CONSTRUCTING ALG. CLOSURES?

Input: a field K

Goal: a field Ω together with a field homomorphism $K \to \Omega$ such that Ω is alg. closed and such that every element of Ω is algebraic/K (i.e. $\forall x \in \Omega\ \exists f \in K[X]_{\text{monic}}$ $f(x) = 0$).

Ex.: In classical mathematics, an alg. closure of $\mathbb{R}$ is $\mathbb{C}$.

Ex.: In all flavors of mathematics, $\overline{\mathbb{Q}}$ is an alg. closure of $\mathbb{Q}$

$$\overline{\mathbb{Q}} = \{z \in \mathbb{C} \mid z \text{ is alg.}/\mathbb{Q}\}$$

Ex.: No finite field is alg. closed: $\prod_{\alpha \in K} (X - \alpha) + 1$

Constr.: Let K be a field. Let $f \in K[X]$ be monic.
Let $\mathfrak{m}$ be a maximal ideal in $K[X]$ with $(f) \subseteq \mathfrak{m}$. Then

$$K' := K[X]/\mathfrak{m}$$

is a field in which f has a zero, namely $[X]$.

Ex.: The ideals of $\mathbb{R}[X]$ in class. mathematics:

$$\mathbb{R}[X]$$

$(X-\pi) = \{f \cdot (X-\pi) \mid f \in \mathbb{R}[X]\}$ $\quad$ $(X-7)$ $\quad$ (X^2+1)

$(X^2-(7+\pi)X+7\pi)$ $\quad$ $((X-7)^{100})$

$$(0) = \{0\}$$

Ex.: $\mathbb{C} \cong \mathbb{R}[X]/(X^2+1)$, $i := [X]$

Ex.: $\mathbb{R}[X]/(X-\alpha) \cong \mathbb{R}$

Constr.: Let K be a countable field.

Let $f_0, f_1, f_2, \ldots \in K[X]$ be an enumeration of all polynomials over K.

Then construct K_0 as the direct limit of

$$\overset{0}{K} \longrightarrow \overset{1}{\text{Kronecker wrt. } f_0} \longrightarrow \overset{2}{\text{Kronecker wrt.}} \longrightarrow \cdots \longrightarrow \text{Kronecker wrt.}$$

$0 \leq 1 \leq 2 \leq \ldots$

$$\longrightarrow \text{Kronecker wrt. } f_1 \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \text{Kronecker wrt.}$$

$$\longrightarrow \;\cdots$$

$$\longrightarrow \;\cdots$$

$$\longrightarrow \;\cdots\cdots$$

Then K_0 is a field extension of K, over which all of the polynomials in $K[X]$ split into linear factors.

If LEM, then K_0 is alg. closed.

In any case, set Ω as the direct limit of

$$K \to K_0 \to K_1 \to K_2 \to \ldots$$

In constructive mathematics:

$\boxed{A}$ Let K be a field which might not be countable. Then pass to a forcing extension in which K appears to be countable. Then in the forcing extension, use the previous construction (utilizing the fact that constructively, we do have maximal ideals for countable rings).

$\boxed{B}$ Use the previous construction, however ~~...~~ replace the Kronecker constr. by a dynamical version of it, and replace the enumeration by a dynamical version of it.

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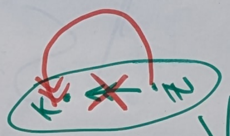

"DS technique"

Dynamical Kronecker construction:

Let K be a field. Let $f \in K[X]$ be a monic poly.

Then propose $K' := K[X]/(f)$.

Given $[g] \in K'$, to check whether $[g] = 0$ in K'

or whether $[g]$ is inv. in K',

do the following:

Compute $d := \gcd(g, f) \in K[X]$.

$(d = \tilde{g} \cdot g + \tilde{f} \cdot f)$ $\quad (g = \hat{g} \cdot d)$

Three cases:

1. $d = 1$. Then $[g]$ is inv. in K', the inverse is given by $[\tilde{g}]$.
2. $d = f$. Then $[g] = 0$ in K'.
3. Else d is a nontrivial factor of f. Then restart with $K' := K[X]/(d)$.

$\Omega := K$