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Problem sheet 4
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The Zeta function and the Riemann Hypothesis
Solution

Problem 13

Let $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$. First note that w.l.o.g. we may assume $s_0 = 0$: $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^{s_0}} \frac{1}{n^{s-s_0}} = \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{n^{s_0}}$ with $\tilde{a}_n = \frac{a_n}{n^{s-s_0}}$. Then $\sum_{n=1}^{\infty} a_n$ converges and by possibly changing a_1, we may further assume that $A(x) = \sum_{n \leqslant x} a_n \to 0$.
By the Abel Summation Theorem

$$\sum_{n \leqslant x} \frac{a_n}{n^s} = \frac{A(x)}{x^s} + s \int_1^x \frac{A(u)}{u^{s+1}} \,\mathrm{d}\, s,$$

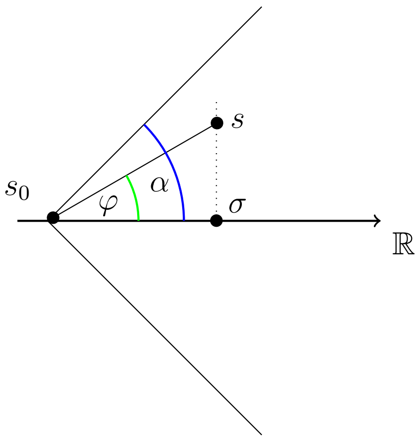
$$\sum_{x<n \leqslant y} \frac{a_n}{n^s} = \frac{A(y)}{y^s} - \frac{A(x)}{x^s} + s \int_x^y \frac{A(u)}{u^{s+1}} \,\mathrm{d}\, u,$$

$$\left| \sum_{x<n \leqslant y} \frac{a_n}{n^s} \right| \leqslant |A(y)| + |A(x)| + |s| \int_x^y \frac{|A(u)|}{u^{\sigma+1}} \,\mathrm{d}\, u,$$

for $s \in \mathrm{Ang}(0, \alpha)$. Let $\varepsilon > 0$ and choose $\varepsilon' := \varepsilon(2 + \cos(\alpha)^{-1})^{-1}$. If x_0 is sufficiently large, than $|A(z)| \leqslant \varepsilon'$ for all $z \geqslant x_0$. Hence for all $s \in \mathrm{Ang}(0, \alpha)$ and $x_0 \leqslant x < y$

$$\left| \sum_{x<n \leqslant y} \frac{a_n}{n^s} \right| \leqslant 2\varepsilon' + \begin{cases} \frac{|s|\varepsilon'}{\sigma x^{\sigma}}, & \text{falls } s \in \mathrm{Ang}(0, \alpha) \setminus \{0\} \\ 0, & \text{falls } s = 0. \end{cases}$$

But $x^{-dfvdfc\sigma} \leqslant 1$ and hence



$\cos(\alpha) \leqslant \cos(\varphi) = \sigma/|s|$ and $\left| \sum_{x<n \leqslant y} \frac{a_n}{n^s} \right| \leqslant \varepsilon$ for $s \in \mathrm{Ang}(0, \alpha)$.

Problem 14

a) Assume that there is a minimal $k \geqslant 1$ such that $a_k \neq 0$. Write $f(s) = \frac{1}{k^s}\left(a_k + \sum_{n>k} \frac{a_n}{(n/k)^s}\right)$. The series converges for s_0 absolutely. Denote $\sum_{n>k} \frac{|a_n|}{(n/k)^{\sigma_0}} = M < \infty$ and note that

$$\left|\sum_{n=k+1}^{\infty} \frac{a_n}{(n/k)^{s_0+r}}\right| \leqslant \sum_{n=k+1}^{\infty} \frac{|a_n|}{(n/k)^{\sigma_0}} \left(\frac{k}{k+1}\right)^{\mathrm{Re}(r)} \leqslant M \left(\frac{k}{k+1}\right)^{\mathrm{Re}(r)} \xrightarrow{\mathrm{Re}(r)\to\infty} 0.$$

Hence we find a σ^* such that $\left|\sum_{n=k+1}^{\infty} \frac{a_n}{(n/k)^s}\right| < |a_k|$ for all $\mathrm{Re}(s) \geqslant \sigma^*$. But then $f(s) \neq 0$ for $\mathrm{Re}(s) \geqslant \sigma^*$ in contradiction to our assumption.

b) We repeat the proof from above for $a_1 = 1$. Hence there is a $\sigma_0 > \sigma_a$ such that $\sum_{n=2}^{\infty} \frac{|a_n|}{n^\sigma} < \frac{1}{2}$ holds for all $\mathrm{Re}(s) = \sigma > \sigma_0$. Thus the logarithm exists and converges against $\log(1) = 0$. Furthermore note that $\left(\sum_{n=1}^{\infty} \frac{a_n}{n^s}\right)^k$ converges absolutely for $\mathrm{Re}\, s > \sigma_a(f)$ - the convolution of Dirichlet series. Hence we may interchange the limit in

$$\log(f(s)) = \sum_{k\geqslant 1}^{\infty} \frac{(-1)^k}{k} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^s}\right)^k$$

and get another Dirichlet series.
In the special of a multiplicate arithmetic function defining the Dirichlet series, we can use $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{1-a_p p^{-s}}$ and

$$\begin{aligned}\log(f(s)) = \log\left(\prod_p \frac{1}{1-a_p p^{-s}}\right) &= \sum_p \log\left(\frac{1}{1-a_p p^{-s}}\right) \\ = \sum_p \left(\sum_{k=1}^{\infty} \frac{a_p^k}{kp^{ks}}\right) &= \sum_{k=1}^{\infty} \left(\sum_p \frac{a_p^k}{kp^{ks}}\right),\end{aligned}$$

where we interchanged the summation of the absolutely convergent series. Finally note, that $p^k = q^m \Rightarrow p = q, k = m$ for arbitrary prime numbers p, q. Hence this is a Dirichlet series.

Problem 15

a) By Abel Summation for $A(x) = O(x^\alpha)$

$$\sum_{n\leqslant x} \frac{a_n}{n^{\alpha+\varepsilon}} = \frac{A(x)}{x^{\alpha+\varepsilon}} + \alpha \int_1^x \frac{A(u)}{u^{\alpha+\varepsilon+1}} \,\mathrm{d}\, u.$$

Thus $\sum_{n=1}^{\infty} \frac{a_n}{n^{\alpha+\varepsilon}}$ exists for all $\varepsilon > 0$. Hence $\inf\{\alpha \in \mathbb{R} : A(x) = O(x^\alpha)\} \geqslant \sigma_c(f)$.
On the other hand if $\sum_{n=1}^{\infty} \frac{a_n}{n^\alpha}$ exists, than $\left|\sum_{n\leqslant x} \frac{a_n}{n^\alpha}\right| < c$ for some c.

$$A(x) = \sum_{n\leqslant x} \frac{a_n n^\alpha}{n^\alpha} = x^\alpha \sum_{n\leqslant x} \frac{a_n}{n^\alpha} - \alpha \int_1^x u^{\alpha-1} \sum_{n\leqslant u} \frac{a_n}{n^\alpha} \,\mathrm{d}\, u.$$

$$|A(x)| \leqslant x^\alpha \cdot c + \alpha \int_1^x u^{\alpha-1} \cdot c \,\mathrm{d}\, u = O(x^\alpha).$$

b) By part a) it will be enough to define a sequence with $\alpha = \inf\{\beta \in \mathbb{R} : A(x) = O(x^\beta)\}$. Choose the a_k rekursively by the following rule: $a_k = 1$ if $A(k-1) < k^\alpha$ and $a_k = -1$ if $A(k-1) \geqslant k^\alpha$. Then we even have $\lim_{x\to\infty} \frac{A(x)}{x^\alpha} = 1$ and $\frac{A(x)}{x^{\alpha-\varepsilon}}$ unbounded. Obviously $\sigma_a(f) = 1$.

Problem 16

a) We have

$$\sum_{k\geqslant 1}\frac{\mu(k)}{k}\sum_{m\geqslant 1}\frac{1}{m}\pi(x^{1/km})=\sum_{s=1}^{\infty}\frac{\pi(x^{1/s})}{s}\underbrace{\sum_{k|s}\mu(k)}_{\delta_{1s}}=\sum_{s=1}^{\infty}\frac{\pi(x^{1/s})}{s}\delta_{1s}=\pi(x),$$

where we used that $\sum_{k|s}\mu(k)=\delta_{1s}$: $s=1$ is obvious and for $s=\prod_{i=1}^{r}p_i^{e_i}\geqslant 2$ and $s'=\prod_{i=1}^{r}p_i$ (pairwise disjoint factors):

$$\sum_{k|s}\mu(k)=\sum_{k|s'}\mu(k)=\sum_{\alpha\in\{0,1\}^r}(-1)^{\sum_{i=1}^{r}\alpha_i}=0.$$

b) We use $\log\zeta(s)=\sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^s}$ with $\Lambda_1(n)=\begin{cases}\frac{1}{m}, & \text{if } n=p^m, p \text{ prime}\\ 0, & \text{else.}\end{cases}$. We first prove, that $\pi_1(x)=\sum_{n\leqslant x}\Lambda_1(x)$ by induction: It will be enough to check $x\in\mathbb{N}$. The case $x=1$ is clear. Now for $x\to x+1$ note that we have two cases:

$$x=p^m\Rightarrow \pi_1(x+1)-\pi_1(x)=\frac{1}{m}=\Lambda_1(x+1)$$

Note that in this case $\lfloor (x+1)^{1/k}\rfloor-\lfloor x^{1/k}\rfloor\neq 0$ if $k|m$ but $p^{m/k}$ is prime only for $k=m$. Furthermore

$$x\neq p^m\Rightarrow \pi_1(x+1)-\pi_1(x)=0=\Lambda_1(x+1)$$

In this case $\lfloor (x+1)^{1/k}\rfloor-\lfloor x^{1/k}\rfloor\neq 0$ implies $x+1=r^k$ for a natural number r, that is not a prime power.
We use Abel's Summation Theorem for $\operatorname{Re}s\geqslant 1$

$$\begin{aligned}\sum_{n\leqslant x}\frac{\Lambda_1(n)}{n^s}&=\frac{\sum_{n\leqslant x}\Lambda_1(n)}{x^s}+s\int_1^x\frac{\sum_{n\leqslant u}\Lambda_1(n)}{u^{s+1}}\,\mathrm{d}\,u\\&=\frac{\pi_1(x)}{x^s}+s\int_1^x\frac{\pi_1(u)}{u^{s+1}}\,\mathrm{d}\,u.\end{aligned}$$

In the limit $x\to\infty$ the first summand vanished, since $\pi_1(u)=\pi(x)+\frac{1}{2}\pi(\sqrt{x})+...=O(x\log(x))$ for $\pi(x)<x$.
Note that $\pi(x^{1/k})=0$ for $x^{1/k}<2\Leftrightarrow k>\frac{\log(x)}{\log(2)}$. Hence we immediately get

$$\sum_{k=1}^{\lfloor\log(x)/\log(2)\rfloor}\frac{1}{k}\pi(x^{1/k})\leqslant\sum_{k=1}^{\lfloor\log(x)/\log(2)\rfloor}\frac{x^{1/k}}{k}\leqslant\frac{\log(x)x}{\log(2)}=O(x^{1+\varepsilon})$$

for every $\varepsilon>0$. Finally we receive $\log\zeta(s)=s\int_1^{\infty}\frac{\pi_1(u)}{u^{s+1}}\,\mathrm{d}\,u$ for $\operatorname{Re}(s)>1$.

Remark. Moreover observe that for $x\geqslant 2^m$

$$\sum_{k=1}^{\lfloor\log(x)/\log(2)\rfloor}\frac{x^{\frac{1}{k}-1}}{k}-\sum_{k=1}^{\lfloor\log(2x)/\log(2)\rfloor}\frac{(2x)^{\frac{1}{k}-1}}{k}\geqslant\frac{1}{2x^{1/2}}-\frac{1}{2(2x)^{1/2}}-\frac{(2x)^{-\frac{\lfloor\log(x)/\log(2)\rfloor}{\lfloor\log(x)/\log(2)\rfloor+1}}}{1+\left\lfloor\frac{\log(x)}{\log(2)}\right\rfloor}$$

$$\geqslant \frac{1}{2x^{1/2}} - \frac{1}{2(2x)^{1/2}} - \frac{1}{(1+m)(2x)^{1/2}} > 0$$

for m big enough. Hence $\sum_{k=1}^{\lfloor \log(x)/\log(2) \rfloor} \frac{x^{\frac{1}{k}-1}}{k}$ is uniformely bounded in x and therefore $\pi_1(x) = O(x)$. Analogously we can show that $\sum_{k=2}^{\lfloor \log(x)/\log(2) \rfloor} \frac{x^{\frac{1}{k}}}{k} = O(\sqrt{x})$ and $\pi_1(x) = \pi(x) + O(\sqrt{x})$.