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Problem sheet 8
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The Zeta function and the Riemann Hypothesis
Solution

Problem 29

a) $f(x) = \frac{1}{a^2+x^2}$ has poles $\pm ia$. We integrate over $\gamma_R^t(s) = Re^{-i\mathrm{sign}(t)s}$, $s \in [0,\pi]$ and denote $g(z,t) = e^{-2\pi izt}f(z), z \in \mathbb{C}$ for $t \neq 0$. Note that

$$\begin{aligned}\mathrm{Res}_{-\mathrm{sign}(t)ia}\, g(z,t) &= \lim_{z\to-\mathrm{sign}(t)ia} \frac{e^{-2\pi izt}(z+\mathrm{sign}(t)ia)}{a^2+z^2} = \lim_{z\to-\mathrm{sign}(t)ia} \frac{e^{-2\pi izt}}{z-\mathrm{sign}(t)ia} \\ &= \frac{ie^{-2\pi|t|a}}{2\mathrm{sign}(t)a}.\end{aligned}$$

As usual

$$\hat{f}(t) = \lim_{R\to\infty}\left(\int_{[-R,R]\oplus\gamma_R^t} g(x,t)\,\mathrm{d}\,x - \int_{\gamma_R^t} g(x,t)\,\mathrm{d}\,x\right) = \mathrm{sign}(t)\frac{\pi e^{-2\pi|t|a}}{\mathrm{sign}(t)a} = \frac{\pi e^{-2\pi|t|a}}{a}$$

where we used the following lemma by Jordan. By continuity of the Fourier transform we also get the case $t = 0$.

Lemma (Jordan). If a function f converges uniformely to 0 for $|z| \to \infty$ in the upper half plane than $\int_{\gamma_R^{-1}} e^{i\alpha z}f(z)\,\mathrm{d}\,z \xrightarrow{R\to\infty} 0$.
Proof.

$$\begin{aligned}\left|\int_{\gamma_R} e^{i\alpha z}f(z)\,\mathrm{d}\,z\right| &= \left|\int_0^\pi e^{i\alpha Re^{i\varphi}}f(Re^{i\varphi})Rie^{i\varphi}\,\mathrm{d}\,\varphi\right| \leqslant R\sup_{x\in\gamma_R^{-1}}|f(x)|\int_0^\pi e^{-\alpha R\sin(\varphi)}\,\mathrm{d}\,\varphi \\ &\leqslant 2R\sup_{x\in\gamma_R^{-1}}|f(x)|\int_0^{\pi/2} e^{-\alpha R\sin(\varphi)}\,\mathrm{d}\,\varphi \leqslant 2R\sup_{x\in\gamma_R^{-1}}|f(x)|\int_0^{\pi/2} e^{-\alpha R\varphi\cdot 2/\pi}\,\mathrm{d}\,\varphi \\ &= 2R\sup_{x\in\gamma_R^{-1}}|f(x)|\cdot\frac{1-e^{-\alpha R}}{\alpha R\cdot 2/\pi} \to 0.\end{aligned}$$

b) Poisson's summation formula reads

$$\begin{aligned}\sum_{n\in\mathbb{Z}} f(n) &= \sum_{n\in\mathbb{Z}}\hat{f}(n) = \sum_{n\in\mathbb{Z}}\frac{\pi e^{-2\pi|n|a}}{a} = \frac{\pi}{a}\left(-1+2\sum_{n=0}^\infty e^{-2\pi na}\right) = \frac{\pi}{a}\left(-1+\frac{2}{1-e^{-2\pi a}}\right) \\ &= \frac{\pi}{a}\left(\frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}\right) = \frac{\pi}{a}\left(\frac{e^{\pi a}+e^{-\pi a}}{e^{\pi a}-e^{-\pi a}}\right) = \frac{\pi}{a}\coth(\pi a).\end{aligned}$$

Hence $\sum_{n=1}^\infty f(n) = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} f(n) - f(0)\right) = \frac{1}{2}\left(\frac{\pi}{a}\coth(\pi a) - \frac{1}{a^2}\right)$.

Problem 31

Let $n = \prod_{i=1}^{k} p_i^{e_i}$.

a) Note that the case $n = p_1^{e_1}$ is obvious by the definition of the Mangoldt function. By induction on the number of prime factors: $\sum_{d|n} \Lambda(d) = \sum_{i=0}^{e_{k+1}} \sum_{dp_{k+1}^{e_{k+1}}|n} \Lambda(dp_{k+1}^{i}) = \sum_{i=1}^{e_{k+1}} \Lambda(p_{k+1}^{i}) + \log(n/p_{k+1}^{e_{k+1}}) = e_{k+1}\log(p_{k+1}) + \log(n/p_{k+1}^{e_{k+1}}) = \log(p_{k+1}^{e_{k+1}}) + \log(n/p_{k+1}^{e_{k+1}}) = \log(n)$.

b) By part a):

$$\log(\lfloor x \rfloor!) = \sum_{n \leqslant x} \sum_{d|n} \Lambda(d) = \sum_{p \leqslant x} \log p \cdot \sharp\{(n,k) : p^k \leqslant x/n\} = \sum_{n \leqslant x} \sum_{d \leqslant x/n} \Lambda(d) = \sum_{n \leqslant x} \psi\left(\frac{x}{n}\right).$$

Since $\Lambda(d) \neq 0$ for $d = p^k$ and $p^k, 2p^k, \ldots, \lfloor x/p^k \rfloor p^k \leqslant x$.

c) Follows from part b), since $\sharp\{(n,k) : p^k \leqslant x/n\} = \sum_{k=1}^{\infty} \lfloor x/p^k \rfloor$.

e) We will use part c). First note that $\log(\lfloor x \rfloor!) = x\log x - x + O(\log(x))$, since

$$x\log(x) - x + 1 - \log(x) = \int_1^x \log u \,\mathrm{d}\, u - \log(x) \leqslant \int_1^{\lfloor x \rfloor} \log u \,\mathrm{d}\, u \leqslant \sum_{n=1}^{x} \log(n)$$
$$\leqslant \int_1^{\lfloor x \rfloor} \log u \,\mathrm{d}\, u + \log\lfloor x \rfloor \leqslant \int_1^{\lfloor x \rfloor} \log u \,\mathrm{d}\, u + \log x = x\log(x) - x + 1 + \log(x).$$

Next note that $\log(\lfloor x \rfloor!) = \sum_{p \leqslant x} \log p \sum_{k=1}^{\infty} \left\lfloor \frac{x}{p^k} \right\rfloor = \sum_{p \leqslant x} \log p \left\lfloor \frac{x}{p^k} \right\rfloor + \sum_{p \leqslant x} \log p \sum_{k=2}^{\infty} \left\lfloor \frac{x}{p^k} \right\rfloor$ and $\sum_{p \leqslant x} \log p \sum_{k=2}^{\infty} \left\lfloor \frac{x}{p^k} \right\rfloor \leqslant x \sum_{p \leqslant x} \frac{\log p}{p(p-1)} = O(x)$.
Moreover $\sum_{p \leqslant x} \log p \left(\frac{x}{p}\right) - \sum_{p \leqslant x} \log p \left\lfloor \frac{x}{p} \right\rfloor \leqslant \sum_{p \leqslant x} \log p = \vartheta(x) = O(x)$. Hence

$$\sum_{p \leqslant x} \log p \left(\frac{x}{p}\right) + O(x) = x\log x + O(x)$$
$$\Rightarrow \sum_{p \leqslant x} \frac{\log p}{p} = \log x + O(1).$$

d) Note that $\sum_{n \leqslant x} \frac{\Lambda(n)}{n} = \sum_{p \leqslant x} \frac{\log(p)}{p} + \sum_{\substack{p^k \leqslant x \\ k \geqslant 2}} \frac{\log(p)}{p^k}$ and $\sum_{\substack{p^k \leqslant x \\ k \geqslant 2}} \frac{\log(p)}{p^k} \leqslant \sum_{p \leqslant x} \frac{\log(p)}{p^2} \sum_{j=0}^{\infty} p^{-j} = \sum_{p \leqslant x} \frac{\log(p)}{p^2} \frac{p}{p-1} < \infty$.

Problem 32

a) From part a) of the previous problem we deduce $\frac{\log n}{n^s} = \sum_{m \cdot k = n} \frac{\Lambda(m)}{m^s} \cdot \frac{1}{k^s}$ and further

$$\sum_{n \leqslant x} \frac{\log n}{n^s} = \sum_{n \cdot k \leqslant x} \frac{\Lambda(n)}{n^s} \cdot \frac{1}{k^s} = \sum_{n \leqslant x} \frac{\Lambda(n)}{n^s} \sum_{k \leqslant x/n} \frac{1}{k^s}.$$

Let $s = 1 + it, t \neq 0$. By problem 8 part a) we know that

$$\sum_{k \leqslant x} \frac{1}{k^s} = \zeta(s) - \frac{1}{s-1} \cdot \frac{1}{x^{s-1}} + O\left(\frac{1}{x^s}\right)$$

and hence

$$\begin{aligned}\sum_{n\leqslant x}\frac{\log n}{n^s} &= \sum_{n\leqslant x}\frac{\Lambda(n)}{n^s}\left(\zeta(s)-\frac{1}{s-1}\cdot\frac{n^{s-1}}{x^{s-1}}\right)+O\left(\sum_{n\leqslant x}\frac{\Lambda(n)}{n^s}\cdot\frac{n^s}{x^s}\right)\\ &= \zeta(s)\sum_{n\leqslant x}\frac{\Lambda(n)}{n^s}-\left(\frac{1}{s-1}\cdot\frac{1}{x^{s-1}}\right)\sum_{n\leqslant x}\frac{\Lambda(n)}{n}+O\left(\frac{\psi(x)}{x^s}\right)\\ &= \zeta(s)\sum_{n\leqslant x}\frac{\Lambda(n)}{n^s}-\left(\frac{1}{s-1}\cdot\frac{1}{x^{s-1}}\right)\log(x)+O\left(1\right)\end{aligned}$$

where we used that $\psi(x)\sim x$ (by the prime number theorem). Next we use again problem 8, i.e.

$$\sum_{n\leqslant x}\frac{\log n}{n^s}=-\zeta'(s)-\frac{1}{(s-1)^2}\cdot\frac{1}{x^{s-1}}-\frac{1}{s-1}\cdot\frac{\log(x)}{x^{s-1}}+O\left(\frac{\log(x)}{x^s}\right)$$

and receive

$$\zeta(s)\sum_{n\leqslant x}\frac{\Lambda(n)}{n^s}=-\zeta'(s)-\frac{1}{(s-1)^2}\cdot\frac{1}{x^{s-1}}+O(1)=O(1).$$

But $\zeta(s)\neq 0$ for $\mathrm{Re}(s)=1$ and hence $\sum_{n\leqslant x}\frac{\Lambda(n)}{n^s}=O(1)$.

b) As before $\sum_{n\leqslant x}\frac{\Lambda(n)}{n^s}=\sum_{p\leqslant x}\frac{\log p}{p^s}+\sum_{k\geqslant 2}\left(\sum_{p^k\leqslant x}\frac{\log p}{p^{ks}}\right)$ and

$$\left|\sum_{k\geqslant 2}\left(\sum_{p^k\leqslant x}\frac{\log p}{p^{ks}}\right)\right|\leqslant\sum_{k\geqslant 2}\sum_{p^k\leqslant x}\frac{\log p}{p^k}<\infty.$$

Therefore

$$\sum_{p\leqslant x}\frac{\log p}{p^{1+it}}=\sum_{p\leqslant x}\frac{\Lambda(n)}{n^{1+it}}+O(1)=O(1).$$

c) Now choose $g(x):=\sum_{p\leqslant x}\frac{\log p}{p^{1+it}}$. We already know that $g(x)=O(1)$. Using Abel's Summation Theorem we get

$$\sum_{p\leqslant x}\frac{1}{p^{1+it}}=\sum_{p\leqslant x}\frac{\log p}{p^{1+it}\log(p)}=\frac{g(x)}{\log(x)}+\int_2^x\frac{g(u)}{u\log^2(u)}\,\mathrm{d}\,u$$

The integral $\int_2^\infty\frac{1}{u\log^2(u)}\,\mathrm{d}\,u$ converges absolutely and we may rewrite the expression as

$$\begin{aligned}\sum_{p\leqslant x}\frac{1}{p^{1+it}} &= \frac{g(x)}{\log(x)}+\int_2^\infty\frac{g(u)}{u\log^2(u)}\,\mathrm{d}\,u-\int_x^\infty\frac{g(u)}{u\log^2(u)}\,\mathrm{d}\,u\\ &= \int_2^\infty\frac{g(u)}{u\log^2(u)}\,\mathrm{d}\,u+O\left(\frac{1}{\log(x)}\right).\end{aligned}$$

Thus

$$\sum_{p\leqslant x}\frac{1}{p^{1+it}}=\sum_{p}\frac{1}{p^{1+it}}+O\left(\frac{1}{\log(x)}\right).$$