SARA NEGRI'S LABELLLED SEQUENTS CALCULI FOR MODAL LOGICS (S5-CUBE AND GL)

Antonella Bilotta

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AGENDA

Definition 1 (Sequent).

> *A* ***sequent*** *is a formal expression of the form:*
>
> $$\Gamma \Rightarrow \Delta$$
>
> *where* $\Gamma,\ \Delta$ *are finite multiset–finite lists modulo permutations–of modal formulas* $\mathbf{Form}_{\Box}$ *and:*
>
> - $\Rightarrow$ *reflects the deducibility relation at the metalevel in the object language;*
> - Γ *is called antecedent;*
> - Δ *is called consequent.*

The *intended semantic meaning* of this syntactic object is the following:

$$\bigwedge \Gamma \Longrightarrow \bigvee \Delta$$

Definition 2 (**Derivation**).

> *A **derivation** in a G3-style sequent calculus* C *of* $\Gamma \Rightarrow \Delta$ *is a finite rooted tree of sequents such that:*
>
> - *Its leaves are **initial sequents** or **zero-ary rules**;*
> - *Each node of the tree is obtained from the sequents directly above using an **inference rule** of the calculus;*
> - *Its root is the **end sequent*** $\Gamma \Rightarrow \Delta$
>
> *A sequent* $\Gamma \Rightarrow \Delta$ *is **derivable** in a G3-style sequent calculus* C *and we formally write* $\mathsf{C} \vdash \Gamma \Rightarrow \Delta$ *iff there is a derivation of* $\Gamma \Rightarrow \Delta$ *in* C.

To derive a given sequent in a calculus C, we perform a **root-first proof search**.

Definition 3 (**Ammissibility of a rule**).

> *A rule R is **admissible** in a* G3*-style sequent calculus* C *iff its conclusion is derivable in* C *whenever its premises are derivable in* C.

Definition 4 (**Invertibility of a rule**).

> *A rule is **invertible** in a* G3*-style sequent calculus* C *iff its premises are derivable in* C *whenever conclusion is derivable in* C.

Sequent Calculus G3cp

Initial Sequents:

$p, \Gamma \Rightarrow \Delta, p$

Propositional Rules:

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ \mathtt{L}\bot$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ \mathtt{L}\neg \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ \mathtt{R}\neg$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ \mathtt{L}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ \mathtt{R}\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ \mathtt{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ \mathtt{R}\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\ \mathtt{L}\rightarrow \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ \mathtt{R}\rightarrow$$

DESIDERATA FOR SEQUENT CALCULI

A well-designed G3-style calculus satisfies these criteria, ensuring efficient *automation of decision procedures* through a *terminating root-first* proof search:

- **Analyticity**
- **Modularity**
- **Invertibility of all the rules**
- **Admissibility of structural rules**
  Inference rules can be divided into two categories, logical and structural rules.
  - **Logical rules**: enable the introduction of a logical operator $\circ$ either on the left ($\mathtt{L}\circ$) or right ($\mathtt{R}\circ$) side of a sequent.
  - **Structural rules**: do not introduce logical operators but rather manipulate the structure of sequents.
- **Termination**

STRUCTURAL RULES

1. **Cut rule**

$$\frac{\Gamma \Rightarrow \Delta,\ A \qquad A,\ \Gamma' \Rightarrow \Delta'}{\Gamma,\ \Gamma' \Rightarrow \Delta,\ \Delta'}\ \mathtt{Cut}_{\mathsf{G3cp}}$$

2. **Weakening**

$$\frac{\Gamma \Rightarrow \Delta}{A,\ \Gamma \Rightarrow \Delta}\ \mathtt{LW}_{\mathsf{G3cp}} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,\ A}\ \mathtt{RW}_{\mathsf{G3cp}}$$

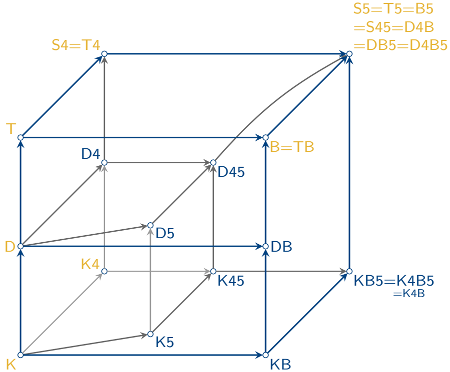
3. **Contraction**

$$\frac{A,\ A,\ \Gamma \Rightarrow \Delta}{A,\ \Gamma \Rightarrow \Delta}\ \mathtt{LC}_{\mathsf{G3cp}} \qquad \frac{\Gamma \Rightarrow \Delta,\ A,\ A}{\Gamma \Rightarrow \Delta,\ A}\ \mathtt{RC}_{\mathsf{G3cp}}$$

We present a **logical variant** of sequent calculi, which does not include primitive structural rules, but for which the *admissibility of structural rules* has been proved.
◁ This both ensures the **flexibility of the system** and the **analyticity** of the calculus. An opposite but equivalent method is to present the general variant and prove the eliminability of the structural rules, e.g. cut elimination.

S5-CUBE AND GL

Axiom schema	Semantic condition
D : $\Box A \to \Diamond A$	Seriality
T : $\Box A \to A$	Reflexivity
B : $A \to \Box\Diamond A$	Symmetry
4 : $\Box A \to \Box\Box A$	Transitivity
5 : $\Diamond A \to \Box\Diamond A$	Euclidianness
GL : $\Box(\Box A \to A) \to \Box A$	Transitivity and Noetherianity



INTERNALISATION TECHNIQUES

A possible approach for **designing sequent calculi for modal systems** is to enriches G3-style calculi by incorporating elements of relational semantics into the proof system. There are two main strategies for internalisation:

(a) **Explicit Internalisation**: enriches the *language* of the calculus itself to explicitly represent semantics elements.
◁ Usually satisfy the basic desiderata of G3-style systems;
▷ The termination of proof search is sometimes difficult to prove and, in many cases, is not optimal in terms of complexity.

(b) **Implicit Internalisation**: enriches the *structure* of the sequents by introducing additional structural connectives beyond the standard '$\Rightarrow$' ','.
◁ Enables efficient decision procedures.
▷ Difficult to design and generally lacks *modularity*.

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Definition 5 (**Alphabet**). $\mathcal{L}_{LS} = \mathcal{L}_{\Box} \cup L \cup \{R\}$

- *L is an **infinite denumerable** set of **labels**: $w, x, y, \dots$;*
- *R is a binary predicate.*

Definition 6 (**Formulas** of $\mathcal{L}_{LS}$). $\varphi \in \mathbf{Form}_{LS} ::= x : A \mid xRy$

- ***labelled formulas*** $x : A \rightarrow \mathcal{M}, x \vDash A$
- ***relational atoms*** $xRy \rightarrow xRy$

Definition 7 (**Labelled Sequent**).

*A **labelled sequent** is a formal expression of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite multiset of formulas in $\mathbf{Form}_{LS}$.*

Labelled Sequent Calculus G3cp

Initial Sequents:

$w : p, \Gamma \Rightarrow \Delta, w : p$

Propositional Rules:

$$\frac{}{w : \bot, \Gamma \Rightarrow \Delta}\ \texttt{L}\bot$$

$$\frac{\Gamma \Rightarrow \Delta, w : A}{w : \neg A, \Gamma \Rightarrow \Delta}\ \texttt{L}\neg \qquad \frac{w : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : \neg A}\ \texttt{R}\neg$$

$$\frac{w : A, w : B, \Gamma \Rightarrow \Delta}{w : A \wedge B, \Gamma \Rightarrow \Delta}\ \texttt{L}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, w : A \qquad \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \wedge B}\ \texttt{R}\wedge$$

$$\frac{w : A, \Gamma \Rightarrow \Delta \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta}\ \texttt{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B}\ \texttt{R}\vee$$

$$\frac{\Gamma \Rightarrow \Delta, w : A \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \to B, \Gamma \Rightarrow \Delta}\ \texttt{L}\to \qquad \frac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \to B}\ \texttt{R}\to$$

Sequent Calculus G3K

Initial Sequents:

$w : p, \Gamma \Rightarrow \Delta, w : p$

Propositional Rules:

$$\frac{}{w : \bot, \Gamma \Rightarrow \Delta}\ \texttt{L}\bot$$

$$\frac{\Gamma \Rightarrow \Delta, w : A}{w : \neg A, \Gamma \Rightarrow \Delta}\ \texttt{L}\neg \qquad \frac{w : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : \neg A}\ \texttt{R}\neg$$

$$\frac{w : A, w : B, \Gamma \Rightarrow \Delta}{w : A \wedge B, \Gamma \Rightarrow \Delta}\ \texttt{L}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, w : A \qquad \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \wedge B}\ \texttt{R}\wedge$$

$$\frac{w : A, \Gamma \Rightarrow \Delta \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta}\ \texttt{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B}\ \texttt{R}\vee$$

$$\frac{\Gamma \Rightarrow \Delta, w : A \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \to B, \Gamma \Rightarrow \Delta}\ \texttt{L}\to \qquad \frac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \to B}\ \texttt{R}\to$$

Modal Rules:

$$\frac{x : A, wRx, w : \Box A, \Gamma \Rightarrow \Delta}{wRx, w : \Box A, \Gamma \Rightarrow \Delta}\ \texttt{L}\Box \qquad \frac{wRx, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, w : \Box A}\ \texttt{R}\Box\ (x!)$$

Why these rules?

All the inference rules are obtained by the inductive the definition of forcing:

- **Propositional rules** are straightforward;
- **Modal rules** governing the $\Box$ operator require a reflection, since the forcing condition for $\Box$ is given by:

$$\mathcal{M}, w \vDash \Box A \text{ iff } \forall x(wRx \implies \mathcal{M}, x \vDash A)^1.$$

This condition leads to the following modal rules:

$$\frac{x : A, wRx, w : \Box A, \Gamma \Rightarrow \Delta}{wRx, w : \Box A, \Gamma \Rightarrow \Delta}\ \mathrm{L}\Box \qquad \frac{wRx, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, w : \Box A}\ \mathrm{R}\Box\ (x!)$$

OSS: The main formula '$x : \Box A$' is repeated in L$\Box$ to make the rule invertible.

Properties of G3K

Logic	Calculus	Str.Ad.	Inv.	Subterm	Termination	Decision	Modular
$\mathbb{K}$	G3K	✓	✓	✓	✓ [1]	✓ [1]	✓

FRAMEWORK STRUCTURE

G3K is a modular system, as demonstrated by Negri and von Plato in [4, §6]. This modularity allows it to be extended with appropriate rules:

	Characteristic Property	Corresponding Rule
D	**seriality**: $\forall w \exists y (wRy)$	$\dfrac{wRy, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ Ser
T	**reflexivity**: $\forall w (wRw)$	$\dfrac{wRw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ Ref
4	**transitivity**: $\forall w, x, z (wRx \wedge xRz \Rightarrow wRz)$	$\dfrac{wRz, wRx, xRz, \Gamma \Rightarrow \Delta}{wRx, xRz, \Gamma \Rightarrow \Delta}$ Trans
5	**euclideanity**: $\forall w, x, y (wRx \wedge wRy \Rightarrow yRx)$	$\dfrac{yRx, wRx, wRy, \Gamma \Rightarrow \Delta}{wRx, wRy, \Gamma \Rightarrow \Delta}$ Eucl
B	**simmetry**: $\forall w, x (wRx \Rightarrow xRw)$	$\dfrac{xRw, wRx, \Gamma \Rightarrow \Delta}{wRx, \Gamma \Rightarrow \Delta}$ Sym
-	**irreflexivity**: $\forall w (\neg wRw)$	$\dfrac{}{wRw, \Gamma \Rightarrow \Delta}$ Irref

Sequent Calculi G3K$*$

Logic	Calculus	Definition
$\mathbb{D}$	G3KD	G3K + `Ser`
$\mathbb{T}$	G3KT	G3K + `Refl`
$\mathbb{K}4$	G3KK4	G3K + `Trans`
$\mathbb{S}4$	G3KS4	G3K + `Refl` + `Trans`
$\mathbb{B}$	G3KB	G3K + `Refl` + `Sym`
$\mathbb{S}5$	G3KS5	G3K + `Refl` + `Eucl`

Properties of G3K$*$

Logic	Calculus	Struct.Ad.	Subterm	Termination	Decision
$\mathbb{D}$	G3K+ `Ser`	✓	✓	✓ [1]	✓ [1]
$\mathbb{T}$	G3K+ `Refl`	✓	✓	✓	✓
$\mathbb{K}4$	G3K+ `Trans`	✓	✓	✓ [1]	✓ [1]
$\mathbb{S}4$	G3K+`Refl`+`Trans`	✓	✓	✓	✓
$\mathbb{B}$	G3K+`Refl`+`Sym`	✓	✓	✓	✓
$\mathbb{S}5$	G3K+ `Refl`+`Eucl`	✓	✓	✓	✓

In [2, §5], Sara Negri provides a modular adaptation of G3K by relying on the following characterisation of the '$\Box$' that holds for models based on finite-irrational-transitive frames (frames characteristic to GL):

$$\mathcal{M}, w \vDash \Box A \text{ iff } \forall y (xRy \textit{ and } \mathcal{M}, y \vDash \Box A \implies \mathcal{M}, x \vDash A)^2.$$

In such a perspective, this condition suggests to modify the $\mathtt{R}\Box$ rule as follows:

$$\frac{w : A, wRx, x : \Box A, \Gamma \Rightarrow \Delta}{wRx, w : \Box A, \Gamma \Rightarrow \Delta}\ \mathtt{L}\Box \qquad \frac{wRx, x : \Box A, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, w : \Box A}\ \mathtt{R}\Box_{Lob}\ (x!)$$

The resulting calculus G3KGL:=(G3K $-$ $\mathtt{R}\Box$) $+$ $\mathtt{R}\Box_{Lob}$ $+$ $\mathtt{Trans}$ $+$ $\mathtt{Irrefl}$ consists of the initial sequents, the propositional rules and $\mathtt{L}\Box$, together with the new modal rule $\mathtt{R}\Box_{Lob}$, transitive and irreflexive inference rules.

Properties of G3K

Logic	**Calculus**	**Str.Ad.**	**Inv.**	**Subterm**	**Termination**	**Decision**
$\mathbb{GL}$	G3KGL	✓	✓	✓	✓ [3]	✓ [3]

[1] Deepak Garg, Valerio Genovese, and Sara Negri.
Countermodels from sequent calculi in multi-modal logics.
In *Proceedings of the 27th Annual IEEE Symposium on Logic in Computer Science, LICS 2012, Dubrovnik, Croatia, June 25-28, 2012*, pages 315–324. IEEE Computer Society, 2012.

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[3] Sara Negri.
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[4] Sara Negri and Jan von Plato.
Structural proof theory.
Cambridge university press, 2008.