

The constructive reverse mathematics of the downward Löwenheim-Skolem theorem

Timothée Huneau¹

¹(soon-to-be) PhD at Inria Paris

The DLS theorem

The downward Löwenheim-Skolem (DLS) [3, 5, 4, 6] theorem is central in first order model theory.

Theorem (DLS theorem)

For any model $\mathcal{M}$ of a signature Σ , there exists a model $\mathcal{N}$ such that:

- *$\mathcal{N}$ is an elementary submodel of $\mathcal{M}$;*
- *$|\mathcal{N}| \leq |\mathbb{N}| + |\Sigma|$.*

We write $\mathbf{DLS}^\kappa$ for the **DLS** theorem up to a signature of size κ .

Reverse mathematics of the DLS theorem

Is **AC** necessary for the **DLS** theorem to hold? Is **LEM** necessary for the **DLS** theorem to hold?

$$\text{LEM} \vdash \mathbf{DLS}^\kappa \leftrightarrow \mathbf{DC} \wedge \mathbf{AC}_\kappa \quad (\text{Karagila [1]})$$

$$\vdash \mathbf{DLS}^{\aleph_0} \leftrightarrow \mathbf{DDC} \wedge \mathbf{BAC}_{\aleph_0} \wedge \mathbf{BDP}_{\aleph_0} \wedge \mathbf{BEP}_{\aleph_0} \quad (\text{Kirst and Zeng [2]})$$

$$\vdash \mathbf{DLS}^\kappa \leftrightarrow \mathbf{DDC} \wedge \mathbf{BAC}_\kappa \wedge \mathbf{BDP}_\kappa \wedge \mathbf{BEP}_\kappa \quad (\text{conjecture})$$

$$\vdash \mathbf{DLS}^\kappa \leftarrow (\text{some interfaces})$$

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