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Summer Term 2016
Problem sheet 8
June 8th, 2016

Riemann Surfaces
Solution

Problem 32

a) Every element in $Y(t_1, t_2)$ can be written as $z = x + iy$ with $x \in \mathbb{R}$, $y \in]t_1, t_2[$. Then $\Phi(z) = e^{2\pi iz} = e^{-2\pi y}e^{2\pi ix}$ and $e^{-2\pi t_2} < |\Phi(z)| = e^{-2\pi y} < e^{-2\pi t_1}$ by monotony of the real exponential function. Hence $\Phi(z) \in A(e^{-2\pi t_2}, e^{-2\pi t_1})$. On the other hand every element in $A(e^{-2\pi t_2}, e^{-2\pi t_1})$ can be written as $a = re^{2\pi ix}$ for $r \in] - e^{-2\pi t_2}, -e^{-2\pi t_1}[$, $x \in \mathbb{R}$. By surjectivity of the exponential function we find a $y \in [t_1, t_2]$ such that $e^{2\pi t_1} < r^{-1} = e^{2\pi y} < e^{2\pi t_2}$. By construction we get $\Phi(x + iy) = a$.
For every $u \in U(t_1, t_2)$ choose a $w \in Y(t_1, t_2)$ such that $u = p(w)$ and define $\varphi(u) = \Phi(w)$. Observe that for $w, w' \in Y(t_1, t_2)$ with $p(w) = p(w') = u$ we get $w - w' \in \mathbb{Z}$: If $w - w' = a + b\tau$, then $\operatorname{Im}(w) - \operatorname{Im}(w') = b\operatorname{Im}(\tau) \Rightarrow \operatorname{Im}(\tau) \geqslant t_2 - t_1 > \operatorname{Im}(w) - \operatorname{Im}(w') = b\operatorname{Im}(\tau) \Rightarrow b = 0$. But then $e^{2\pi iw} = e^{2\pi iw'}$. Thus φ is well-defined. It is surjective since p and Φ are surjective. Injectivity follows by construction, i.e. $\varphi(u) = \varphi(u') \Rightarrow \Phi(w) = \Phi(w') \Rightarrow \operatorname{Im}(w) = \operatorname{Im}(w'), \operatorname{Re}(w) - \operatorname{Re}(w') \in \mathbb{Z} \Rightarrow u = p(w) = p(w') = u'$. Since p is locally biholomorphic, φ is everywhere locally holomorphic and therefore holomorphic everywhere.

b) U_1 and U_2 identify via Φ with annuli and we get by Problem 31 that $H^1(U_1, \mathcal{O}) = H^1(U_2, \mathcal{O}) = 0$. Hence $\mathfrak{U} = (U_1, U_2)$ is a Leray cover. We show that $U_1 \cap U_2$ has two connected components $W_0 = p(Y(0, T/2))$, $W_1 = p(Y(-T/2, 0)) = p(Y(T/2, T))$.[1] W_0 and W_1 are open as image of open sets under the holomorphic non-constant (hence open) map p. W_0 and W_1 are disjoint, since $w \in W_0 \cap W_1$ implies there are $z \in Y(0, T/2), z' \in Y(T/2, T)$ with $p(z) = p(z') \Rightarrow z - z' \in \Lambda \Rightarrow \operatorname{Im}(z) - \operatorname{Im}(z') = b \cdot T$. But there can be no such b, since $\operatorname{Im}(z) - \operatorname{Im}(z') < T$ and $\operatorname{Im}(z) \neq \operatorname{Im}(z')$. We still have to show that $U_1 \cap U_2 = W_0 \dot\cup W_1$. By construction we already know that $U_1 \cap U_2 \supset W_0 \dot\cup W_1$. If $u \in U_1 \cap U_2$, then there are $z \in Y(0, T), z' \in Y(-T/2, T/2)$ with $p(z) = p(z')$. If $z \in Y(0, T/2)$ we are done. If $z \notin Y(0, T/2)$ then $\operatorname{Im}(z) \geqslant T/2$. But $\operatorname{Im}(z) = T/2$ implies $\operatorname{Im}(z') = T/2 + T \cdot \mathbb{Z}$ in contradiction to $z' \in Y(-T/2, T/2)$. Hence $\operatorname{Im}(z) > T/2$ and $z \in Y(T/2, T) \Rightarrow p(z) \in W_1$.

c) By Riemann-Roch the result follows easily, since $H^0(X, \mathcal{O}) \simeq \mathbb{C}$ on a compact Riemann surfaces we get for a genus 1 surface and $D = 0$: $\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg(D) \Rightarrow \dim H^1(X, \mathcal{O}_D) = 1$.
Instead of using the Riemman-Roch theorem we can for example use the charts of part a). Consider $U_1 \simeq A(e^{-2\pi T}, 1)$, $U_2 \simeq A(e^{-\pi T}, e^{\pi T})$, $W_0 \simeq A(e^{-\pi T}, 1)$ and $W_1 \simeq A(1, e^{\pi T})$. Then a holomorphic function f on $W_0 \dot\cup W_1$ looks like $f|_{W_0} = \sum_{n=-\infty}^{\infty} c_n z^n, f|_{W_1} = \sum_{n=-\infty}^{\infty} d_n z^n$. A holomorphic function on U_1 may be written as $g = \sum_{n=-\infty}^{\infty} a_n z^n$. For a

[1] $f : Y(-T/2, 0) \to Y(-T/2, 0), z \mapsto z + \tau$ is a bijection and $p \circ f = p$.

holomorphic function h on U_2 we get $h|_{W_0} = \sum_{n=-\infty}^{\infty} b_n z^n$. Since $\Phi(z-\tau) = \Phi(z) \cdot e^{-2\pi i \tau}$ we must have $h|_{W_1} = \sum_{n=-\infty}^{\infty} b_n (e^{-2\pi i \tau} z)^n = \sum_{n=-\infty}^{\infty} b_n e^{-2n\pi i \tau} z^n$.[2]This Hence

$$\delta(g,h)|_{W_0} = g - h = \sum_{n=-\infty}^{\infty} (a_n - b_n) z^n, \delta(g,h)|_{W_1} = g - h = \sum_{n=-\infty}^{\infty} (a_n - b_n e^{-2n\pi i \tau}) z^n$$

If $c_0 = d_0$, then $a_n - b_n = c_n, a_n - b_n e^{-2n\pi i \tau} = d_n$ has a solution[3]

$$b_n = (1 - e^{-2n\pi i \tau})^{-1} (d_n - c_n), a_n = c_n - b_n \text{ for } n \geqslant 1, a_0 + b_0 = c_0.$$

If $c_0 \neq d_0$ there is no solution. Hence $\delta(C^0(\mathfrak{U}, \mathcal{O})) = \{(f_1, f_2) \in \mathcal{O}(W_0) \times \mathcal{O}(W_1) :$ f_1, f_2 have the same zero coefficient in their Laurent expansion$\}$. We get $H^1(X, \mathcal{O}) \simeq \mathbb{C}$ where every class is represented by a constant $d_0 - c_0$ and f_0 with $f_0|_{W_0} = 0, f_0|_{W_1} = 1$ represents a basis.

[2] $z \mapsto e^{-2\pi i \tau} z$ is the transition function from one chart to the other.

[3] Note that $n\tau \in \mathbb{Z} \Leftrightarrow n = 0$.