Ludwig-Maximilians-
Universität München
Dr. Pascal Reisert

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Problem sheet 2
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The Zeta function and the Riemann Hypothesis
Solution

Problem 5

a) By Abel's Summation Theorem:

$$\begin{aligned}
\sum_{n=0}^{k} a_n r^n e^{itn} &= r^k \sum_{n=0}^{k} a_n e^{itn} - \int_0^k \sum_{n=0}^{u} a_n \log(r) r^u e^{itn} \,\mathrm{d}\, u \\
&= r^k \sum_{n=0}^{k} a_n e^{itn} - \sum_{0\leqslant j\leqslant k-1} \int_j^{j+1} \left(\sum_{n=0}^{j} a_n e^{int} \right) \cdot \log(r) r^u \,\mathrm{d}\, u \\
&= r^k \sum_{n=0}^{k} a_n e^{itn} - \sum_{0\leqslant j\leqslant k-1} \sum_{n=0}^{j} a_n e^{int} \int_j^{j+1} \cdot \log(r) r^u \,\mathrm{d}\, u \\
&= r^k \sum_{n=0}^{k} a_n e^{itn} - \sum_{0\leqslant j\leqslant k-1} \sum_{n=0}^{j} a_n e^{int} \left(r^{j+1} - r^j \right) \\
&= r^k \sum_{n=0}^{k} a_n e^{itn} + (1-r) \sum_{0\leqslant j\leqslant k-1} r^j \sum_{n=0}^{j} a_n e^{int}
\end{aligned}$$

We get $\sum_{n=0}^{\infty} a_n r^n e^{itn} = (1-r) \sum_{j=0}^{\infty} r^j \sum_{n=0}^{j} a_n e^{int}$ and hence using the explicit term for the geometric series

$$\sum_{n=0}^{\infty} a_n r^n e^{itn} - \sum_{n=0}^{\infty} a_n e^{itn} = (1-r) \sum_{j=0}^{\infty} r^j \left(\sum_{n=0}^{j} a_n e^{int} - \sum_{n=0}^{\infty} a_n e^{itn} \right).$$

Denote $s_j = \sum_{n=0}^{j} a_n e^{int}$. Now choose $\varepsilon > 0$ and $m \geqslant 1$ large enough such that $|s_j - s_\infty| < \frac{\varepsilon}{2}$ for all $j \geqslant m$. We get

$$\begin{aligned}
\left| \sum_{n=0}^{\infty} a_n r^n e^{itn} - \sum_{n=0}^{\infty} a_n e^{itn} \right| &\leqslant (1-r) \sum_{j=0}^{m-1} r^j |s_j - s_\infty| + (1-r) \sum_{j=m}^{\infty} r^j |s_j - s_\infty| \\
&\leqslant (1-r) \sum_{j=0}^{m-1} r^j |s_j - s_\infty| + (1-r) \sum_{j=m}^{\infty} r^j \cdot \frac{\varepsilon}{2} \\
&= (1-r) \sum_{j=0}^{m-1} r^j |s_j - s_\infty| + (1-r) \cdot \frac{\varepsilon}{2} \cdot \frac{r^m}{1-r} \\
&\leqslant (1-r) \sum_{j=0}^{m-1} r^j |s_j - s_\infty| + \frac{\varepsilon}{2}
\end{aligned}$$

Now for $1-r < \frac{\varepsilon}{2} \cdot \left(\sum_{j=0}^{m-1} r^j \left|s_j - s_\infty\right|\right)^{-1}$ we get $|\sum_{n=0}^{\infty} a_n r^n e^{itn} - \sum_{n=0}^{\infty} a_n e^{itn}| < \varepsilon$ and therefore convergence.

b) Let $a_n = 1$ for alle n, then $f(z) = \frac{1}{1-z}$ inside the unit disc. For $z \to -1$ we get $f(z) \to \frac{1}{2}$ but $\sum_{n=0}^{k} e^{i\pi n}$ jumps between 0 and 1.

Remark.

1.) There are series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which all radial limits $\lim_{r\to 1} f(re^{it})$ exist but the $f(e^{it_0})$ is not convergent for at least one t_0.[1]

2.) There are analytic functions on the (open) unit disk that are nowhere analyticly continuable, but their power series converges uniformly (e.g. $\sum_{n=1}^{\infty} z^{n!} n^{-2}$).

3.) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in the unit disk and assume that $\lim_{|z|\to 1} |f(z)| = \infty$ (everywhere). Then the zeros of f have to accumulation point on the closed unit disk - hence there are only finitely many of them. We find a polynomial p such that f/p is a holomorphic function without zeros, and hence $g = p/f$ is a holomorphic function with $\lim_{|z|\to 1} |g| = 0$. By the maximum modulus principle, we get $g = 0$ in contradiction to our assumptions on f. Thus there is a sequence $z_k \to e^{it_0}$ such that $\{|f(z_k)| : k \in \mathbb{N}\}$ is bounded. Then $f(z_k)$ has a convergent subsequence.

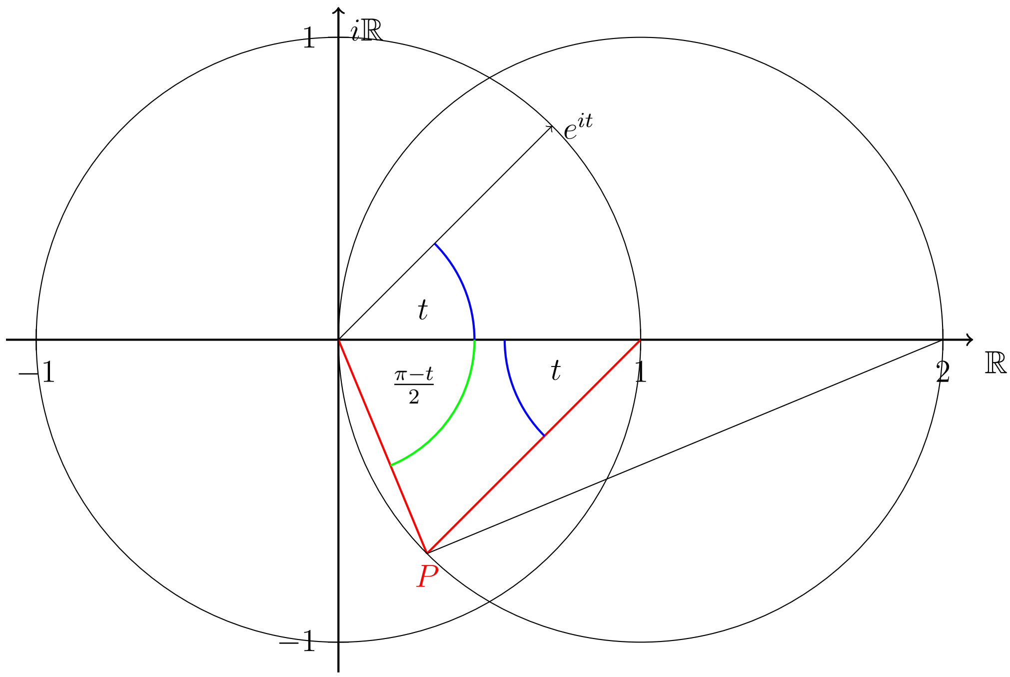
Problem 6

Fix $t \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ and choose δ small enough such that $t \bmod 2\pi \in [\delta, 2\pi - \delta]$. Problem 3 tells us that $S(k,t) = \sum_{n=1}^{k} \frac{e^{int}}{n}$ converges for $k \to \infty$. The series converges inside the unit disk .[2] We conclude

$$S(\infty, t) = -\lim_{r\to 1} \log\left(1 - re^{it}\right) = -\log\left|1 - e^{it}\right| - i \arg\left(1 - e^{it}\right) = -\log\left|1 - e^{it}\right| + i \cdot \frac{\pi - t}{2}$$

[1] Paul Du Bois-Reymond: Eine neue Theorie der Konvergenz und Divergenz von Reihen mit positiven Gliedern in *Journal für die reine und angewandte Mathematik / Zeitschriftenband (1873) / Artikel / S. 63 - 91.*

[2] $\sum_{n=1}^{k} \frac{r^n e^{int}}{n} = -\log\left(1 - re^{it}\right)$.



Let $P = 1-e^{it}$. We used that $-e^{it} = e^{i(\pi+t)}$ and the angle $\angle((0,0),(0,1),P) = 2\angle((0,0),(0,2),P)$. Thus $\angle(P,(0,0),(0,1)) = \pi - \frac{\pi}{2} - \frac{t}{2} = \frac{\pi-t}{2}$.

Problem 7

First note that $\zeta(s,a) = \zeta(s,a+1) + a^{-s}$. Hence it will be enough to consider $a < 1$ ($\zeta(s,1) = \zeta(s)$). For the zeta function we know that

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \sum_{k=1}^{r} \frac{B_{2k}}{2k} \cdot \binom{s+2k-2}{2k-1} - \binom{s+2r-1}{2r} \int_1^\infty \frac{\tilde{B}_{2r}}{x^{s+2r}} \,\mathrm{d}\, x.$$

There is a similar formula for the Hurwitz zeta function for $\mathrm{Re}(s) > 1-2r$

$$\begin{aligned}\zeta(s,a) =& \frac{1}{a^s} + (1+a)^{-s}\left(\frac{1}{2} + \frac{1+a}{s-1}\right) + \sum_{k=1}^{r} \frac{B_{2k}}{2k(1+a)^{s+2k-1}} \cdot \binom{s+2k-2}{2k-1} \\ &- \binom{s+2r-1}{2r} \int_1^\infty \frac{\tilde{B}_{2r}}{(x+a)^{s+2r}} \,\mathrm{d}\, x.\end{aligned}$$

To prove the formula and therewith the meromorphic continuation we will use the Euler-Maclaurin theorem:

Theorem (Euler-Maclaurin). Let x_0 be a real number and $f : [x_0, \infty[\to \mathbb{C}$ a $2r$-times continuously differentiable function. Then we have for all integers $n \geqslant m \geqslant x_0$ and all $r \geqslant 1$:

$$\begin{aligned}\sum_{k=m}^{n} f(k) =& \frac{1}{2}(f(m)+f(n)) + \int_m^n f(x) \,\mathrm{d}\, x \\ &+ \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n) - f^{(2k-1)}(m)\right) - \int_m^n \frac{\tilde{B}_{2r}(x)}{(2r)!} f^{(2r)}(x) \,\mathrm{d}\, x\end{aligned}$$

Recall that $\tilde{B}_{2r}(x) = B_{2r}(x - \lfloor x \rfloor)$, $B_n(x) = n!\beta_n(x)$, $B_n = B_n(0)$ and $\beta_{2k}(x) = (-1)^{k-1} \cdot 2 \cdot \sum_{n=1}^\infty \frac{\cos(2\pi n x)}{(2\pi n)^{2k}}$ and $\beta_{2k+1}(x) = (-1)^{k-1} \cdot 2 \cdot \sum_{n=1}^\infty \frac{\sin(2\pi n x)}{(2\pi n)^{2k+1}}$ for $k \geqslant 1$.

Let $f(k) = (k+a)^{-s}$, then

$$\sum_{k=1}^{n} f(k) = \frac{1}{2}((1+a)^{-s} + (n+a)^{-s}) + \left(\frac{1}{(1-s)(n+a)^{s-1}} - \frac{1}{(1-s)(1+a)^{s-1}}\right)$$
$$- \sum_{k=1}^{r} \binom{s+2k-2}{2k-1} \frac{B_{2k}}{2k} \left(\frac{1}{(n+a)^{s+2k-1}} - \frac{1}{(1+a)^{s+2k-1}}\right)$$
$$- \binom{s+2r-1}{2r} \int_1^n \frac{\tilde{B}_{2r}(x)}{(x+a)^{s+2r}} \,\mathrm{d}\,x$$

and in the limit $n \to \infty$ we get the claimed formula for the Hurwitz zeta function. Finally observe that the last integral defines a holomorphic function for $\mathrm{Re}(s) > 1 - 2r$ since the function $\tilde{B}_{2r}(x)$ is bounded and

$$\left|\frac{1}{(x+a)^{s+2r}}\right| \leqslant \frac{1}{(x+a)^{1+\delta}}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geqslant 1 - 2r + \delta$.

Problem 8

a) We use $\{t\} := t - \lfloor t \rfloor$ respectively $\lfloor t \rfloor = t - \{t\}$. By Abel's Summation:

$$\sum_{n \leqslant x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{\lfloor t \rfloor}{t^{s+1}} \,\mathrm{d}\,t$$

with

$$\int_1^x \frac{\lfloor t \rfloor}{t^{s+1}} \,\mathrm{d}\,t = \int_1^x \frac{t}{t^{s+1}} \,\mathrm{d}\,t - \int_1^\infty \frac{\{t\}}{t^{s+1}} \,\mathrm{d}\,t + \int_x^\infty \frac{\{t\}}{t^{s+1}} \,\mathrm{d}\,t$$
$$= \frac{1}{1-s}\left(\frac{1}{x^{s-1}} - 1\right) - \int_1^\infty \frac{\{t\}}{t^{s+1}} \,\mathrm{d}\,t + O(x^{-s}).$$

Hence

$$\sum_{n \leqslant x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^{s-1}} + \frac{s}{1-s}\left(\frac{1}{x^s} - 1\right) - s\int_1^\infty \frac{\{t\}}{t^{s+1}} \,\mathrm{d}\,t + O(x^{-s})$$
$$= x^{1-s} + O(x^{-s}) + \frac{s}{1-s}\left(\frac{1}{x^{s-1}} - 1\right) - s\int_1^\infty \frac{\{t\}}{t^{s+1}} \,\mathrm{d}\,t + O(x^{-s})$$
$$= \frac{1-s+s}{1-s} \cdot x^{1-s} + \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}} \,\mathrm{d}\,t + O(x^{-s})$$
$$= \frac{1}{1-s} \cdot x^{1-s} + \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}} \,\mathrm{d}\,t + O(x^{-s})$$

For $\mathrm{Re}(s) > 1$ both sides converge in the limit $x \to \infty$:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}} \,\mathrm{d}\,t.$$

However the right-hand side is holomorphic in all points of $\mathrm{Re}(s) > 0$ apart from $s = 1$. Hence the equality holds even for $\mathrm{Re}(s) > 0, s \neq 1$. We get

$$\begin{aligned}\sum_{n\leqslant x}\frac{1}{n^s} - \zeta(s) &= \frac{1}{1-s}\cdot x^{1-s} + O(x^{-s}) \\ \Rightarrow \zeta(s) &= \sum_{n\leqslant x}\frac{1}{n^s} + \frac{1}{s-1}\cdot x^{1-s} + O(x^{-s}).\end{aligned}$$

b) We repeat the strategy from part a):

$$\begin{aligned}\sum_{n\leqslant x}\frac{\log(n)}{n^s} &= \frac{\lfloor x\rfloor \log(x)}{x^s} - \int_1^x \lfloor t\rfloor \cdot \frac{\mathrm{d}(t^{-s}\log(t))}{\mathrm{d}\,t}\,\mathrm{d}\,t \\ &= \frac{\lfloor x\rfloor \log(x)}{x^s} - \int_1^x \lfloor t\rfloor \cdot t^{-s-1}\,\mathrm{d}\,t + \int_1^x \lfloor t\rfloor \cdot s \cdot t^{-s-1}\cdot \log(t)\,\mathrm{d}\,t \\ &= \frac{(x - \{x\}\log(x)}{x^s} + \int_1^x (s\log(t)-1)t^{-s}\,\mathrm{d}\,t + \int_1^x (1 - s\log(t))\frac{\{t\}}{t^{s+1}}\,\mathrm{d}\,t.\end{aligned}$$

Furthermore observe that using integration by parts we get

$$\begin{aligned}\int_1^x (s\log(t)-1)t^{-s}\,\mathrm{d}\,t &= \left[(s\log(t)-1)\cdot\frac{t^{1-s}}{1-s}\right]_1^x - \frac{s}{1-s}\int_1^x t^{-s}\,\mathrm{d}\,t \\ &= \frac{1}{1-s} + (s\log(x)-1)\frac{x^{1-s}}{1-s} + \frac{s}{(1-s)^2}(1-x^{1-s}) \\ &= \frac{1-s+s}{(1-s)^2} + s\log(x)\cdot\frac{x^{1-s}}{1-s} - \frac{s+1-s}{(1-s)^2}\cdot x^{1-s}\end{aligned}$$

Adding up we receive

$$\begin{aligned}\sum_{n\leqslant x}\frac{\log(n)}{n^s} =& \log(x)x^{1-s} + \frac{1}{(1-s)^2} + s\log(x)\cdot\frac{x^{1-s}}{1-s} \\ & - \frac{1}{(1-s)^2}\cdot x^{1-s} + O(\log(x)x^{-s}) + \int_1^x (1-s\log(t))\frac{\{t\}}{t^{s+1}}\,\mathrm{d}\,t. \\ =& \frac{1}{(1-s)^2} + \log(x)\cdot\frac{x^{1-s}}{1-s} - \frac{1}{(1-s)^2}\cdot x^{1-s} \\ & + O(\log(x)x^{-s}) + \int_1^x (1-s\log(t))\frac{\{t\}}{t^{s+1}}\,\mathrm{d}\,t.\end{aligned}$$

Take the limit for $\mathrm{Re}(s) > 1$ to get

$$-\zeta(s)' = \frac{1}{(1-s)^2} + \int_1^\infty (1-s\log(t))\frac{\{t\}}{t^{s+1}}\,\mathrm{d}\,t.$$

The right-hand side exists for $s \neq 1$ and $\mathrm{Re}(s) > 0$ and hence the equality holds for $\mathrm{Re}(s) > 0, s \neq 1$. We get

$$\zeta(s)' = -\sum_{n\leqslant x}\frac{\log(n)}{n^s} + \log(x)\cdot\frac{x^{1-s}}{1-s} - \frac{1}{(1-s)^2}\cdot x^{1-s} + O(\log(x)x^{-s})$$