(0) (???) If $S \models A$, then $S \vdash A$.

~~If A holds in the (standard) model of S, then $S \vdash A$.~~

~~If A is true, then $S \vdash A$.~~

(1) (IPRA) There is A such that

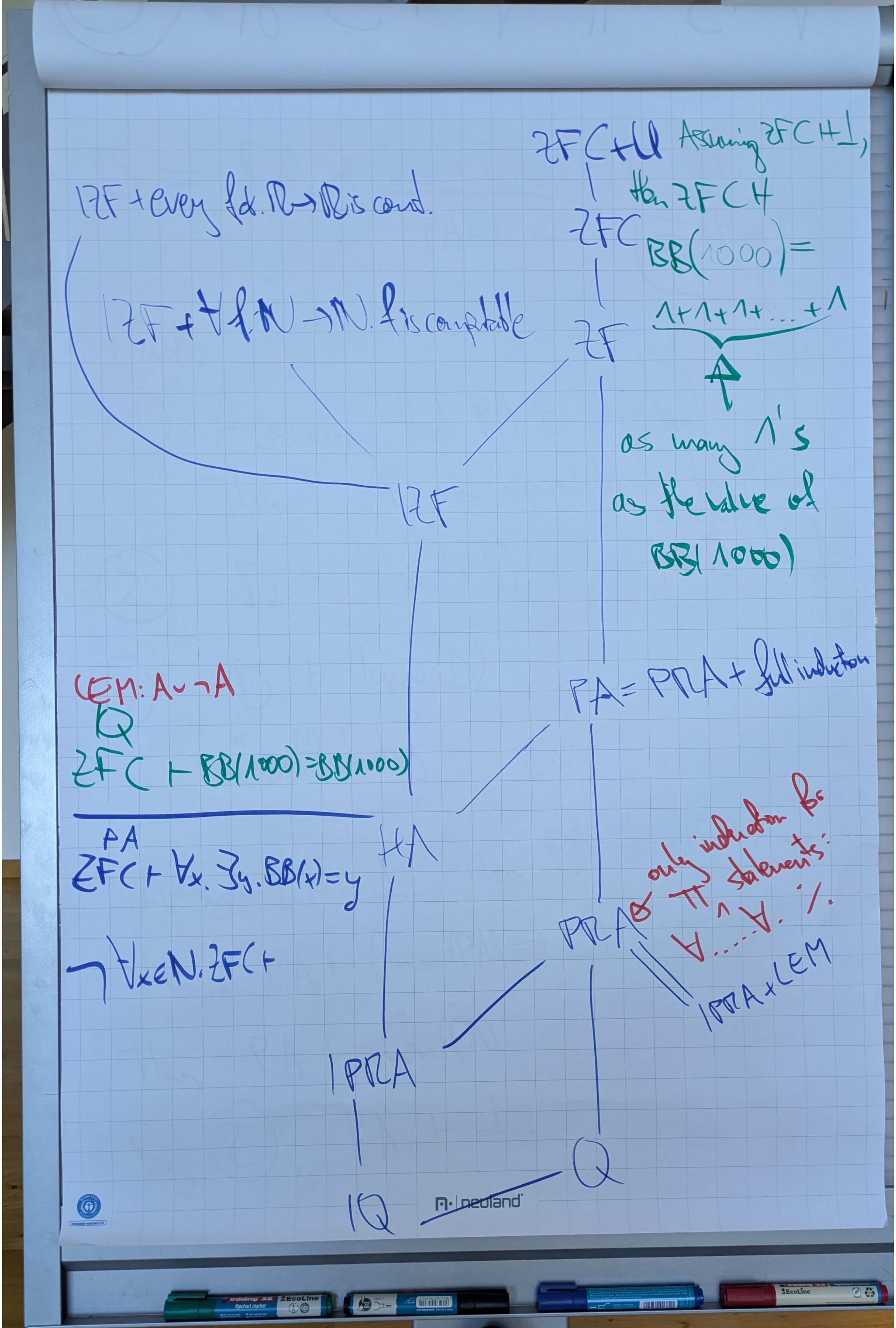
$$(S \vdash A) \Rightarrow (S \vdash \bot)$$

and such that $(S \vdash \neg A) \Rightarrow (S \vdash \bot)$.

(2) (IPRA) If $S \vdash \underbrace{\mathrm{Con}(S)}_{\text{"S is consistent"}}$, then $S \vdash \bot$.

- S: a formal system, e.g. PA, ZFC, HA, IZF, ...
- S recursively axiomatizable
- S needs to encompass IQ
- S needs to encompass intuitionistic logic

S needs to satisfy the HBL condition



$ZFC + \mathcal{U}$
Assuming $ZFC \nvdash \bot$, then $ZFC \nvdash$ $BB(1000) = \underbrace{1+1+1+\ldots+1}$
as many 1's as the value of BB(1000)
ZFC
ZF
IZF + every fct. $\mathbb{R} \to \mathbb{R}$ is cont.
IZF + $\forall f : \mathbb{N} \to \mathbb{N}$. f is computable
IZF
PA = PRA + full induction
LEM: $A \vee \neg A$
$ZFC \vdash BB(1000) = BB(1000)$
$ZFC \vdash_{PA} \forall x. \exists y. BB(x) = y$
$\neg \forall x \in \mathbb{N}. ZFC \vdash$
HA
PRA
only induction for Π statements: $\forall \ldots \forall.$ %
IPRA + LEM
IPRA
Q
IQ
neuland

$$D = PA + \underbrace{\mathrm{Incon}(PA)}_{PA \vdash \bot}$$

$\neg P$

$\equiv (P \to \bot)$

Facts:

1. Assuming that PA is consistent.
   Then: D is consistent. In other words: $\neg (D \vdash \bot)$

2. $D \vdash \mathrm{Incon}(PA)$

3. $D \vdash \mathrm{Incon}(D)$
   "self-hating"

Assume $D \cancel{M} \vdash \bot$. Then $PA + \mathrm{Incon}(PA) \vdash \bot$.

Hence $PA \vdash (\mathrm{Incon}(PA) \to \bot)$.

Thus $PA \vdash \mathrm{Con}(PA)$.

By 2, $PA \vdash \bot$.

Def.: $S \models A$ $:\Longleftrightarrow$

$\forall$ model M of S, A holds in M.

Rem.: (??)
If $S \nvdash \bot$
iff $\neg(S \vdash \bot)$,
then $\exists M$, M
model of S.

Def.: A model of HA is a
- a set N,
- an element $z \in N$,
- a function $s : N \to N$
- a function $+ : N \times N \to N$
- a function $\cdot : N \times N \to N$

such that the axioms of HA are satisfied.
(For instance: $\forall x \in N.\ x + \overset{z}{\cancel{0}} = x$)

Ex.: (In ZFC) Models of PA:
- $(\mathbb{N}, 0, \mathrm{succ}, +, \cdot)$ "standard model"
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