From Sharp to Unsharp
Exploring the Frontiers of Quantum Logic
Lecture I

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Setting the stage

- In these three lectures we shall move back and forth between **logic** and **physics**.
- Our aim is to understand how **algebraic structures** can model reasoning as well as physical phenomena.

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From algebraic models to physical theories

- **Algebraic structures** play a central role both in the formalization of **logic** and in the description of **physical systems**.
- Boolean algebras provide the standard model of **classical reasoning** and **classical physics** (deterministic, two-valued).
- Łukasiewicz algebras (Multi-valued algebras (MV-algebras)) generalize the classical setting to **many values**, while still retaining a part of the “classical” flavour (**distributivity**).
- **Quantum theory** suggests going further: it naturally gives rise to **non-Boolean lattices**, where the link between logic and physics becomes much less straightforward.
- Which non-Boolean structures should be “logicized” as the most suitable candidates for the **logic of quantum mechanics**?

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The Leitmotiv

- To address our guiding question, we will start from a more fundamental distinction that permeates both **classical** and **quantum** domains: the contrast between **sharp** and **unsharp**.
- This distinction will act as our **Leitmotiv**, recurring throughout the three lectures.

SHARP vs. UNSHARP

non-contradiction law	violation of the non-contradiction law

⇓

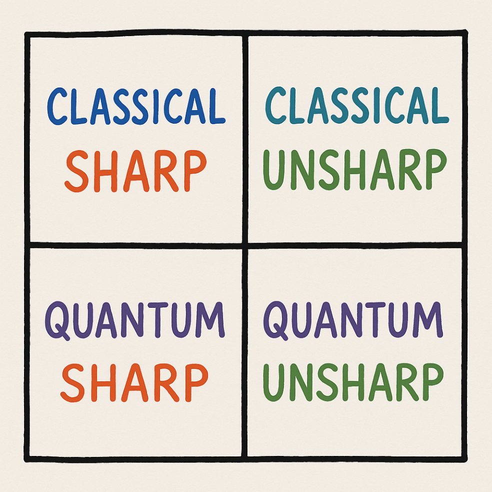
Classical / classical-like

- *Sharp facet:* crisp boundaries.
- *Unsharp facet:* Emerging from vagueness, approximation, and the gradual nature of information.

Quantum / Quantum-like

- *Sharp facet:* ideal yes/no events; precise tests in principle.
- *Unsharp facet:* effects of noise, disturbance, and limited resolution.

The Sharp and the Unsharp Universes



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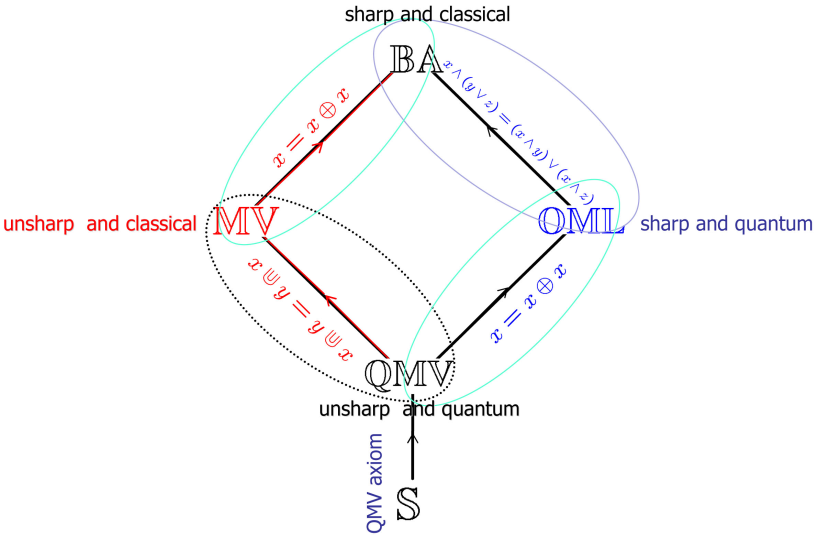

- **LECTURE I**
  - Introduction
  - Mathematical interlude: Lattice theory and Universal algebra
  - The classical sharp universe
  - The classical unsharp universe
- **LECTURE II**
  - The classical unsharp universe (continued)
  - The quantum sharp universe
- **LECTURE III**
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References (Lecture I)

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Automatic reasoning and algebraic software

In these lectures I will also make use of tools coming from **automated theorem proving** and **universal algebra**:

A. **Prover9 and Mace4** (**W. McCune**).
Prover9 is a resolution/paramodulation automated theorem prover for first-order and equational logic. Mace4 searches for finite structures satisfying first-order and equational statements (the same kind of statement that Prover9 accepts).
https://www.cs.unm.edu/ mccune/mace4/

B. **Prover9–Mace4 interface (P. Jipsen & M. Maróti)**
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Mathematical interlude: Lattice theory

Definition (poset)

A partially ordered set (poset) is a structure

$$\mathcal{B} = (B\,,\,\leq),$$

where: $B \neq \emptyset$ and $\leq$ is a partial order relation on B. In other words, $\leq$ satisfies the following conditions for all $x, y, z \in B$:

(i) $x \leq x$ (reflexivity);

(ii) $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry);

(iii) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity).

⇢ **Poset Notebook**

Mathematical interlude: Lattice Theory

Definition (Bounded poset)

A bounded poset is a structure

$$\mathcal{B} = (B\,,\,\leq\,,\,0\,,1),$$

where:

(i) $(B\,,\,\leq)$ is a poset;

(ii) 0 and 1 are special elements of B: the *minimum* and the *maximum* with respect to $\leq$. In other words, for all $x \in B$:

$$0 \leq x \text{ and } x \leq 1.$$

⇢ **Poset Notebook**

Mathematical interlude: Lattice Theory

Definition (Lattice)

A lattice is a poset $\mathcal{B} = (B, \leq)$ in which any pair of elements x, y has a meet (infimum) $x \wedge y$ and a join (supremum) $x \vee y$ such that:

(I) $x \wedge y \leq x, y$, and $\forall z \in B$: $z \leq x, y$ implies $z \leq x \wedge y$ ($\wedge$ is the *greatest lower bound* (*inf*));

(II) $x, y \leq x \vee y$, and $\forall z \in B$: $x, y \leq z$ implies $x \vee y \leq z$ ($\vee$ is the *lowest upper bound* (*sup*)).

In any lattice the following condition holds:

$$x \leq y \text{ iff } x \wedge y = x \text{ iff } x \vee y = y.$$

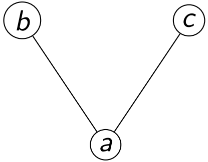
Mathematical interlude: Lattice Theory

Definition (Lattice [equivalent definition])

A lattice is a structure $\mathcal{B} = (B, \wedge, \vee)$ of type $(2,2)$ s.t. $\forall x, y, z \in B$:

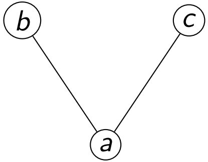
1. $x \wedge x = x; \quad x \vee x = x$ (idempotent);
2. $x \wedge y = y \wedge x \; ; x \vee y = y \vee x$ (commutative);
3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z;$
   $x \vee (y \vee z) = (x \vee y) \vee z$ (associative)
4. $x \wedge (y \vee x) = x;$
   $x \vee (y \wedge x) = x$ (absorption).

Mathematical interlude: Lattice Theory



The pair $\{b, c\}$ has **meet** (a) but no common upper bound in the set, so their join does not exist.

Mathematical interlude: Lattice Theory



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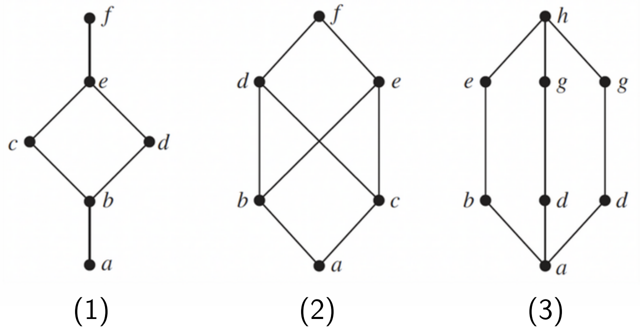
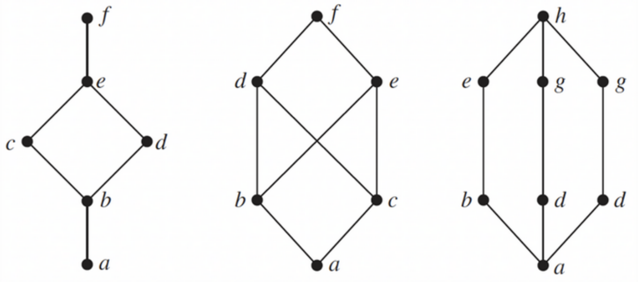
MATHEMATICAL INTERLUDE: LATTICE THEORY



(1) and (3) are lattices; (2) is a poset that is not a lattice.

Mathematical interlude: Lattice Theory



(1) (2) (3)

(1) and (3) are lattices; (2) is a poset that is not a lattice.

Mathematical interlude: Lattice Theory

# of elements	# of non-isomorphic lattices
1	1
2	1
3	1
4	2
5	5
6	15
7	53
8	222
9	1078
10	5994
11	37622
12	262776
13	2018305
14	16873364
15	152233518
16	1471613387
17	15150569446
18	165269824761
19	1901910625578
20	23003059864006

- - →**Lattice Notebook**

Mathematical interlude: Lattice Theory

Definition (involutive bounded poset)

A involutive bounded poset (lattice) is a structure $\mathcal{B} = (B\,, \leq\,, '\,, 0\,, 1)$ where:

(I) $(B\,, \leq\,, 0\,, 1)$ is a bounded poset (lattice);

(II) $'$ is a 1-ary operation (involution) s.t. $\forall x, y \in B$:

1. $x = x''$ (double negation);
2. $x \leq y$ implies $y' \leq x'$ (contraposition).

Contraposition ($x \leq y$ implies $y' \leq x'$) is equivalent to the equation $(x \vee y)' = x' \wedge y'$.

$\Downarrow$

The class of all involutive bounded lattices $\mathbb{IBL}$ is an equational class (= variety).

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$$\Downarrow$$

The class of all involutive bounded lattices $\mathbb{IBL}$ is an equational class ($=$ variety).

Mathematical interlude: Lattice Theory

We will consider only regular involutive bounded posets.

Definition (regularity)

A involutive bounded poset (lattice) $\mathcal{B} = (B\,,\, \leq\,,\, '\,,\, 0\,,1)$ is regular iff $\forall x, y \in B$:

$$x \leq x' \ \text{ and } \ y \leq y' \ \text{ implies } \ x \leq y'.$$

A lattice is **regular** iff

$$\forall x, y \in B: \ \ x \wedge x' \leq y \vee y'.$$

Thus, the regularity condition is equational.

Mathematical interlude: Lattice Theory

(Regular) Involutive bounded lattices

# of elements	# of non-isomorphic involutive bounded lattices
1	1
2	1
3	1
4	2
5	2
6	6
7	7
8	24
9	31
10	120
11	171
12	746

⇢ **(Regular) Involutive bounded lattices**

Mathematical interlude: Lattice Theory

Definition (ortholattice)

An ortholattice is a bounded involutive lattice $\mathcal{B} = (B\,,\wedge\,,\vee\,,{}'\,,0\,,1)$ s.t. $\forall x \in B$:

1. $x \wedge x' = 0$ (non-contradiction principle);
2. $x \vee x' = 1$ (excluded-middle principle).

The operation $'$ of an ortholattice is called orthocomplementation (shortly orthocomplement).

non-contradiction = sharpness

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non-contradiction = sharpness

Mathematical interlude: Lattice Theory

The class of all orthololattices $\mathbb{OL}$ is an equational class (variety).

$$\mathbb{OL} = \mathbb{IBL} \; + \; \text{noncontradiction}$$

Mathematical interlude: Lattice Theory

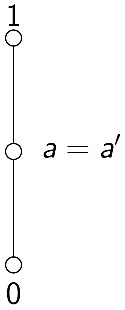
Ortholattices

# of elements	# of non-isomorphic ortholattices
1	1
2	1
3	0
4	1
5	0
6	2
7	0
8	5
9	0
10	15
11	0
12	60

--→**Ortholattices**

Mathematical interlude: Lattice Theory

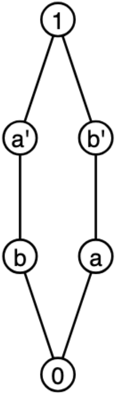
$\mathcal{D}_3$



$\mathcal{D}_3 \in \mathbb{IBL}$ but $a \wedge a' = a \neq 0$.

Mathematical interlude: Lattice Theory

The benzene ring $\mathcal{OL}_6$



MATHEMATICAL INTERLUDE: LATTICE THEORY

DEFINITION

An orthomodular lattice is an ortholattice $\mathcal{A}$ s.t.:

$$\forall x, y \in A: \quad \text{if } x \leq y \text{ then } y = x \vee (x' \wedge y)$$

For all $x, y \in B$:

$$x \Cap y := (x \vee y') \wedge y.$$

$x \Cap y$ is called the Sasaki projector of x onto y.

MATHEMATICAL INTERLUDE: LATTICE THEORY

THEOREM

Let $\mathcal{A}$ be an ortholattice. The following conditions are equivalent:

1. $\mathcal{A}$ *is orthomodular;*
2. $\forall x, y \in A:$ *if* $x \leq y$ *then* $x = x \Cap y$;
3. $\forall x, y \in A:$ *if* $x \leq y$ *and* $x' \wedge y = 0$, *then* $x = y$;
4. $\forall x, y \in A: \ x \vee y = ((x \vee y) \wedge y') \vee y$; (*orthomodular equation*)
5. *The benzene ring is* **not** *a sub-ortholattice of* $\mathcal{A}$.

By (4), the class $\mathbb{OML}$ of all orthomodular lattices is a **variety**.

Mathematical interlude: Lattice Theory

Orthomodular lattices

# of elements	# of non-isomorphic ortholattices
1	1
2	1
3	0
4	1
5	0
6	1
7	0
8	2
9	0
10	2
11	0
12	3

⇢**Orthomodular lattices** ⇢**Orthomodular lattices Prover9**

Mathematical interlude: Lattice Theory

Definition (Boolean algebra)

A Boolean algebra is an ortholattice $\mathcal{B} = (B\,,\, \wedge\,,\, \vee\,,\, '\,,\, 0\,,\, 1)$ s.t. $\forall x, y, z \in B$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad (\textit{distributivity})$$

Both **non-contradiction** and **distributivity** hold. The class of all Boolean algebras $\mathbb{BA}$ is an equational class (variety).

$$\mathbb{BA} = \mathbb{OL} \; + \; \text{distributivity}$$

Mathematical interlude: Lattice Theory

$\mathbb{BA} \subset \mathbb{OML} \subset \mathbb{OL}$

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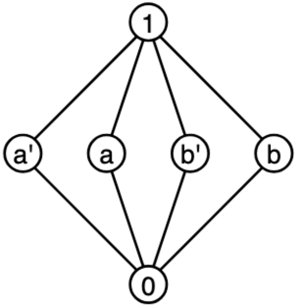

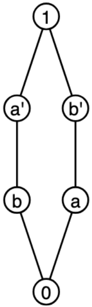

Mathematical interlude: Lattice Theory

The smallest Boolean algebra:

$$\mathfrak{B}_2 = (\{0,1\}\,,\wedge\,,\vee\,,{}'\,,0\,,1),$$

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

x	x'
0	1
1	0

Mathematical interlude: Lattice Theory

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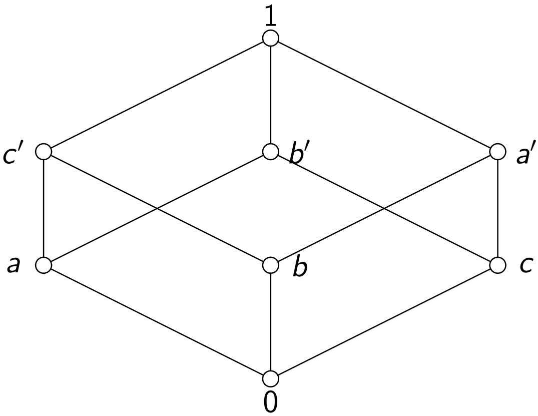
x	x'
0	1
1	0

Mathematical interlude: Lattice Theory



Mathematical interlude: Lattice Theory

Given a set U, we can define for any $A \in \mathcal{P}(U)$, the characteristic function (crisp set) associated to A:

$$\chi_A : U \to \{0,1\}$$

such that $\forall x \in U$:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

Let $U^{\{0,1\}}$ be the set of all crisp sets of U. $U^{\{0,1\}}$ can be endowed with the following pointwise operations:

- $\forall x \in U : \quad (\chi_A \wedge \chi_B)(x) := \chi_A(x) \wedge \chi_B(x)$
- $\forall x \in U : \quad (\chi_A)'(x) := 1 - \chi_A(x)$
- $\forall x \in U : \quad f_0(x) := 0.$

It turns out that $(U^{\{0,1\}}, \wedge, ', f_0)$ is a Boolean algebra that is isomorphic to the set-Boolean algebra based on $\mathcal{P}(U)$.

Mathematical interlude: Universal Algebra

Definition

Fix a signature τ (a set of operation symbols, each with an assigned arity).
An algebra of type τ is a structure

$$\mathcal{A} = \langle A, (f^{\mathcal{A}})_{f \in \tau} \rangle$$

where:

- A is a nonempty set (the universe);
- for each $f \in \tau$ of arity n, $f^{\mathcal{A}} : A^n \to A$ is an n-ary operation.

Example

- Involutive bounded lattices, Ortholattices and Boolean algebras are all algebras of the same type.

Mathematical interlude: Universal Algebra

Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of algebras of the same type.

Definition

The direct product

$$\prod_{i \in I} \mathcal{A}_i$$

is the algebra with universe

$$\prod_{i \in I} A_i = \{(a_i)_{i \in I} : a_i \in A_i\},$$

and with each basic operation f defined coordinatewise:

$$f^{\prod \mathcal{A}_i}((a_i^1)_{i \in I}, \ldots, (a_i^n)_{i \in I}) = \left(f^{\mathcal{A}_i}(a_i^1, \ldots, a_i^n)\right)_{i \in I}.$$

Mathematical interlude: Universal Algebra

Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of algebras and $\prod_{i\in I}\mathcal{A}_i$ their direct product.

Definition

For each $j \in I$, the canonical projection is the homomorphism

$$\pi_j : \prod_{i\in I}\mathcal{A}_i \to \mathcal{A}_j, \qquad \pi_j\big((a_i)_{i\in I}\big) = a_j.$$

Remark.

- π_j simply "forgets all coordinates except the j-th."
- Each π_j is a surjective homomorphism.

Mathematical interlude: Universal Algebra

Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of algebras of the same type.

Definition

An algebra $\mathcal{B}$ is a subdirect product of $\{\mathcal{A}_i : i \in I\}$ iff there exists an embedding

$$f : \mathcal{B} \hookrightarrow \prod_{i\in I} \mathcal{A}_i$$

such that, for every $i \in I$, the composition

$$\pi_j \circ f : \mathcal{B} \to \mathcal{A}_j$$

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Mathematical interlude: Universal Algebra

Intuition.

- The direct product $\prod_{i\in I}\mathcal{A}_i$ satisfies exactly the equations that hold in each factor $\mathcal{A}_i$.
- A subdirect product $B \hookrightarrow \prod_{i\in I}\mathcal{A}_i$ is a *subalgebra* that still projects onto every $\mathcal{A}_i$ via the canonical projections.

Preservation of equations.

- If an identity $s \approx t$ holds in all factors $\mathcal{A}_i$, then it also holds in the product.
- Since $\mathcal{B}$ covers every factor (all $\pi_i \circ f$ are surjective),

Key idea: A subdirect product is "smaller" than the product but is an *equational mirror* of the factors.

Mathematical interlude: Universal Algebra

Varieties

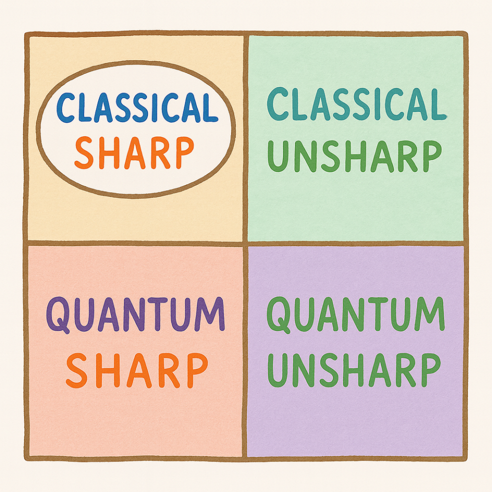
- A *variety* is a class of algebras of the same type defined by equations.
- By Birkhoff's HSP Theorem, varieties are closed under:
  - Homomorphic images (H),
  - Subalgebras (S),
  - Direct products (P).

Subdirect products.

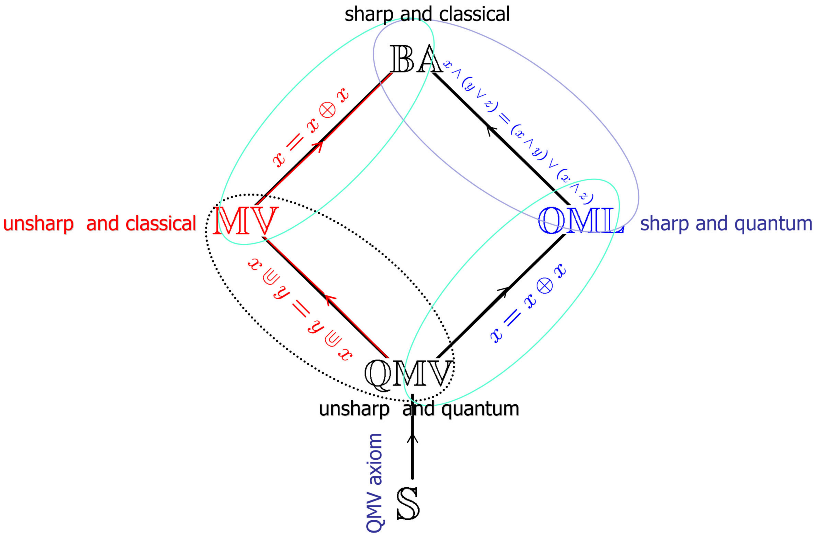
- Subdirect products are special subalgebras of products that "project" onto each factor.
- They preserve exactly the same equations as the factors.

Birkhoff's Subdirect Representation Theorem. Every algebra in a variety is a subdirect product of *subdirectly irreducible* algebras of the same variety.

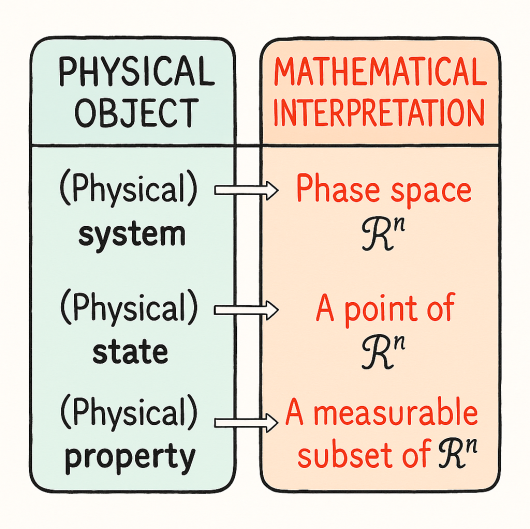
Part I: The classical sharp universe



The classical sharp universe



Classical Mechanics



The structure of classical sharp properties

Classical pure states represent *pieces of information* (about the physical system) that are maximal and logically complete.
They are:

- maximal because they represent a *maximum of information* that cannot be consistently extended to a richer knowledge in the framework of the theory;
- logically complete because they semantically decide any property. For any p and X,

$$p \in X \text{ or } p \in X^c.$$

Classical sharp properties

Let S be a (classical) physical system.

$$S \Rightarrow \mathfrak{R}^n$$

Let P be an *experimental proposition* about S, asserting that a given *physical quantity* (*observable*) has a certain value: For instance:
"the value of position in the x-direction lies in a certain interval"

$$\mathsf{P} \Rightarrow X_{\mathsf{P}} := \text{the set of all states for which P } \textbf{holds}.$$

Classical sharp properties

- The (measurable) subsets of $\mathfrak{R}^n$ are good mathematical representatives of experimental propositions (properties).

When a state $p \in \mathfrak{R}^n$ belongs to a property X, we can say that the system in state p verifies both X and the corresponding property.

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What about the **structure** of all properties?

The **power set** of any set gives rise to a Boolean algebra and the set of all **measurable subsets** of $\mathfrak{R}^n$ is a Boolean algebra as well.

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The structure of classical sharp properties

$$(P(\mathfrak{R}^n), \cap, \cup, {}^c, 0, 1),$$

where:

- $\cap, \cup, {}^c$ are the set-theoretic operations intersection, union, relative complement, respectively;
- 0 is the empty set ($\emptyset$), while 1 is the total set ($\mathfrak{R}^n$).
- $P(\mathfrak{R}^n)$ is partially ordered by the relation of inclusion ($\subseteq$).

The structure of classical sharp properties

Theorem (Stone's Theorem, 1936)

Every Boolean algebra is isomorphic to a **field of sets** *(i.e. a Boolean algebra of subsets of some set, with the usual set-theoretic operations of union, intersection, and complement).*

Intuition.

Every Boolean algebra can be thought of as a set-theoretic Boolean algebra, without any loss of generality.

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- $\cap \Longrightarrow$ AND ($\wedge$)
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THE IMPORTANCE OF BEING $\mathfrak{B}_2$

THEOREM

Every Boolean algebra is the subdirect product of two-element Boolean algebras $\mathfrak{B}_2$.

In other words:

$$\models_{\mathbb{BA}} s \approx t \qquad \text{iff} \qquad \models_{\mathfrak{B}_2} s \approx t,$$

where $\mathbb{BA}$ is the equational class (= variety) of *all* Boolean algebras.

The equational theory of Boolean algebras only needs $\mathfrak{B}_2$!

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Classical propositional logic (CPL) and Boolean algebras

A sentence α is a classical tautology iff α is valid in $\mathfrak{B}_2$.

Halmos: a (classical) logician is the dual of a (Boolean) algebraist.

Dunn and Hardegree: "by duality we obtain that the (Boolean) algebraist is the dual of the (classical) logician."

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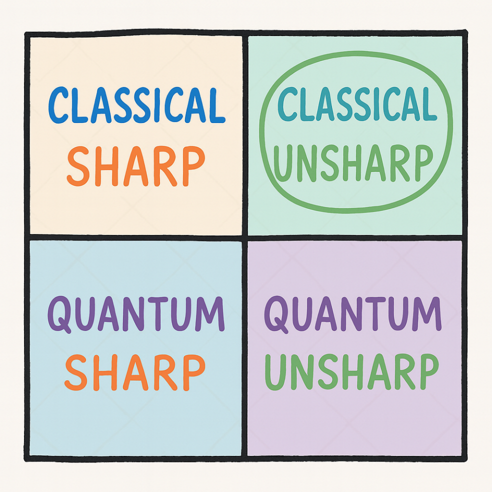
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SHARP
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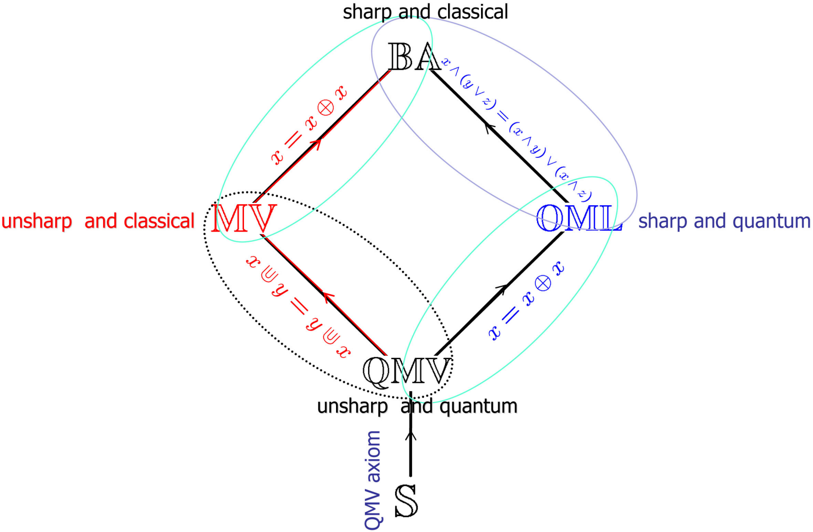
The classical unsharp universe



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- In 1920, Łukasiewicz published his two-page article *On three-valued logic*.
- Motivation: escape the determinism implied by bivalence.
    - If every sentence is either true or false, then the future is already determined.
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From Prophecy to Reality

- Łukasiewicz foresaw only a *theoretical* role for many-valued logics.
- Yet, two surprising developments occurred:
  1. **Fuzzy logics** (natural heirs of Łukasiewicz' logics) entered technology: washing machines, cameras, control systems.
  2. **Quantum theory** gave empirical meaning to indeterminism: no-go theorems and experimental tests (Bell, Aspect, . . .).
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- **MV-algebras:** (Chang, 1958) fuzzy (non-idempotent) generalization of Boolean algebras.
  - Capture Łukasiewicz infinite-valued logic
  - Provide **algebraic semantics** for graded truth-values.

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References (Lecture I)

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...TO BE CONTINUED

in LECTURE II