum and the classical worlds.

The Quantum Sharp Universe

Why are the *mere* subsets of $\mathcal{H}$ not adequate as mathematical representatives of quantum properties, unlike in the phase-space case?

The reason lies in the superposition principle, one of the fundamental dividing lines between the quantum and the classical worlds.

The quantum sharp universe: pure states

- QM: pure states are (represented by) unit vectors of $\mathcal{H}$.
- Unlike CM, in QM any unit vector, that is a linear combination of pure states, gives rise to a new pure state (superposition principle).

The quantum sharp universe: the superposition principle

Suppose two pure states $|\psi_1\rangle, |\psi_2\rangle$ are orthogonal and suppose that a pure state $|\psi\rangle$ is a linear combination of $|\psi_1\rangle, |\psi_2\rangle$.

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle,$$

where $c_1, c_2 \in \mathbb{C}$ and $|c_1|^2 + |c_2|^2 = 1$.

The quantum sharp universe: the superposition principle

According to the QM formalism, this means that a quantum system in state $|\psi\rangle$ might verify with probability $|c_1|^2$ those properties that are certain for state $|\psi_1\rangle$ (and are not certain for $|\psi\rangle$) and might verify with probability $|c_2|^2$ those events that are certain for state $|\psi_2\rangle$ (and are not certain for $|\psi\rangle$).

The quantum sharp universe: the superposition principle

Suppose $\{|\psi_i\rangle\}_{i\in I}$ is a set of pairwise orthogonal pure states, where each $|\psi_i\rangle$ assigns probability 1 to a given property.

Consider the linear combination

$$|\psi\rangle = \sum_i c_i|\psi_i\rangle \quad (c_i \neq 0 \text{ and } \sum_i |c_i|^2 = 1).$$

$|\psi\rangle$ is a pure state.

The quantum (sharp) universe: closed subspaces

- $|\psi\rangle$ will assign probability 1 to the same property!

$$\Downarrow$$

The mathematical representatives of properties of a quantum physical system should be closed under finite and infinite linear combinations.

The quantum (sharp) universe: closed subspaces

The closed subspaces of $\mathcal{H}$ are just the mathematical objects that can realize such a role.

- The physical properties of a quantum system are identified with the class of all closed subspaces of $\mathcal{H}$ or, equivalently, with
- the class of all projection operators of $\mathcal{H}$.

The quantum (sharp) universe: closed subspaces

The closed subspaces of $\mathcal{H}$ are just the mathematical objects that can realize such a role.

- The physical properties of a quantum system are identified with the class of all closed subspaces of $\mathcal{H}$ or, equivalently, with
- the class of all projection operators of $\mathcal{H}$.

The quantum (sharp) universe: closed subspaces

The closed subspaces of $\mathcal{H}$ are just the mathematical objects that can realize such a role.

- The physical properties of a quantum system are identified with the class of all closed subspaces of $\mathcal{H}$ or, equivalently, with
- the class of all projection operators of $\mathcal{H}$.

Projection operators

- A projection operator (projector) is a self-adjoint and idempotent linear operator P of $\mathcal{H}$.

$$\Downarrow$$

$$P^2 = P = P^*.$$

$$\Downarrow$$

$$\text{Spectrum}(P) \subseteq \{0, 1\}.$$

Projection operators

- A projection operator (projector) is a self-adjoint and idempotent linear operator P of $\mathcal{H}$.

$$\Downarrow$$

$$P^2 = P = P^*.$$

$$\Downarrow$$

$$\text{Spectrum}(P) \subseteq \{0, 1\}.$$

Projection operators

- A projection operator (projector) is a self-adjoint and idempotent linear operator P of $\mathcal{H}$.

$$\Downarrow$$

$$P^2 = P = P^*.$$

$$\Downarrow$$

$$\text{Spectrum}(P) \subseteq \{0, 1\}.$$

Projection operators

- A projection operator (projector) is a self-adjoint and idempotent linear operator P of $\mathcal{H}$.

$$\Downarrow$$

$$P^2 = P = P^*.$$

$$\Downarrow$$

$$\text{Spectrum}(P) \subseteq \{0, 1\}.$$

Closed subspaces and Projection operators

- $C(\mathcal{H})$:= the set of all closed subspaces of $\mathcal{H}$
- $\Pi(\mathcal{H})$:= the set of all projections of $\mathcal{H}$

$$C(\mathcal{H}) \simeq \Pi(\mathcal{H})$$

$$X \in C(\mathcal{H}) \leftrightarrows P_X \in \Pi(\mathcal{H})$$

Closed subspaces and Projection operators

- $C(\mathcal{H})$:= the set of all closed subspaces of $\mathcal{H}$
- $\Pi(\mathcal{H})$:= the set of all projections of $\mathcal{H}$

$$C(\mathcal{H}) \simeq \Pi(\mathcal{H})$$

$$X \in C(\mathcal{H}) \quad \leftrightarrows \quad P_X \in \Pi(\mathcal{H})$$

THE QUANTUM SHARP UNIVERSE: THE BORN RULE

Let X be a closed subspace of $\mathcal{H}$ and ρ be a quantum state, i.e., a density operator of $\mathcal{H}$:

$$\mathrm{Prob}_{\rho}(X) = \mathrm{tr}(\rho P_X),$$

where tr is the trace-functional.

$\mathrm{Prob}_{\rho}(X)$ represents the probability that the physical system in the quantum state ρ satisfies the property X (equivalently, P_X).

The quantum sharp universe: the Born rule

Let X be a closed subspace of $\mathcal{H}$ and ρ be a quantum state, i.e., a density operator of $\mathcal{H}$:

$$\mathrm{Prob}_\rho(X) = \mathrm{tr}(\rho P_X),$$

where tr is the trace-functional.

$\mathrm{Prob}_\rho(X)$ represents the probability that the physical system in the quantum state ρ satisfies the property X (equivalently, P_X).

The quantum sharp universe: the Born rule

In particular, if $|\psi\rangle$ and $|\phi\rangle$ are pure states:

$$\text{Prob}_{P_{|\psi\rangle}}(P_{|\phi\rangle}) = |\ \langle\psi|\phi\rangle\ |^2 .$$

The quantum sharp universe

Why quantum properties generalize classical properties?

- CM: properties are (measurable) subsets of $\mathfrak{R}^n$. Subsets X are in 1:1 correspondence with characteristic functions $f_X : \mathfrak{R}^n \to \{0, 1\}$.
- QM: properties are closed subspaces of $\mathcal{H}$. Closed subspaces are in 1:1 correspondence with projection operators of $\mathcal{H}$. The set of all eigenvalues of a projection operator is contained in $\{0, 1\}$.

The quantum sharp universe

Why quantum properties generalize classical properties?

- CM: properties are (measurable) subsets of $\mathfrak{R}^n$. Subsets X are in 1:1 correspondence with characteristic functions $f_X : \mathfrak{R}^n \to \{0, 1\}$.
- QM: properties are closed subspaces of $\mathcal{H}$. Closed subspaces are in 1:1 correspondence with projection operators of $\mathcal{H}$. The set of all eigenvalues of a projection operator is contained in $\{0, 1\}$.

The quantum sharp universe

Why quantum properties generalize classical properties?

- CM: properties are (measurable) subsets of $\mathfrak{R}^n$. Subsets X are in 1:1 correspondence with characteristic functions $f_X : \mathfrak{R}^n \to \{0, 1\}$.
- QM: properties are closed subspaces of $\mathcal{H}$. Closed subspaces are in 1:1 correspondence with projection operators of $\mathcal{H}$. The set of all eigenvalues of a projection operator is contained in $\{0, 1\}$.

The Quantum Sharp Universe: Quantum Properties

What is the meaning of negation, conjunction, and disjunction in the realm of quantum properties, i.e. closed subspaces of a Hilbert space?

TOWARDS SHARP QUANTUM LOGIC: NEGATION

Birkhoff and von Neumann (1936):

> *The mathematical representative of the negation of any experimental proposition is the orthogonal complement (orthocomplement) of the representative of the proposition itself.*

Let $X \in C(\mathcal{H})$ (a closed subspace). Then

$$X^{\perp} := \{ |\psi\rangle \in \mathcal{H} \mid \forall |\phi\rangle \in X : \ |\psi\rangle \perp |\phi\rangle \}$$
$$= \{ |\psi\rangle \in \mathcal{H} \mid \forall |\phi\rangle \in X : \ \langle \psi \mid \phi \rangle = 0 \}.$$

Thus $X^{\perp} \in C(\mathcal{H})$, i.e. the family $C(\mathcal{H})$ is closed under the operation $^{\perp}$.

Towards sharp quantum logic: negation

A pure state $|\psi\rangle$ assigns to a property X probability 1 (0, resp.) iff $|\psi\rangle$ assigns to the orthocomplement of X probability 0 (1, resp.).

$$\Downarrow$$

$^{\perp}$ is an operation that *inverts* the two extreme probability-values, which naturally correspond to the truth-values *truth* and *falsity* (as in the classical truth-table of negation).

Towards sharp quantum logic: negation

A pure state $|\psi\rangle$ assigns to a property X probability 1 (0, resp.) iff $|\psi\rangle$ assigns to the orthocomplement of X probability 0 (1, resp.).

$$\Downarrow$$

$^{\perp}$ is an operation that *inverts* the two extreme probability-values, which naturally correspond to the truth-values *truth* and *falsity* (as in the classical truth-table of negation).

Towards Sharp Quantum Logic: Conjunction

- Birkhoff and von Neumann (1936): The mathematical representative of the conjunction of two propositions X, Y is the set-theoretic intersection $X \cap Y$ of the representatives of the two propositions.
- The intersection $X \cap Y$ of two closed subspaces is again a closed subspace. Hence, the connective and behaves as expected:

$$|\psi\rangle \text{ verifies } X \cap Y \quad \text{iff} \quad |\psi\rangle \text{ verifies both } X \text{ and } Y.$$

Towards Sharp Quantum Logic: Conjunction

- Birkhoff and von Neumann (1936): The mathematical representative of the conjunction of two propositions X, Y is the set-theoretic intersection $X \cap Y$ of the representatives of the two propositions.
- The intersection $X \cap Y$ of two closed subspaces is again a closed subspace. Hence, the connective `and` behaves as expected:

$$|\psi\rangle \text{ verifies } X \cap Y \quad \text{iff} \quad |\psi\rangle \text{ verifies both } X \text{ and } Y.$$

Towards Sharp Quantum Logic: Disjunction

Disjunction, however, cannot be represented by a set-theoretic union. Indeed, the union $X \cup Y$ of two closed subspaces is in general **not** a closed subspace.

The disjunction of two propositions X, Y is therefore defined as the smallest closed subspace containing $X \cup Y$, i.e. the supremum (closed linear span) of X and Y.

Towards Sharp Quantum Logic: Disjunction

Disjunction, however, cannot be represented by a set-theoretic union. Indeed, the union $X \cup Y$ of two closed subspaces is in general **not** a closed subspace.

The disjunction of two propositions X, Y is therefore defined as the smallest closed subspace containing $X \cup Y$, i.e. the supremum (closed linear span) of X and Y.

The lattice of closed subspaces

$$\mathcal{C}(\mathcal{H}) = (C(\mathcal{H}), \wedge, \vee, {}', 0, 1),$$

where :

- $\wedge$ is the set-theoretic intersection;
- $\vee$ is the closure of the set-theoretic union;
- $'$ is the orthogonal complement $^{\perp}$;
- 0 and 1 represent, respectively, the null subspace (the singleton consisting of the null vector, which is the smallest possible subspace) and the total space $\mathcal{H}$.

The standard model of the quantum sharp universe

Theorem

$\mathcal{C}(\mathcal{H}) := (C(\mathcal{H}), \wedge, \vee, ', 0, 1)$ *is an orthomodular lattice, i.e.:* $\forall X, Y \in C(\mathcal{H})$:

- $X \wedge X' = 0$ *and* $X \vee X' = 1$.
- $X \leq Y$, *then* $X = (X \vee Y') \wedge Y$.

The orthomodular lattices $\mathcal{C}(\mathcal{H})$ are called Hilbert lattices.

The standard model of the quantum sharp universe

Theorem

$\mathcal{C}(\mathcal{H}) := (C(\mathcal{H}), \wedge, \vee, ', 0, 1)$ is an orthomodular lattice, i.e.:
$\forall X, Y \in C(\mathcal{H})$:

- $X \wedge X' = 0$ *and* $X \vee X' = 1$.
- $X \leq Y$, *then* $X = (X \vee Y') \wedge Y$.

The orthomodular lattices $\mathcal{C}(\mathcal{H})$ are called Hilbert lattices.

The standard model of the quantum sharp universe

Theorem

$\mathcal{C}(\mathcal{H}) := (C(\mathcal{H}), \wedge, \vee, ', 0, 1)$ *is an orthomodular lattice, i.e.:* $\forall X, Y \in C(\mathcal{H})$:

- $X \wedge X' = 0$ *and* $X \vee X' = 1$.
- $X \leq Y$, *then* $X = (X \vee Y') \wedge Y$.

The orthomodular lattices $\mathcal{C}(\mathcal{H})$ are called Hilbert lattices.

The lattice of projection operators

Because of the 1:1 correspondence between closed subspaces and projections, the Theorem can be rephrased in terms of projections:

Theorem

$\mathcal{P}(\mathcal{H}) := (\Pi(\mathcal{H}), \wedge, \vee, ', \mathbb{O}, \mathbb{I})$ *is an orthomodular lattice.*

where:

The lattice-theoretic structure of projections

- $\forall P, Q \in \Pi(\mathcal{H})$:
  - $P \wedge$ Q is the projection onto the closed subspace associated to the intersection of the closed subspaces that are associated to the projections P and Q;
  - $P \vee$ Q is the projection onto the smallest closed subspace associated to the union of the closed subspaces that are associated to the projections P and Q.
  - $P' = \mathbb{I} - P$, where $\mathbb{I}$ is the identity operator. Equivalently, if X is the closed subspace associated to P, then P' is the projection that is associated to X'.

The Lattice-Theoretic Structure of Projections

It turns out that for all $P, Q \in \Pi(\mathcal{H})$ (orthogonal projections):

$$P \preceq Q \quad \text{(i.e., } P \wedge Q = P\text{)}$$

iff

$$\text{for any density operator } \rho : \quad \mathrm{tr}(\rho P) \leq \mathrm{tr}(\rho Q).$$

Interpretation. By Born's rule, $\mathrm{tr}(\rho P)$ is the probability that a physical system in the state ρ verifies the property represented by P. Thus, $P \preceq Q$ means that whenever P holds, Q holds with at least as high probability.

The Lattice-Theoretic Structure of Projections

It turns out that for all $P, Q \in \Pi(\mathcal{H})$ (orthogonal projections):

$$P \preceq Q \quad \text{(i.e., } P \wedge Q = P\text{)}$$

iff

$$\text{for any density operator } \rho: \quad \mathrm{tr}(\rho P) \ \leq \ \mathrm{tr}(\rho Q).$$

Interpretation. By Born's rule, $\mathrm{tr}(\rho P)$ is the probability that a physical system in the state ρ verifies the property represented by P. Thus, $P \preceq Q$ means that whenever P holds, Q holds with at least as high probability.

The failure of distributivity

For any Hilbert space $\mathcal{H}$, the structure $\mathcal{C}(\mathcal{H})$ turns out to simulate a "quasi-Boolean behavior"; however, it is **not** a Boolean algebra.

Distributivity fails:
conjunction and disjunction are **not** distributive.
Generally,

$$X \wedge (Y \vee Z) \neq (X \wedge Y) \vee (X \wedge Z).$$

THE FAILURE OF DISTRIBUTIVITY

For any Hilbert space $\mathcal{H}$, the structure $\mathcal{C}(\mathcal{H})$ turns out to simulate a "quasi-Boolean behavior"; however, it is **not** a Boolean algebra.

Distributivity fails:
conjunction and disjunction are **not** distributive.
Generally,

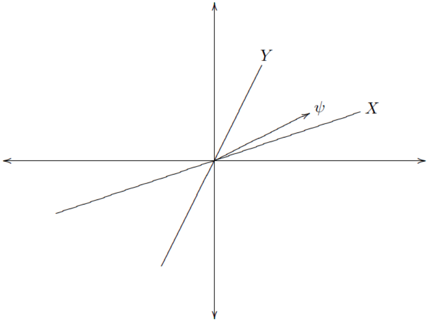
$$X \wedge (Y \vee Z) \neq (X \wedge Y) \vee (X \wedge Z).$$

The role of quantum disjunction

$$|\psi\rangle \in \mathbb{R}^2 = X \vee Y$$

but

$$|\psi\rangle \notin X \text{ and } |\psi\rangle \notin Y.$$



The role of quantum disjunction

In QM one is often dealing with alternatives that are semantically determined and true, while both members are, in principle, indeterminate.

The role of quantum disjunction

In QM one is often dealing with alternatives that are semantically determined and true, while both members are, in principle, indeterminate.

The role of quantum disjunction

Consider spin one-half particle whose spin in a certain direction may assume only two possible values: either up or down. Now, according to one of the *uncertainty principles*, the spin in the x direction ($spin_x$) and the spin in the y direction ($spin_y$) represent two *incompatible* quantities that cannot be simultaneously measured. Suppose an electron in state ψ verifies the proposition "$spin_x$ is up". By the uncertainty principle both propositions "$spin_y$ is up" and "$spin_y$ is down" shall be indeterminate. However the disjunction "either $spin_y$ is up or $spin_y$ is down" must be true.

OML and Hilbert lattices

Let $\mathbb{CH}$:= the variety generated by the class of all Hilbert lattices $\mathcal{C}(\mathcal{H})$.

Theorem

$$\mathbb{CH} \subset \mathbb{OML}.$$

There exists an equation (the so called orthoarguesian law) that holds in $\mathbb{CH}$ but fails in a particular (finite) orthomodular lattice.

OML AND HILBERT LATTICES

Let $\mathbb{CH}$:= the variety generated by the class of all Hilbert lattices $\mathcal{C}(\mathcal{H})$.

THEOREM

$$\mathbb{CH} \subset \mathbb{OML}.$$

There exists an equation (the so called orthoarguesian law) that holds in $\mathbb{CH}$ but fails in a particular (finite) orthomodular lattice.

OML and Hilbert lattices

Let $\mathbb{CH}$:= the variety generated by the class of all Hilbert lattices $\mathcal{C}(\mathcal{H})$.

Theorem

$$\mathbb{CH} \subset \mathbb{OML}.$$

There exists an equation (the so called orthoarguesian law) that holds in $\mathbb{CH}$ but fails in a particular (finite) orthomodular lattice.

OML and Hilbert lattices: the orthoarguesian law

$$\models_{\mathbb{PH}} x \approx x \wedge (y \vee ((x \Cap y') \wedge ((x \Cap y') \vee ((y \vee z) \wedge ((x \Cap y') \vee (x \Cap z'))))))$$

but

$$\not\models_{G_{30}} a \approx a \wedge (b \vee ((a \Cap b') \wedge ((a \Cap c') \vee ((b \vee c) \wedge ((a \Cap b') \vee (a \Cap c'))))))$$

OML and Hilbert lattices: the orthoarguesian law

$$\models_{\mathbb{PH}} x \approx x \wedge (y \vee ((x \Cap y') \wedge ((x \Cap y') \vee ((y \vee z) \wedge ((x \Cap y') \vee (x \Cap z'))))))$$

but

$$\not\models_{G_{30}} a \approx a \wedge (b \vee ((a \Cap b') \wedge ((a \Cap c') \vee ((b \vee c) \wedge ((a \Cap b') \vee (a \Cap c'))))))$$

The two main open problem of (sharp) quantum logic

- Is $\mathbb{CH}$ (finitely) axiomatizable?
- Does the logic algebraically characterized by $\mathbb{OML}$ (the so called Orthomodular Quantum Logic) have the finite model property?
- If not, is Orthomodular Quantum Logic decidable?

The quantum sharp universe

- Universe $\Longrightarrow$ The set $\mathcal{C}(\mathcal{H})$ of all closed subspaces (projection operators) of $\mathcal{H}$
- Algebra $\Longrightarrow$ Orthomodular Lattices (cannot be reduced to $\mathcal{C}(\mathcal{H})$).
- Logic $\Longrightarrow$ Orthomodular Quantum Logic

The quantum sharp universe

- Universe $\Longrightarrow$ The set $\mathcal{C}(\mathcal{H})$ of all closed subspaces (projection operators) of $\mathcal{H}$
- Algebra $\Longrightarrow$ Orthomodular Lattices (cannot be reduced to $\mathcal{C}(\mathcal{H})$).
- Logic $\Longrightarrow$ Orthomodular Quantum Logic

The quantum sharp universe

- Universe $\Longrightarrow$ The set $\mathcal{C}(\mathcal{H})$ of all closed subspaces (projection operators) of $\mathcal{H}$
- Algebra $\Longrightarrow$ Orthomodular Lattices (cannot be reduced to $\mathcal{C}(\mathcal{H})$).
- Logic $\Longrightarrow$ Orthomodular Quantum Logic

The quantum sharp universe

- Universe $\Longrightarrow$ The set $\mathcal{C}(\mathcal{H})$ of all closed subspaces (projection operators) of $\mathcal{H}$
- Algebra $\Longrightarrow$ Orthomodular Lattices (cannot be reduced to $\mathcal{C}(\mathcal{H})$).
- Logic $\Longrightarrow$ Orthomodular Quantum Logic

The quantum sharp universe

- Universe $\Longrightarrow$ The set $\mathcal{C}(\mathcal{H})$ of all closed subspaces (projection operators) of $\mathcal{H}$
- Algebra $\Longrightarrow$ Orthomodular Lattices (cannot be reduced to $\mathcal{C}(\mathcal{H})$).
- Logic $\Longrightarrow$ Orthomodular Quantum Logic

The quantum sharp universe

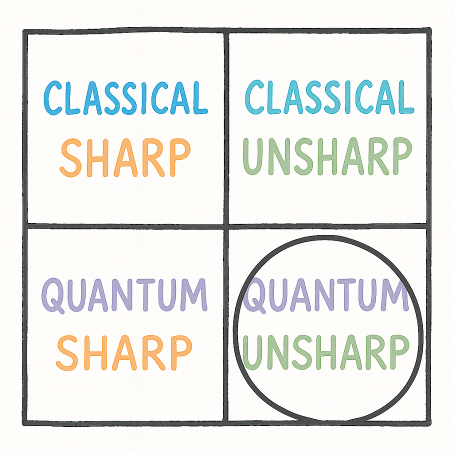
- Universe $\Longrightarrow$ The set $\mathcal{C}(\mathcal{H})$ of all closed subspaces (projection operators) of $\mathcal{H}$
- Algebra $\Longrightarrow$ Orthomodular Lattices (cannot be reduced to $\mathcal{C}(\mathcal{H})$).
- Logic $\Longrightarrow$ Orthomodular Quantum Logic

The quantum sharp universe

- Universe $\Longrightarrow$ The set $\mathcal{C}(\mathcal{H})$ of all closed subspaces (projection operators) of $\mathcal{H}$
- Algebra $\Longrightarrow$ Orthomodular Lattices (cannot be reduced to $\mathcal{C}(\mathcal{H})$).
- Logic $\Longrightarrow$ Orthomodular Quantum Logic

References (Lecture I)

- E. Beltrametti, G. Cassinelli, *The Logic of Quantum Mechanics*, Cambridge University Press, 1981.
- R. Cignoli, I. D'Ottaviano, D. Mundici, *Algebraic Foundations of Many-Valued Reasoning*, Kluwer, Dordrecht, 2000.
- M. Luisa Dalla Chiara, R. G., R. Greechie,*Reasoning in Quantum Theory*, Kluwer, 2004.
- G. Kalmbach, *Orthomodular Lattices*, Academic Press, 1983.


CLASSICAL
SHARP
CLASSICAL
UNSHARP
QUANTUM
SHARP
QUANTUM
UNSHARP

...TO BE CONTINUED

in LECTURE III