PC2025: Modal Sequent Calculi

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What is Sequent Calculus?

Proof Search Challenge. No systematic way of doing this in Hilbert calculi

- both for provability or non-provability
- can always use modus ponens
- no bound on formulae that can appear in a proof

Sequent Calculus repairs this:

- each proof rule deconstructs one connective
- strong structural relationship between premiss and conclusion
- often: premisses structurally simpler (i.e. backwards proof search terminates)

Sequent Calculus 101

Notation. Let Γ, Δ be multisets of formulae, and A a formula.

- We write comma for union: Γ, Δ means $\Gamma \cup \Delta$
- We elide braces of singleton sets: Γ, A means $\Gamma, \{A\}$
- We elide the empty multiset: $\Longrightarrow \Gamma$ means $\emptyset \Longrightarrow \Gamma$ and $\Gamma \Longrightarrow$ means $\Gamma \Longrightarrow \emptyset$.
- We apply operators elementwise: $\heartsuit\Gamma$ means $\heartsuit A_1, \dots, \heartsuit A_n$ if $\Gamma = A_1, \dots, A_n$.

Definition.

A *sequent* is a pair (Γ, Δ), written $\Gamma \Longrightarrow \Delta$ of multisets Γ, Δ of formulae.

- $[\![\Gamma \Longrightarrow \Delta]\!] = \bigvee \neg\Gamma \vee \Delta$ is the formula associated with the sequent $\Gamma \Longrightarrow \Delta$
- $[\![A]\!] = (\Longrightarrow A)$ is the sequent associated with formula A.

A sequent $\Gamma \Longrightarrow \Delta$ is *propositional* if Γ, Δ are propositional formulae.

Propositional Sequent Rules

$$\frac{}{\Gamma, p \Longrightarrow p, \Delta} \qquad \frac{}{\bot, \Gamma \Longrightarrow \Delta}$$

$$\frac{A, B, \Gamma \Longrightarrow \Delta}{A \wedge B, \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow A, \Delta \quad \Gamma \Longrightarrow B, \Delta}{\Gamma \Longrightarrow A \wedge B, \Delta}$$

$$\frac{A, \Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \neg A, \Delta} \qquad \frac{\Gamma \Longrightarrow B, \Delta}{\Gamma, \neg B \Longrightarrow \Delta}$$

Definition.

If Φ is a set of sequence, $\Phi \vdash_{\mathsf{PL}}$ is the least set of sequents that contains Φ and is closed under the above rules.

Notation: $\Phi \vdash_{\mathsf{PL}} \Gamma \Longrightarrow \Delta$ means $\Gamma \Longrightarrow \Delta \in \Phi \vdash_{\mathsf{PL}}$.

Soundness and Completeness of Propositional Logic

Soundness

If $\vdash_{\mathsf{PL}} \Gamma \implies \Delta$ for a propositional sequent $\Gamma \implies \Delta$, then $[\![\Gamma \implies \Delta$ is a propositional tautology.

(Proof by induction on the derivation)

The Swiss Army Knife of Structural Proof Theory:

$$\Gamma, \Delta < \Gamma, A \qquad \text{(if every } B \in \Delta \text{ is a proper subformula of } A\text{)}$$

generates a well-founded partial order, the *Dershowitz-Manna ordering*

Completeness

If $[\![\Gamma \implies \Delta]\!]$ is a propositional tautology, then $\vdash_{\mathsf{PL}} \Gamma \implies \Delta$.

(Proof by well-founded induction on the Dershowitz-Manna ordering)

Admissible Rules

Inversion Rules are depth-preserving admissible:

$$\frac{\Gamma, A \wedge B, \Gamma \Longrightarrow \Delta}{\Gamma, A, B \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow A \wedge B, \Delta}{\Gamma \Longrightarrow A, \Delta} \qquad \frac{\Gamma \Longrightarrow A \wedge B, \Delta}{\Gamma \Longrightarrow B, \Delta}$$

$$\frac{\Gamma, \neg B \Longrightarrow \Delta}{\Gamma \Longrightarrow B, \Delta} \qquad \frac{\Gamma \Longrightarrow \neg A, \Delta}{\Gamma, A \Longrightarrow \Delta}$$

Admissible Structural Rules: Weakening, left and right contraction and cut.

$$(\mathrm{W})\frac{\Gamma \Longrightarrow \Delta}{\Gamma, \Gamma' \Longrightarrow \Delta, \Delta'} \qquad (\mathrm{CL})\frac{\Gamma, A, A \Longrightarrow \Delta}{\Gamma, A \Longrightarrow \Delta} \qquad (\mathrm{CR})\frac{\Gamma \Longrightarrow B, B, \Delta}{\Gamma \Longrightarrow B, \Delta} \qquad (\mathrm{Cut})\frac{\Gamma \Longrightarrow \Delta, A \quad \Sigma, A \Longrightarrow \Pi}{\Gamma, \Sigma \Longrightarrow \Delta, \Pi}$$

Admissible rules.

- admissible rules can always be eliminated from a proof tree
- eliminating depth-preserving admissible rules doesn't increase height of proof tree.

Contraction

Example.

$$(\mathrm{CL})\ \dfrac{\dfrac{\Gamma, \neg A \Longrightarrow \Delta, A}{\Gamma, \neg A, \neg A \Longrightarrow \Delta}}{\Gamma, \neg A \Longrightarrow \Delta} \quad \rightsquigarrow \quad (\mathrm{CL})\ \dfrac{\dfrac{\dfrac{\Gamma, \neg A \Longrightarrow \Delta, A}{\Gamma \Longrightarrow A, A, \Delta}}{\Gamma \Longrightarrow A, \Delta}}{\Gamma, \neg A \Longrightarrow \Delta}$$

Cut Elimination: Two Cases

Cut on Principal / Non-Principal Formula

$$
(\text{Cut})\ \dfrac{\dfrac{\Gamma, A, B \Longrightarrow \Delta, \neg C}{\Gamma, A \wedge B \Longrightarrow \Delta, \neg C} \qquad \dfrac{\Sigma \Longrightarrow C, \Pi}{\Sigma, \neg C \Longrightarrow \Pi}}{\Gamma, A \wedge B, \Sigma \Longrightarrow \Delta, \Pi}
\quad \rightsquigarrow \quad
(\text{Cut})\ \dfrac{\dfrac{\dfrac{\Sigma \Longrightarrow C, \Pi}{\Sigma, \neg C \Longrightarrow \Pi} \qquad \Gamma, A, B \Longrightarrow \Delta, \neg C}{\Gamma, A, B, \Sigma \Longrightarrow \Delta, \Pi}}{\Gamma, A \wedge B, \Sigma \Longrightarrow \Delta, \Pi}
$$

Cut on Principal / Principal Formula

$$
(\text{Cut})\ \dfrac{\dfrac{\Gamma \Longrightarrow A, \Delta \qquad \Gamma \Longrightarrow B, \Delta}{\Gamma \Longrightarrow A \wedge B, \Delta} \qquad \dfrac{A, B, \Sigma \Longrightarrow \Pi}{A \wedge B, \Sigma \Longrightarrow \Pi}}{\Gamma, \Sigma \Longrightarrow \Delta, \Pi}
\rightsquigarrow
(\text{Cut})\ \dfrac{\dfrac{\dfrac{\Gamma \Longrightarrow A, \Delta \qquad A, B, \Sigma \Longrightarrow \Pi}{B, \Gamma, \Sigma \Longrightarrow \Delta, \Pi}\ (\text{Cut}) \qquad \Gamma \Longrightarrow B, \Delta}{\Gamma, \Gamma, \Sigma \Longrightarrow \Pi, \Delta, \Delta}}{\Gamma, \Sigma \Longrightarrow \Delta, \Pi}
$$

Adding Modal Operators

Modal Rules.

$$\frac{\Gamma_1 \Longrightarrow \Delta_1 \quad \dots \quad \Gamma_n \Longrightarrow \Delta_n}{\Gamma_0 \Longrightarrow \Delta_0}$$

- $\Gamma_i, \Delta_j \subseteq V \cup \{\heartsuit(a_1, \dots, a_n) \mid \heartsuit \in \Lambda \text{ } n\text{-ary}, A_1, \dots, A_n \in V\}$ for $i \geq 0$ – only introduce modalities
- $\Gamma_i \cup \Delta_i \subseteq \mathrm{subf}(\Gamma_0 \cup \Delta_0)$ for $i > 0$ – no new literals in the premisses.

Notation.

- $\Phi \vdash_{\mathrm{Rl}} \Gamma \Longrightarrow \Delta$ if $\Gamma \Longrightarrow \Delta$ can be derived from Φ using propositional and modal rules.
- $\vdash_{\mathrm{Rl}}^{\mathrm{Cut}}$ and $\vdash_{\mathrm{Rl}}^{\mathrm{Cut,W}}$ additionally uses cut and weakening.

Soundness and Completeness: Road Map

Goal.

Given axions Ax, find sequent rules Rl such that

- Rl is sound wrt. Ax: if $\vdash^{\mathsf{Cut},W}_{\mathsf{Rl}} \Gamma \Longrightarrow \Delta$ then $[\![\Gamma \Longrightarrow \Delta]\!] \in \mathsf{L}(\mathsf{Ax})$
- Rl complete wrt. Ax: if $A \in \mathsf{L}(\mathsf{Ax})$ then $\vdash_{\mathsf{Rl}} \Longrightarrow A$.

Trivial Observation.

- Soundness will also hold if Cut and W are *not* used, and
- Completeness will still be valid if we allow Cut and W.

Example: Modal Logic K

Rule Set.

$$\frac{A_1, \ldots, A_n \Longrightarrow A_0}{\Gamma, \Box A_1, \ldots, \Box A_n \Longrightarrow \Delta, \Box A_0} (n \geq 0)$$

Axiomatisation. tautologies, (MP), (US) plus:

$$\frac{A}{\Box A} \qquad \Box(A \to B) \to \Box A \to \Box B$$

Soundness: If $\vdash^{\mathrm{Cut,W}}_{\mathrm{RI}} \Gamma \Longrightarrow \Delta$ then $[\![\Gamma \Longrightarrow \Delta]\!] \in \mathrm{K}$ by induction

- Modal case: if $A_1 \wedge \cdots \wedge A_n \to A_0 \in \mathrm{K}$ then $\Box A_1 \wedge \cdots \wedge \Box A_n \to \Box A_0 \in \mathrm{K}$.
- Key Lemma: $\Box A \wedge \Box B \to \Box(A \wedge B)$
  - Have $A \to (B \to A \wedge B)$, a tautology
  - Hence $\Box(A \to (B \to A \wedge B))$ and $\Box(A \to (B \to A \wedge B)) \to \Box A \to \Box(B \to (A \wedge B))$
  - Using modus ponens, $\Box A \to \Box(B \to A \wedge B)$.
  - On the other hand, also $\Box(B \to (A \wedge B)) \to \Box B \to \Box(A \wedge B)$
  - Using tautologies and modus ponens, $\Box A \to \Box B \to \Box(A \wedge B)$.

Completeness

First Goal. If $\vdash^{\mathsf{Cut}}_{\mathsf{RI}} \Longrightarrow A$, then $A \in \mathsf{K}$.

As K is the least set

- that contains propositional tautologies and $\Box(a \to b) \to \Box a \to \Box b$.
- is closed under modus ponens, substitution, necessitation

we just need to show that $\{A \in \mathcal{L}(\Lambda) \mid \vdash \Longrightarrow A\}$ has these closure properties.

Closure under ...

- propositional tautologies: follows from propositional completeness
- modus ponens: follows from Cut and inversion
- necessitation: is an instance of the K-rule
- distribution axiom: build derivation backwards

Closure under Substitution

Generalised Axioms

Let $\Gamma \Longrightarrow \Delta$ be a sequent. Then $\vdash \Gamma, A \Longrightarrow A, \Delta$ for all formulae $A \in \mathcal{L}(\Lambda)$.
(By induction on the formula where $A = p$ is the case of axiom.)

Substitution

Suppose that $\vdash_{\mathsf{RI}} \Gamma \Longrightarrow \Delta$. Then $\vdash_{\mathsf{RI}} \Gamma\sigma \Longrightarrow \Delta\sigma$ for all substitutuions $\sigma : \mathsf{V} \to \mathcal{L}(\Lambda)$.
(By induction on the derivation, using Generalised Axiom for the base case.)

Both hold more generally if the *Congruence Rule*

$$\frac{A \Longrightarrow B \qquad B \Longrightarrow A}{\Gamma, \Box A \Longrightarrow \Delta, \Box B}$$

is admissible.

Cut Elimination

Second Goal. If $A \in \mathsf{K}$ then $\vdash_{\mathsf{RI}} \Longrightarrow A$.

Show more generally that $\vdash^{\mathsf{Cut}}_{\mathsf{RI}} \Gamma \Longrightarrow \Delta$, then $\vdash_{\mathsf{RI}} \Gamma \Longrightarrow \Delta$

Modal / Propositional Rule Example.

$$\frac{A_1, \dots, A_n \Longrightarrow A_0}{\Gamma, \Box A_1, \dots, \Box A_n \Longrightarrow \Delta, \Box A_0} \qquad \frac{\Sigma \Longrightarrow A, \Pi \quad \Sigma \Longrightarrow B, \Pi}{\Sigma \Longrightarrow A \land B, \Pi}$$

- cuts on elements of Γ, Δ are trivial, cuts on elements of Σ, Π can be permuted upwards

Modal / Modal Rule Example

$$\frac{\dfrac{A_1, \dots, A_n \Longrightarrow A_0}{\Box A_1 \dots, \Box A_n \Longrightarrow \Box A_0} \qquad \dfrac{A_0, B_1, \dots, B_k \Longrightarrow B_0}{\Box A_0, \Box B_1, \dots, \Box B_k \Longrightarrow \Box B_0}}{\Box A_1, \dots \Box A_n, \Box B_1, \dots, \Box B_k \Longrightarrow \Box B_0}$$

- cuts between modal rules can be replaced by different instance of modal rules.

Application I: Complexity and Consistency

Theorem.
The modal logic K is consistent.

Proof. The empty sequent is not derivable.

Theorem.
Validity in K is decidable in polynomial space.

Proof. Generalise to provability of sequents and use PSPACE = APTIME.

- Existential states: non-deterministically choose proof rule given a sequent
- Universal states: non-deterministically choose premiss given proof rule
- This machine runs in polynomial time, as branches are polynomially long.

Application II: Craig Interpolation

Definition.

A logic L has the *Craig Interpolation Property* (CIP) if for all implications $A \to B \in \mathsf{L}$ there is I such that both $A \to I, I \to B \in \mathsf{L}$ and I only has variables common to A and B.

Maehara's Method: Generalise to Sequents

- $\Gamma_1 \mid \Gamma_2 \Longrightarrow \Delta_1 \Longrightarrow \Delta_2$ is a *splitting* of $\Gamma_1, \Gamma_2 \Longrightarrow \Delta_1, \Delta_2$
- I is an interpolant of $\Gamma_1 \mid \Gamma_2 \Longrightarrow \Delta_1 \mid \Delta_2$ if $\vdash \Gamma_1 \Longrightarrow I, \Gamma_2$ and $\Gamma_2, I \Longrightarrow \Delta_2$ (and I only uses variables common to Γ_1, Γ_2 and Δ_1, Δ_2)

Main Idea. Construct interpolant of splitting of *rule conclusion* assuming that every splitting of premisses have interpolant.

Theorem. The modal logic K has the CIP.

Construction of Proof Rules

Starting Point. Necessitation and distribution axiom textbook

$$\frac{p}{\Box p} \qquad \Box(p \to q) \to \Box p \to \Box q$$

As Sequent Rules, i.e. applying inversion

$$\frac{\Longrightarrow A}{\Longrightarrow \Box A} \qquad \frac{}{\Box(A \to B), \Box A \Longrightarrow \Box B}$$

Find Occurrences of Cut

$$\frac{\dfrac{\Longrightarrow A \to B}{\Longrightarrow \Box(A \to B)} \qquad \dfrac{}{\Box(A \to B), \Box A \Longrightarrow \Box B}}{\Box A \Longrightarrow \Box B}$$

Idea. Add this as a new rule

$$\frac{A \Longrightarrow B}{\Box A \Longrightarrow \Box B}$$

More Cuts

Extended Rule Set.

$$\frac{\Longrightarrow A}{\Longrightarrow \Box A} \qquad \frac{A \Longrightarrow B}{\Box A \Longrightarrow \Box B} \qquad \frac{}{\Box(A \to B), \Box A \Longrightarrow \Box B}$$

More cuts.

$$\frac{\dfrac{A \Longrightarrow B \to C}{\Box A \Longrightarrow \Box(B \to C)} \qquad \dfrac{}{\Box(B \to C), \Box B \Longrightarrow \Box C}}{\Box A, \Box B \Longrightarrow \Box C}$$

New Rule.

$$\frac{\neg A, \neg B, \neg C}{\neg \Box A, \neg \Box B, \Box C}$$

After finitely many steps . . .

$$(K_n)\frac{A_1, \ldots, A_n \Longrightarrow A_0}{\Box A_1, \ldots, \Box A_n \Longrightarrow \Box A_0}$$

General Idea. Add cuts between modal rules until this process terminates.

Ingredients of Cut Elimination

Admissibility of Weakening.

$$\frac{A_1, \dots, A_n \Longrightarrow A_0}{\Box A_1, \dots, \Box A_n \Longrightarrow \Box A_0} \quad \rightsquigarrow \quad \frac{A_1, \dots, A_n \Longrightarrow A_0}{\Gamma, \Box A_1, \dots, \Box A_n \Longrightarrow \Delta, \Box A_0}$$

Inversion and Contraction.

- hold for *this* example – easy to see.

Cuts Between Modal Rules.

$$\frac{\dfrac{A_1, \dots, A_n \Longrightarrow A_0}{\Gamma, \Box A_1 \dots, \Box A_n \Longrightarrow \Box A_0, \Delta} \qquad \dfrac{A_0, B_1, \dots, B_k \Longrightarrow B_0}{\Sigma, \Box A_0, \Box B_1, \dots, \Box B_k \Longrightarrow \Box B_0, \Pi}}{\Gamma, \Sigma, \Box A_1, \dots \Box A_n, \Box B_1, \dots, \Box B_k \Longrightarrow \Box B_0, \Delta, \Pi}$$

- Can be permuted upwards

Example: The modal Logic $T = K + \Box A \to A$

Admissibility of Cuts between (K) and $(T) = \Box A \Longrightarrow A$

$$\dfrac{\dfrac{A \Longrightarrow B}{\Box A \Longrightarrow \Box B} \qquad \dfrac{}{\Box B \Longrightarrow B}}{\Box A \Longrightarrow B} \qquad \leadsto \qquad \dfrac{A \Longrightarrow B}{\Box A \Longrightarrow B}$$

Admissibility of Weakening and Inversion (e.g. $B = \neg B_0$)

$$\dfrac{A \Longrightarrow B}{\Box A \Longrightarrow B} \qquad \leadsto \qquad \dfrac{\Gamma, A \Longrightarrow \Delta}{\Gamma, \Box A \Longrightarrow \Delta}$$

Admissibility of Conteraction (e.g. $\Gamma = \Box A$)

$$\dfrac{\Gamma, A \Longrightarrow \Delta}{\Gamma, \Box A \Longrightarrow \Delta} \qquad \leadsto \qquad \dfrac{\Gamma, A, \Box A \Longrightarrow \Delta}{\Gamma, \Box A \Longrightarrow \Delta}$$

Admissibility of Structural Rules

Let $\mathsf{S}(\Gamma \Longrightarrow \Delta)$ be the closure of $\Gamma \Longrightarrow \Delta$ under structural rules and inversion:

$$\frac{\Gamma, A \wedge B, \Gamma \Longrightarrow \Delta}{\Gamma, A, B \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow A \wedge B, \Delta}{\Gamma \Longrightarrow A, \Delta} \qquad \frac{\Gamma \Longrightarrow A \wedge B, \Delta}{\Gamma \Longrightarrow B, \Delta} \qquad \frac{\Gamma, \neg B \Longrightarrow \Delta}{\Gamma \Longrightarrow B, \Delta} \qquad \frac{\Gamma \Longrightarrow \neg A, \Delta}{\Gamma, A \Longrightarrow \Delta}$$

$$\frac{\Gamma \Longrightarrow \Delta}{\Gamma, \Gamma' \Longrightarrow \Delta, \Delta'} \qquad \frac{\Gamma, A, A \Longrightarrow \Delta}{\Gamma, A \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow B, B, \Delta}{\Gamma \Longrightarrow B, \Delta}$$

Easy Theorem.

Let Rl be a set of rules, and suppose that Rl admits structural rules and inversion, i.e.

- for all $\Gamma_1 \Longrightarrow \Delta_1, \dots, \Gamma_n \Longrightarrow \Delta_n / \Gamma_0 \Longrightarrow \Delta_0 \in \mathsf{Rl}$, every $\Gamma \Longrightarrow \Delta \in \mathsf{S}(\Gamma_0 \Longrightarrow \Delta_0)$
- $\Gamma \Longrightarrow \Delta$ is derivable from $\bigcup\{\mathsf{S}(\Gamma_i \Longrightarrow \Delta_i) \mid i = 1, \dots, n\}$.

Then the structural and inversion rules are admissible in $\vdash_{\mathsf{Rl}}$.

Admissibility of Cut

Let $\mathrm{Cut}(R_1, R_2, A)$ be the least set of sequents that contains:

- cuts between premisses of R_1, R_2, and cuts between a premiss of R_i and a conclusion of R_{3-i}

and is closed under:

- cuts on formulae $< A$, the structural rules, propositional rules, and rules in Rl.

where R_1 and R_2 are rules with conclusion $\Gamma, A \Longrightarrow \Delta$ and $\Sigma \Longrightarrow A, \Pi$.

Local Cut Elimination Theorem.

If $\mathrm{Cut}(R_1, R_2, A)$ contains $\Gamma, \Sigma \Longrightarrow \Delta, \Pi$ for all choices of $R_1, R_2 \in \mathrm{Rl}$ and A as above, and Rl admits the structural rules, then (Cut) is admissible in $\vdash_{\mathrm{Rl}}$.

Proof. This is set up in such a way that Gentzen's double induction proof applies.