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Problem sheet 1
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The Zeta function and the Riemann Hypothesis
Solution

Problem 1

a) Let $p_1, \ldots, p_k$ be the prime numbers $\leqslant x$. Note that $\prod_{p\leqslant x}\left(1-\frac{1}{p}\right) = \prod_{j=1}^{k}\left(1-\frac{1}{p_j}\right)$ Now consider the following equation

$$\log\left(\prod_{j=1}^{k}\left(1-\frac{1}{p_j}\right)\right) = \sum_{j=1}^{k}\log\left(1-\frac{1}{p_j}\right) = -\sum_{j=1}^{k}\sum_{i=1}^{\infty}\frac{1}{ip_j^i} < -\sum_{j=1}^{k}\frac{1}{p_j} \xrightarrow{k\to\infty} -\infty.$$

Instead of the power series expansion of the logarithm used above, we could equally use $\log(x) \leqslant x-1$.[1]
Hence

$$\lim_{k\to\infty}\prod_{j=1}^{k}\left(1-\frac{1}{p_j}\right) = \lim_{k\to\infty}\exp\left(\log\left(\prod_{j=1}^{k}\left(1-\frac{1}{p_j}\right)\right)\right) \to 0$$

b) W.l.o.g. it is enough to prove $\prod_{j=1}^{k}\left(1-\frac{1}{p_j}\right) = \sum_{n|\prod_{j=1}^{k}p_j}\frac{\mu(n)}{n}$. The case $k=1$ is trivial. Assume that the claim holds for all $i \leqslant k$.

Induction $k \mapsto k+1$: We have

$$\sum_{n|\prod_{j=1}^{k+1}p_j}\frac{\mu(n)}{n} = \sum_{n|\prod_{j=1}^{k}p_j}\frac{\mu(n)}{n} + \sum_{n|\prod_{j=1}^{k}p_j}\frac{\mu(np_{k+1})}{np_{k+1}} = \left(\sum_{n|\prod_{j=1}^{k}p_j}\frac{\mu(n)}{n}\right)\left(1-\frac{1}{p_{k+1}}\right)$$
$$= \prod_{j=1}^{k}\left(1-\frac{1}{p_j}\right)$$

For the first equation we used, that $n|\prod_{j=1}^{k+1}p_j$ implies either $n|\prod_{j=1}^{k}p_j$ or $(n|\prod_{j=1}^{k+1}p_j \wedge p_{k+1}|n)$.

Problem 2

By direct calculation we receive

$$\gamma = \lim_{k\to\infty}\left(\sum_{n=1}^{k}\frac{1}{n} - \log(k)\right)$$

[1] $\log(1) = 0 = 1-1$ and $\partial log(x)/\partial x = x^{-1} \lesseqqgtr 1 = \partial x/\partial x$ for $x \gtreqqless 1$.

$$
\begin{aligned}
&= \lim_{k\to\infty}\left(\sum_{n=1}^{k}\frac{1}{n}\underbrace{-\log(k)+\log(k-1)}_{\log\left(\frac{k-1}{k}\right)}-\log(k-1)+\ldots-\log(2)+\underbrace{\log(1)}_{=0}\right)\\
&= 1+\sum_{n=2}^{\infty}\left(\frac{1}{n}+\log\left(\frac{n-1}{n}\right)\right) = 1+\sum_{n=2}^{\infty}\left(\frac{1}{n}+\log\left(1-\frac{1}{n}\right)\right)\\
&= 1+\sum_{n=2}^{\infty}\left(\frac{1}{n}-\sum_{j=1}^{\infty}\frac{n^{-j}}{j}\right) = 1+\sum_{n=2}^{\infty}\left(-\sum_{j=2}^{\infty}\frac{n^{-j}}{j}\right) = 1+\sum_{l=2}^{\infty}\left(-\sum_{n=2}^{\infty}\frac{n^{-j}}{j}\right)\\
&= 1+\sum_{j=2}^{\infty}\left(\frac{1}{j}-\sum_{n=1}^{\infty}\frac{n^{-j}}{j}\right) = 1-\sum_{j=2}^{\infty}\left(\frac{\zeta(j)-1}{j}\right)
\end{aligned}
$$

Since the series representing the zeta function converges absolutely for $\mathrm{Re}(s)>1$, interchanging the order of summation is justified.

Problem 3

a) Choose $0<\delta<\pi$. Hence for $x\in\mathbb{R}_+, k=\lfloor x\rfloor$ and $t\in[\delta,2\pi-\delta]$.

$$
\begin{aligned}
|S(x,t)| &= \left|\sum_{1\leqslant n\leqslant x} e^{int}\right| = \left|\frac{e^{i(k+1)t}-1}{e^{it}-1}-1\right| = \left|\frac{e^{i(k+1)t}-1-e^{it}+1}{e^{it}-1}\right| = \left|\frac{e^{it}(e^{ikt}-1)}{e^{it}-1}\right|\\
&= \frac{|e^{ikt}-1|}{|e^{it}-1|} \leqslant \frac{2}{|e^{it/2}-e^{-it/2}|} = |\sin(t/2)|^{-1} \leqslant \sin(\delta/2)^{-1}
\end{aligned}
$$

since $t/2\in[\delta/2,\pi-\delta/2]$ and $\inf_{[\delta/2,\pi-\delta/2]}\sin(y)=\sin(\delta/2)$.

b) Let $a_n=e^{int}$ and $f_t:[1,\infty[\to\mathbb{C}, x\mapsto x^{-1}$. Set $S(x,t):=\sum_{1\leqslant n\leqslant x}a_n$ as before. By Abel's summation formula

$$
\begin{aligned}
\sum_{n=1}^{k}a_n f_t(n) = \sum_{n=1}^{k}\frac{e^{int}}{n} &= S(k,t)f(k)-\int_1^k S(u,t)f'(u)\ \mathrm{d}\,u\\
&= S(k,t)\cdot\frac{1}{k}+\int_1^k\left(\sum_{1\leqslant n\leqslant u}e^{int}\right)\cdot\frac{1}{u^2}\,\mathrm{d}\,u\\
&= S(k,t)\cdot\frac{1}{k}+\sum_{1\leqslant j\leqslant k-1}\int_j^{j+1}\left(\sum_{n=1}^{j}e^{int}\right)\cdot\frac{1}{u^2}\,\mathrm{d}\,u\\
&= S(k,t)\cdot\frac{1}{k}-\sum_{1\leqslant j\leqslant k-1}\left[\left(\sum_{n=1}^{j}e^{int}\right)\cdot\frac{1}{u}\right]_j^{j+1}\\
&= S(k,t)\cdot\frac{1}{k}-\sum_{1\leqslant j\leqslant k-1}\left(\sum_{n=1}^{j}e^{int}\right)\left(\frac{1}{j+1}-\frac{1}{j}\right)\\
&= S(k,t)\cdot\frac{1}{k}+\sum_{1\leqslant j\leqslant k-1}S(j,t)\left(\frac{1}{(j+1)j}\right)
\end{aligned}
$$

The last term obviously converges uniformely since $S(k,t)$ is uniformely bounded.[2]

[2] Recall $\sum_{n\geqslant 1}n^{-2}=\pi/6$.

Problem 4

a) Let $s = 1 + it, t \neq 0$ and recall that $\log\left(1 + \frac{m}{k}\right) = \log(k+m) - \log(k) < \frac{m}{k}$ for all $k, m \in \mathbb{N}_+$ (cf. problem 1). Let $k \leqslant n \leqslant k+m$, then

$$\begin{aligned}
\left|e^{-it\log n} - e^{-it\log k}\right| &= \left|e^{-it\left(\log n - \frac{\log n + \log k}{2}\right)} - e^{-it\left(\log k - \frac{\log n+\log k}{2}\right)}\right| \\
&= \left|e^{it\left(\frac{\log k - \log n}{2}\right)} - e^{-it\left(\frac{\log k - \log n}{2}\right)}\right| = 2\left|\sin\left(t\cdot\frac{\log k - \log n}{2}\right)\right| \\
&= 2\left|\sin\left(t\cdot\frac{\log(n-k+k) - \log k}{2}\right)\right| < 2\sin\left(\frac{|t|m}{2k}\right) < \frac{|t|m}{k}.
\end{aligned}$$

for $|t|\cdot\frac{m}{k} < \pi$. We conclude

$$\begin{aligned}
\left|\sum_{n=1}^{k+m}\frac{1}{n^{1+it}} - \sum_{n=1}^{k}\frac{1}{n^{1+it}}\right| &= \left|\sum_{n=k+1}^{k+m}\frac{e^{-it\log(n)}}{n}\right| = \left|\sum_{n=k+1}^{k+m}\frac{e^{-it\log(k)} + e^{-it\log n} - e^{-it\log k}}{n}\right| \\
&\geqslant \left|\sum_{n=k+1}^{k+m}\frac{e^{-it\log(k)}}{n}\right| - \left|\sum_{n=k+1}^{k+m}\frac{e^{-it\log n} - e^{-it\log k}}{n}\right| \\
&\geqslant \sum_{n=k+1}^{k+m}\frac{1}{n}\left(1 - \frac{|t|m}{k}\right) \geqslant \frac{m}{k+m}\cdot\frac{1}{2} \geqslant \frac{\frac{k}{2|t|} - 1}{k + \frac{k}{2|t|}}\cdot\frac{1}{2} \\
&= \frac{\frac{1}{2|t|} - k^{-1}}{2 + \frac{1}{|t|}} \geqslant \frac{\frac{1}{4|t|}}{\frac{8|t|+4}{4|t|}} \geqslant \frac{1}{8|t|+4}
\end{aligned}$$

where we used $\frac{k}{2|t|} \geqslant m \geqslant \frac{k}{2|t|} - 1$ and $k > 4|t|$. Observe that $|t|\cdot\frac{m}{k} \leqslant \left(\frac{k}{2|t|} + 1\right)\frac{|t|}{k} = \frac{1}{2} + \frac{1}{4} < \frac{\pi}{2}$.
The series is not Cauchy and hence does not converge.

b) Postponed.