

Bachelor's Thesis

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# Symmetry Reduction in the Newtonian Three-Body Problem

Symmetriereduktion im Newton'schen Dreikörperproblem

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# 1 Introduction

The Newtonian three-body problem is one of the classical models of celestial mechanics. It describes the motion of three point masses moving under their mutual gravitational attraction and gives rise to a rich variety of dynamical behaviours, ranging from simple periodic motions to chaotic trajectories. The Newtonian three-body problem has played a central role in the development of modern dynamical systems theory. Finding a general solution to this problem is extremely difficult.

A key difficulty lies in the fact that the configuration space contains degrees of freedom that do not affect the physical interaction between the bodies. Translations and rotations change only the position or orientation of the system, but not the relative arrangement of the masses. Removing these redundant degrees of freedom leads to a reduced configuration space in which each point represents a geometric configuration of the three bodies. This reduced space, known as *shape space*, provides a far clearer view of the qualitative features of the problem.

The aim of this thesis is to explain the construction of shape space and to show how it can be used to study the planar three-body problem. Beginning with the full physical formulation, we examine the symmetries of the system and apply Noether's theorem to identify the conserved quantities associated with them. These observations allow us to restrict the motion to a plane and to perform a step-by-step symmetry reduction of the configuration space. The resulting coordinates reveal a natural geometric structure, in particular the *shape sphere*, which serves as a compact representation of all oriented similarity classes of planar triangles.

The shape sphere provides a powerful representation of the dynamics. Important configurations such as equilateral triangles, collinear states, and binary collisions appear as distinguished points on the sphere. Periodic and non-periodic solutions can likewise be identified as trajectories in this space. This geometric point of view allows us to encode certain three-body motions by symbolic sequences and offers a simple and visual framework for understanding the qualitative behaviour of solutions.

The purpose of this thesis is therefore twofold: to present a clear derivation of shape space, and to illustrate how this reduced geometry can be used to analyse the dynamics of the three-body problem.

## 2 Physical Foundations of the Three-Body Problem

Before carrying out the symmetry reduction that leads to shape space, we begin by formulating the problem in its full physical setting. In this chapter we introduce the Lagrangian description of the system, identify its fundamental symmetries, and apply Noether's theorem to obtain the associated conserved quantities. In particular, the conservation of angular momentum implies that any solution with zero total angular momentum evolves in a fixed plane, a fact that justifies the planar formulation used in the subsequent chapters.

### 2.1 Lagrangian Formulation

In classical mechanics, the evolution of a system is governed by a function

$$\mathcal{L} : TQ \longrightarrow \mathbb{R},$$

called the *Lagrangian*. Here  $Q$  denotes the configuration space and  $TQ$  its tangent bundle; an element of  $TQ$  is a pair  $(x, \dot{x})$ , where  $x \in Q$  represents the positions of the bodies and  $\dot{x} \in T_x Q$  their velocities.

For the Newtonian three-body problem, the configuration space is

$$Q = (\mathbb{R}^3)^3 \cong \mathbb{R}^9,$$

where each body is represented by its position vector in space.

Let  $x_i(t) \in \mathbb{R}^3$  denote the position of the  $i$ -th mass  $m_i$  with  $i = 1, 2, 3$  and  $r_{ij} = \|x_i - x_j\|$  the mutual distances. The Lagrangian takes the classical form

$$\mathcal{L}(x, \dot{x}) = K(\dot{x}) - U(x),$$

with kinetic energy

$$K(\dot{x}) = \frac{1}{2} \sum_{i=1}^3 m_i \|\dot{x}_i\|^2,$$

and potential energy

$$U = - \left( \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right),$$

The *action functional* associated to a trajectory  $x(t)$  is defined by

$$\mathcal{I}[x] = \int_{t_0}^{t_1} \mathcal{L}(x(t), \dot{x}(t)) dt.$$

In the Lagrangian formulation of mechanics, the physical trajectories are those that satisfy the Euler–Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) - \frac{\partial \mathcal{L}}{\partial x_i} = 0, \quad i = 1, \dots, 9.$$

These equations encode the full dynamics of the system.

## 2.2 Translational and Rotational Invariance of the Equations of Motion

A fundamental feature of the Newtonian three-body problem is that the dynamics do not depend on where the system is located in space nor on how it is oriented. This invariance under translations and rotations is built directly into the structure of the Lagrangian.

The potential

$$U(x_1, x_2, x_3) = - \left( \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right)$$

depends only on the relative distances  $r_{ij} = \|x_i - x_j\|$ , since if we translate all positions by the same vector  $a \in \mathbb{R}^3$ ,

$$x_i \mapsto x'_i = x_i + a,$$

we obtain

$$r'_{ij} = \|x'_i - x'_j\| = \|(x_i + a) - (x_j + a)\| = r_{ij},$$

and the velocities satisfy  $\dot{x}'_i = \dot{x}_i$ . Thus both the kinetic and potential energies remain unchanged, and hence the Lagrangian is invariant under translations:

$$\mathcal{L}(x_1 + a, x_2 + a, x_3 + a, \dot{x}) = \mathcal{L}(x, \dot{x}).$$

The same reasoning applies to rotations. For any  $R \in \text{SO}(3) := \{A \in \text{GL}(3) : A^\top A = AA^\top = I, \det A = 1\}$ , the transformation

$$x_i \mapsto x'_i = R x_i$$

preserves Euclidean norms:

$$\|R(x_i - x_j)\| = \|x_i - x_j\|, \quad \|R\dot{x}_i\| = \|\dot{x}_i\|.$$

Consequently,

$$r'_{ij} = r_{ij}, \quad K' = K, \quad U' = U,$$

and therefore

$$\mathcal{L}(Rx, R\dot{x}) = \mathcal{L}(x, \dot{x}).$$

Hence the Lagrangian, and therefore the Euler–Lagrange equations, are invariant under the full group of rigid motions of space  $\text{SE}(3) = \mathbb{R}^3 \rtimes \text{SO}(3)$ .

These symmetries will later be factored out in order to obtain the desired reduced configuration space.

## 2.3 Noether's Theorem

One of the central ideas in theoretical physics is that continuous symmetries of a system are always linked to conserved quantities. This connection is made precise by Noether's theorem, which explains how a symmetry of the Lagrangian leads to a quantity that remains constant along every solution of the equations of motion.

Before turning to Noether's theorem, we introduce the notions of a continuous symmetry of the Lagrangian and of a conserved quantity, since these are the key ingredients in the theorem.

**Definition 2.1** (Continuous symmetry). *Let*

$$(x, \dot{x}) \longmapsto (x + \delta x, \dot{x} + \delta \dot{x})$$

*be an infinitesimal transformation of the coordinates and velocities. The corresponding variation of the Lagrangian is defined by*

$$\delta \mathcal{L} := \mathcal{L}(x + \delta x, \dot{x} + \delta \dot{x}, t) - \mathcal{L}(x, \dot{x}, t).$$

*The transformation is called a continuous symmetry of the system if the variation of the Lagrangian is a total time derivative, that is, if there exists a smooth function  $F(x, \dot{x}, t)$  such that*

$$\delta \mathcal{L} = \frac{dF}{dt}.$$

**Definition 2.2** (Conserved quantity). *Let  $L$  be a Lagrangian, and let  $x(t)$  be any solution of the associated Euler–Lagrange equations. A smooth function*

$$Q: TQ \times \mathbb{R} \longrightarrow \mathbb{R}$$

is called a conserved quantity if

$$\frac{d}{dt}Q(x(t), \dot{x}(t), t) = 0 \quad \text{for all solutions } x(t).$$

With these notions in place, we can now state Noether's theorem.

**Theorem 2.3** (Noether). *Let  $x \mapsto x + \delta x$  be a continuous symmetry of the Lagrangian. Then the quantity*

$$Q = \sum_i p_i \delta x_i - F, \quad p_i = \frac{\partial L}{\partial \dot{x}_i},$$

*is conserved along all solutions of the Euler–Lagrange equations.*

Noether's theorem was first proved by Emmy Noether in 1918 in her work *Invariante Variationsprobleme* [1]. Since then, it has played a central role in modern theoretical physics and analytical mechanics. The proof presented below follows the modern exposition given in [2].

*Proof.* We first compute the variation of the Lagrangian using the chain rule:

$$\delta \mathcal{L} = \sum_{i=1}^n \left( \frac{\partial \mathcal{L}}{\partial x_i} \delta x_i + \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \delta \dot{x}_i \right).$$

Introducing the generalized momenta  $p_i := \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$ , this becomes

$$\delta \mathcal{L} = \sum_{i=1}^n \left( \frac{\partial \mathcal{L}}{\partial x_i} \delta x_i + p_i \delta \dot{x}_i \right).$$

If we differentiate  $p_i$  with respect to time, we obtain

$$\dot{p}_i = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right).$$

According to the Euler-Lagrange equations, we have

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) = \frac{\partial \mathcal{L}}{\partial x_i}.$$

Thus along any physical trajectory,

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial x_i}.$$

Substituting this identity into the expression for  $\delta \mathcal{L}$  yields

$$\delta \mathcal{L} = \sum_{i=1}^n (\dot{p}_i \delta x_i + p_i \delta \dot{x}_i).$$

Since the expression inside the sum is the time derivative of  $p_i \delta x_i$  (by the product rule), we obtain

$$\delta \mathcal{L} = \sum_{i=1}^n \left( \frac{d}{dt} p_i \delta x_i \right) = \frac{d}{dt} \left( \sum_{i=1}^n p_i \delta x_i \right).$$

By hypothesis, the variation satisfies

$$\delta \mathcal{L} = \frac{dF}{dt}.$$

Equating both expressions gives

$$\frac{d}{dt} \left( \sum_{i=1}^n p_i \delta x_i - F \right) = 0.$$

Thus the quantity

$$Q = \sum_{i=1}^n p_i \delta x_i - F$$

has vanishing time derivative and is therefore conserved.  $\square$

Noether's theorem shows that each symmetry of the Lagrangian corresponds to a specific conserved quantity.

## 2.4 Linear and Angular Momentum in the Three-Body Problem

The symmetries of the Lagrangian considered in this thesis are exact, that is, the variation of the Lagrangian vanishes,

$$\delta \mathcal{L} = 0.$$

Hence the function  $F$  appearing in Noether's theorem satisfies  $\frac{dF}{dt} = 0$ , so  $F$  is constant and can be chosen to be zero without loss of generality. In this case, the conserved quantity reduces to

$$Q = \sum_i p_i \delta x_i.$$

**Translational invariance:** For an infinitesimal translation by a constant vector  $\varepsilon$ , the symmetry is given by

$$x_i \longmapsto x_i + \varepsilon, \quad \delta x_i = \varepsilon.$$

Substituting this variation into the expression for  $Q$  yields

$$Q = \sum_i p_i \varepsilon.$$

Since  $\varepsilon$  is a constant vector, translational invariance of the Lagrangian leads to the conservation of

$$P := \sum_i p_i,$$

which is precisely the total linear momentum of the system.

For the Newtonian three-body problem,  $p_i$  satisfies

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i.$$

Hence the conserved quantity  $P$  coincides with the classical definition of the total linear momentum,

$$P = \sum_{i=1}^3 m_i \dot{x}_i.$$

**Rotational invariance:** An infinitesimal rotation about a unit vector  $n$  by an angle  $\delta\theta$  acts on the position vectors as

$$x_i \longmapsto x_i + \delta x_i, \quad \delta x_i = \delta\theta n \times x_i.$$

Substituting this variation into the Noether quantity gives

$$Q = \sum_i p_i \cdot (\delta\theta n \times x_i).$$

Using the cyclic property of the scalar triple product, we may rewrite this as

$$Q = \sum_i \delta\theta n \cdot (x_i \times p_i) = \delta\theta n \cdot \sum_i (x_i \times p_i).$$

This implies the conservation of the vector

$$L := \sum_i x_i \times p_i.$$

Since  $p_i = m_i \dot{x}_i$ , we obtain the conserved quantity for three masses

$$L = \sum_{i=1}^3 x_i \times (m_i \dot{x}_i),$$

which is the classical definition of the total angular momentum.

Since both the total linear momentum  $P$  and the total angular momentum  $L$  are conserved, their values remain constant along any solution of the equations of motion. For

simplicity and without loss of generality, we restrict attention to the case

$$P = 0 \quad \text{and} \quad L = 0,$$

which will be assumed throughout the remainder of this chapter.

## 2.5 Reduction to the Planar Problem

The conservation of total angular momentum has a direct geometric consequence for the motion of the three bodies. If the total angular momentum  $L$  vanishes, then the positions and velocities of the bodies at any time span a two-dimensional subspace of  $\mathbb{R}^3$ . Consequently, the motion remains confined to a fixed plane for all time.

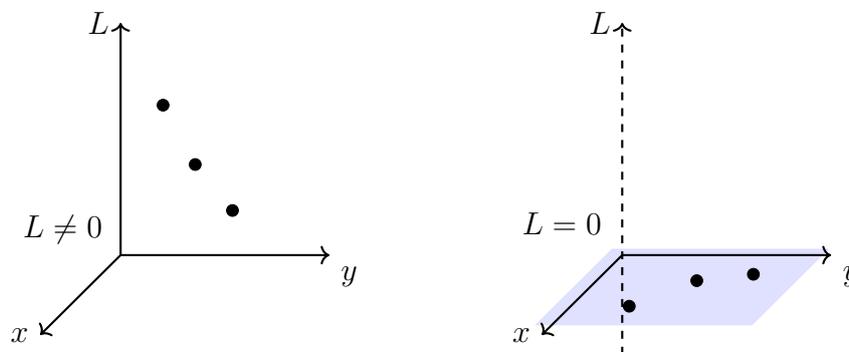


Figure 2.1: Visualization of the planar reduction induced by vanishing total angular momentum. If  $L \neq 0$ , the motion may be three-dimensional. If  $L = 0$ , all configurations remain confined to the horizontal plane  $L = 0$ .

Source: Created by me

Saari explains this planar reduction through a geometric decomposition of the velocity of a configuration. The velocity can be written as the sum of four independent components: translation, rotation, scaling, and shape change. Among these, rotation is the only component that can move the configuration out of its plane. If the total angular momentum vanishes, the rotational part of the velocity is identically zero. Consequently, the configuration plane does not rotate, and the motion remains confined to a fixed two-dimensional subspace of  $\mathbb{R}^3$  for all times.

For a detailed proof of the velocity decomposition, see [3], in particular pages 51–56.

## 3 Symmetry Reduction

This chapter deals with the symmetry reduction of the configuration space of the three-body problem. We start with the configuration space  $Q = \mathbb{R}^6$ , since each of the three bodies has two coordinates in the plane. As explained in the previous chapter, the three-body problem is invariant under translations and rotations. These symmetries correspond to degrees of freedom that are redundant for the problem itself, since they do not affect the mutual dynamics of the three bodies. It is therefore natural to divide out these components and pass to a reduced configuration space, the so-called shape space.

The two relevant symmetries for this reduction are the rigid motions of the plane, which form the special Euclidean group

$$\text{SE}(2) = \mathbb{R}^2 \rtimes S^1.$$

Its action on the configuration space translates and rotates the entire system without affecting the mutual geometry of the three bodies. Our goal in this chapter is to remove these symmetries explicitly by performing the quotient by  $\text{SE}(2)$  in two successive stages. First, we eliminate translations. In Jacobi coordinates the configuration naturally separates into a translation-invariant part and a center-of-mass coordinate. The last one represents the pure translational degree of freedom, which we will set to zero. Next, we eliminate rotations. Using Hopf coordinates, the system again splits into rotationally invariant quantities and a single angular parameter encoding the overall orientation, which we likewise fix.

Together, these two reductions yield a three-dimensional space that records only the intrinsic geometry of the configuration, the shape space.

### 3.1 Reduction by Translation

We begin the symmetry reduction by removing the freedom of translating the entire configuration in the plane. A simultaneous shift of all three bodies changes only their absolute location, not their mutual distances or the geometry of the triangle they form. Thus translations contribute no meaningful information to the dynamics and can be factored out. In this section we describe the translation group, its action on the configuration space, and how passing to Jacobi coordinates isolates the translation-invariant part from the center-of-mass coordinate.

### 3.1.1 Translation Symmetry

Planar translations form an abelian group, which we denote by

$$\mathbb{R}^2 := (\mathbb{R}^2, +).$$

Its elements are vectors  $t \in \mathbb{R}^2$ , and the group operation is ordinary vector addition. The configuration space is

$$Q = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^6,$$

where each  $x_i \in \mathbb{R}^2$  represents the position of the  $i$ -th body in the plane. The translation group acts on the configuration space  $\mathbb{R}^2 \curvearrowright Q$  via the map

$$\alpha : \mathbb{R}^2 \times Q \longrightarrow Q, \quad (t, (x_1, x_2, x_3)) \longmapsto (t + x_1, t + x_2, t + x_3).$$

Below, I include a short proof written by myself showing that  $\alpha$  is indeed a well-defined group action.

**Lemma 3.1** (The translation action is a group action). *Let  $\alpha : \mathbb{R}^2 \times Q \longrightarrow Q$  be the map defined above. Then  $\alpha$  is a left group action of the translation group  $\mathbb{R}^2$  on the configuration space  $Q$ .*

*Proof.* Let  $0 \in \mathbb{R}^2$  be the neutral element and  $(x_1, x_2, x_3) \in Q$ , with  $x_i \in \mathbb{R}^2$  for  $i = 1, 2, 3$ . Then

$$\alpha(0, (x_1, x_2, x_3)) = (0 + x_1, 0 + x_2, 0 + x_3) = (x_1, x_2, x_3),$$

so the identity axiom is satisfied.

For the compatibility axiom, let  $t_1, t_2 \in \mathbb{R}^2$ . Then

$$\begin{aligned} \alpha(t_1 + t_2, (x_1, x_2, x_3)) &= (t_1 + t_2 + x_1, t_1 + t_2 + x_2, t_1 + t_2 + x_3) \\ &= \alpha(t_1, (t_2 + x_1, t_2 + x_2, t_2 + x_3)) = \alpha(t_1, \alpha(t_2, (x_1, x_2, x_3))), \end{aligned}$$

which proves the claim. □

**Remark 3.1.** *The verification for a right group action is completely analogous.*

Moreover, the translation action is free, i.e. the only translation that fixes a configuration is the trivial one. If  $\alpha(t, (x_1, x_2, x_3)) = (x_1, x_2, x_3)$ , then the equalities  $t + x_i = x_i$  for  $i = 1, 2, 3$  imply  $t = 0$ . This property will be important later.

### 3.1.2 Jacobi Coordinates

Taking this algebraic framework into account, we now describe the system in such a way that we can separate, on the one hand, the coordinates that encode the relative positions

of the bodies and, on the other hand, the coordinate that describes the overall position of the system in the plane. To achieve this, we introduce the *Jacobi coordinates* [4], defined as

$$\xi_1 = x_2 - x_1, \quad \xi_2 = x_3 - \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

The first coordinate  $\xi_1$  is the vector from the first body  $x_1$  to the second  $x_2$ . The second Jacobi coordinate  $\xi_2$  is the vector from the center of mass of the first two bodies  $\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$  to the third body  $x_3$ .

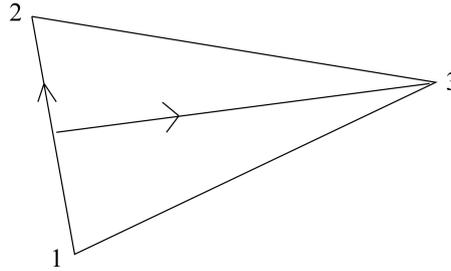


Figure 3.1: Geometric interpretation of the Jacobi coordinates  
Source: Montgomery, *Infinitely Many Syzygies* (2002).

The Jacobi coordinates describe the configuration of the three-body system in terms of *relative positions*, completely independent of its absolute location in the plane. In contrast, the *absolute position* of the system in the plane is encoded by the center of mass of all three bodies, given by

$$c = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}.$$

Thus, we obtain a map

$$J : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2,$$

$$(x_1, x_2, x_3) \longmapsto (\xi_1, \xi_2, c),$$

which separates the configuration into the translation-invariant part  $(\xi_1, \xi_2)$  and the overall position  $c$  of the system.

**Lemma 3.2.** *The Jacobi coordinates  $\xi_1, \xi_2$  defined above are invariant under translations. That is, for any  $a \in \mathbb{R}^2$ , if  $x'_i = x_i + a$ ,  $i = 1, 2, 3$ , then the transformed coordinates*

$$\xi'_1 = x'_2 - x'_1 \quad \text{and} \quad \xi'_2 = x'_3 - \frac{m_1 x'_1 + m_2 x'_2}{m_1 + m_2}$$

*satisfy  $\xi'_1 = \xi_1$  and  $\xi'_2 = \xi_2$ .*

In the following, I present the proof written by myself.

*Proof.* For the first coordinate we compute

$$\xi'_1 = (x_2 + a) - (x_1 + a) = x_2 - x_1 = \xi_1.$$

Hence,  $\xi_1$  is invariant under translations.

For the second coordinate we compute

$$\xi'_2 = (x_3 + a) - \frac{m_1(x_1 + a) + m_2(x_2 + a)}{m_1 + m_2}.$$

Expanding the numerator, we obtain

$$\xi'_2 = (x_3 + a) - \frac{m_1x_1 + m_2x_2 + (m_1 + m_2)a}{m_1 + m_2}.$$

Writing both terms over the common denominator gives

$$\xi'_2 = \frac{(m_1 + m_2)(x_3 + a) - (m_1x_1 + m_2x_2 + (m_1 + m_2)a)}{m_1 + m_2}.$$

After canceling the terms involving  $a$ , we obtain

$$\xi'_2 = \frac{(m_1 + m_2)x_3 - (m_1x_1 + m_2x_2)}{m_1 + m_2} = x_3 - \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \xi_2.$$

Hence both  $\xi'_1$  and  $\xi'_2$  remain unchanged under translations, which proves the claim.  $\square$

### 3.1.3 Moment of Inertia

Having established the translation invariance of the Jacobi map, we now examine how these coordinates affect the moment of inertia. Recall that for three point masses  $m_1, m_2, m_3$  with planar positions  $x_1, x_2, x_3 \in \mathbb{R}^2$ , the moment of inertia is

$$I = m_1|x_1|^2 + m_2|x_2|^2 + m_3|x_3|^2.$$

Following Montgomery [4], since the Jacobi coordinates are invariant under translations, it is natural to express  $I$  in terms of  $\xi_1$  and  $\xi_2$ . Remarkably, this change of variables diagonalizes the moment of inertia, as the following lemma shows.

**Lemma 3.3** (Moment of inertia in Jacobi coordinates). *Let  $\xi_1, \xi_2$  be the Jacobi coordinates. If the center-of-mass condition  $\sum_i m_i x_i = 0$  holds, then the moment of inertia takes the form*

$$I = \mu_1 |\xi_1|^2 + \mu_2 |\xi_2|^2,$$

where the coefficients  $\mu_i$  satisfy

$$\frac{1}{\mu_1} = \frac{1}{m_1} + \frac{1}{m_2}, \quad \frac{1}{\mu_2} = \frac{1}{m_3} + \frac{1}{m_1 + m_2}.$$

**Remark 3.2.** A brief calculation, carried out by myself, yields the explicit formula

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}, \quad \mu_2 = \frac{(m_1 + m_2) m_3}{m_1 + m_2 + m_3}.$$

Next, I present the proof written by myself.

*Proof.* Let the system satisfy the center-of-mass condition  $\sum_i m_i x_i = 0$ . We now aim to express the coordinates  $x_i$  through the Jacobi variables, removing any dependence on the original coordinates.

Starting from the definition of  $\xi_2$  we write

$$x_3 = \xi_2 + \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

Multiplying by  $(m_1 + m_2)$  and using the center-of-mass condition  $m_1 x_1 + m_2 x_2 = -m_3 x_3$ , we obtain

$$(m_1 + m_2) x_3 = (m_1 + m_2) \xi_2 - m_3 x_3.$$

Rearranging gives

$$(m_1 + m_2 + m_3) x_3 = (m_1 + m_2) \xi_2,$$

and hence

$$x_3 = \frac{m_1 + m_2}{m_1 + m_2 + m_3} \xi_2.$$

Next, we determine  $x_1$  and  $x_2$  in terms of  $\xi_1$  and  $\xi_2$ .

Since  $\xi_1 = x_2 - x_1$ , we have  $x_2 = x_1 + \xi_1$ . Substituting this into the definition of  $\xi_2$  gives

$$\xi_2 = x_3 - \frac{m_1 x_1 + m_2 (x_1 + \xi_1)}{m_1 + m_2} = x_3 - \frac{(m_1 + m_2) x_1 + m_2 \xi_1}{m_1 + m_2} = x_3 - x_1 - \frac{m_2}{m_1 + m_2} \xi_1.$$

Hence,

$$x_1 = x_3 - \xi_2 - \frac{m_2}{m_1 + m_2} \xi_1.$$

Using the previously obtained expression for  $x_3$ , we find

$$x_1 = \frac{m_1 + m_2}{m_1 + m_2 + m_3} \xi_2 - \xi_2 - \frac{m_2}{m_1 + m_2} \xi_1.$$

After simplification, this yields

$$x_1 = -\frac{m_3}{m_1 + m_2 + m_3} \xi_2 - \frac{m_2}{m_1 + m_2} \xi_1.$$

Finally, using  $x_2 = x_1 + \xi_1$ , we obtain

$$x_2 = -\frac{m_3}{m_1 + m_2 + m_3} \xi_2 - \frac{m_2}{m_1 + m_2} \xi_1 + \xi_1.$$

By expanding and simplifying, we find

$$x_2 = -\frac{m_3}{m_1 + m_2 + m_3} \xi_2 + \frac{m_1}{m_1 + m_2} \xi_1.$$

We now substitute the expressions for  $x_i$  into the formula for the moment of inertia, using

$$\begin{aligned} x_1 &= -\frac{m_3}{m_1 + m_2 + m_3} \xi_2 - \frac{m_2}{m_1 + m_2} \xi_1, \\ x_2 &= -\frac{m_3}{m_1 + m_2 + m_3} \xi_2 + \frac{m_1}{m_1 + m_2} \xi_1, \\ x_3 &= \frac{m_1 + m_2}{m_1 + m_2 + m_3} \xi_2. \end{aligned}$$

The moment of inertia is then given by

$$\begin{aligned} I &= m_1 \left| -\frac{m_3}{m_1 + m_2 + m_3} \xi_2 - \frac{m_2}{m_1 + m_2} \xi_1 \right|^2 + m_2 \left| -\frac{m_3}{m_1 + m_2 + m_3} \xi_2 + \frac{m_1}{m_1 + m_2} \xi_1 \right|^2 \\ &\quad + m_3 \left| \frac{m_1 + m_2}{m_1 + m_2 + m_3} \xi_2 \right|^2. \end{aligned}$$

Expanding and reorganizing gives

$$\begin{aligned} I &= m_1 \left[ \left( -\frac{m_3}{m_1 + m_2 + m_3} \right)^2 |\xi_2|^2 - 2 \left( -\frac{m_3}{m_1 + m_2 + m_3} \right) \left( \frac{m_2}{m_1 + m_2} \right) \langle \xi_2, \xi_1 \rangle \right. \\ &\quad \left. + \left( \frac{m_2}{m_1 + m_2} \right)^2 |\xi_1|^2 \right] \\ &\quad + m_2 \left[ \left( -\frac{m_3}{m_1 + m_2 + m_3} \right)^2 |\xi_2|^2 + 2 \left( -\frac{m_3}{m_1 + m_2 + m_3} \right) \left( \frac{m_1}{m_1 + m_2} \right) \langle \xi_2, \xi_1 \rangle \right. \\ &\quad \left. + \left( \frac{m_1}{m_1 + m_2} \right)^2 |\xi_1|^2 \right] + m_3 \left[ \left( \frac{m_1 + m_2}{m_1 + m_2 + m_3} \right)^2 |\xi_2|^2 \right]. \end{aligned}$$

Thus, we obtain

$$\begin{aligned}
I = & \left[ m_1 \left( \frac{m_2}{m_1 + m_2} \right)^2 + m_2 \left( \frac{m_1}{m_1 + m_2} \right)^2 \right] |\xi_1|^2 \\
& + 2 \left( - m_1 \left( \frac{m_3}{m_1 + m_2 + m_3} \right) \left( \frac{m_2}{m_1 + m_2} \right) + m_2 \left( \frac{m_3}{m_1 + m_2 + m_3} \right) \left( \frac{m_1}{m_1 + m_2} \right) \right) \langle \xi_2, \xi_1 \rangle \\
& + \left[ m_1 \left( \frac{m_3}{m_1 + m_2 + m_3} \right)^2 + m_2 \left( \frac{m_3}{m_1 + m_2 + m_3} \right)^2 + m_3 \left( \frac{m_1 + m_2}{m_1 + m_2 + m_3} \right)^2 \right] |\xi_2|^2.
\end{aligned}$$

After simplification, the cross terms vanish and we obtain

$$I = \frac{m_1 m_2}{m_1 + m_2} |\xi_1|^2 + \frac{(m_1 + m_2) m_3}{m_1 + m_2 + m_3} |\xi_2|^2.$$

We set

$$\mu_1 := \frac{m_1 m_2}{m_1 + m_2}, \quad \mu_2 := \frac{(m_1 + m_2) m_3}{m_1 + m_2 + m_3}.$$

Thus, the moment of inertia can be expressed as

$$I = \mu_1 |\xi_1|^2 + \mu_2 |\xi_2|^2.$$

□

**Remark 3.3.** *In order to eliminate the mass factors that appear in the expressions for  $I$ , we introduce the rescaled Jacobi coordinates*

$$z_1 = \sqrt{\mu_1} \xi_1, \quad z_2 = \sqrt{\mu_2} \xi_2.$$

*This change of variables transforms the moment of inertia into the standard form*

$$I = |z_1|^2 + |z_2|^2.$$

### 3.1.4 Forming the Quotient $Q/\mathbb{R}^2$

As emphasized throughout this chapter, our goal is to eliminate those degrees of freedom that are redundant for the system; in this section, the translational one. To achieve this, we construct the quotient space  $Q/\mathbb{R}^2$ , which identifies all configurations that differ only by a global translation.

After passing to Jacobi coordinates, every configuration is represented by a triple  $(z_1, z_2, c)$ , where  $(z_1, z_2)$  are translation-invariant and  $c$  encodes the absolute position of the system

in the plane. Under a translation,  $t \in \mathbb{R}^2$ , these coordinates transform as

$$(z_1, z_2, c) \mapsto (z_1, z_2, c + t).$$

Thus each translation orbit has the form

$$\mathcal{O}_{(z_1, z_2, c)} = \{(z_1, z_2, c + t) : t \in \mathbb{R}^2\}.$$

Since the action of  $\mathbb{R}^2$  on  $Q$  is free, no two distinct translations produce the same point in the orbit. Moreover, given any point written in Jacobi coordinates as  $(z_1, z_2, c)$ , we can always eliminate its translational component by applying the translation  $t = -c$ , which shifts the center-of-mass coordinate to  $c + t = c + (-c) = 0$ . In other words, fixing the center of mass at the origin provides a canonical representative of each translation orbit. To remove translations, we introduce the equivalence relation

$$(x_1, x_2, x_3) \sim (y_1, y_2, y_3) \iff \exists t \in \mathbb{R}^2 \text{ such that } x_i = y_i + t \ (i = 1, 2, 3).$$

Each class consists of all configurations differing only by a common translation.

Under the Jacobi map

$$J' : (x_1, x_2, x_3) \mapsto (z_1, z_2, c),$$

the coordinate  $c$  encodes exactly this translational freedom.

Thus the quotient by translations identifies all points with identical  $(z_1, z_2)$ , and we may represent each class by choosing  $c = 0$ . This gives the map

$$\pi : Q \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2, \quad (x_1, x_2, x_3) \mapsto (z_1, z_2),$$

which assigns to each translation orbit its canonical representative in Jacobi coordinates. Its image is naturally isomorphic to  $\mathbb{R}^4$

Since  $\mathbb{R}^4 \cong \mathbb{C}^2$ , it follows that  $Q/\mathbb{R}^2 \cong \mathbb{C}^2$ . Working in  $\mathbb{C}^2$  is particularly advantageous, as it allows us to use the algebraic and geometric structure of the complex plane, especially the action of planar rotations, which will be the focus of the next reduction step.

### 3.1.5 Fiber bundle

Geometrically, the projection  $\pi : Q \rightarrow Q/\mathbb{R}^2$  defines a fiber bundle with fiber  $\mathbb{R}^2$ . Indeed, since the action of  $\mathbb{R}^2$  on  $Q = \mathbb{R}^6$  is free and proper, each fiber of  $\pi$  is exactly one translation orbit,

$$\pi^{-1}([(z_1, z_2, c)]) = \{(z_1, z_2, c + t) : t \in \mathbb{R}^2\},$$

which is naturally homeomorphic to  $\mathbb{R}^2$ . The base space is the quotient  $Q/\mathbb{R}^2 \cong \mathbb{R}^4$ .

To describe the bundle structure, we choose an open neighborhood  $U \subset Q/\mathbb{R}^2$ . Over such a set, the projection  $\pi$  simply separates each point in  $\pi^{-1}(U)$  into its translation-invariant part and its translational degree of freedom. This relation is expressed by the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{R}^2 \\ \pi \downarrow & \swarrow \text{proj}_1 & \\ U & & \end{array}$$

where  $\text{proj}_1$  denotes the projection onto the first factor.

In our case the bundle has an especially simple structure. The projection  $\pi : Q \rightarrow Q/\mathbb{R}^2$  separates each configuration into its translation-invariant part and its center-of-mass coordinate, so that  $Q$  itself decomposes as a product

$$Q \cong (Q/\mathbb{R}^2) \times \mathbb{R}^2.$$

Thus the projection is not only locally a product, but *globally* one. In the terminology of fiber bundles, this means that  $\pi$  defines a *trivial bundle* with fiber  $\mathbb{R}^2$ .

## 3.2 Reduction by Rotation

We proceed to the second step in the process of constructing shape space, which is to divide  $Q/\mathbb{R}^2 \cong \mathbb{C}^2$  by the action of rotations. After eliminating translations, each configuration of the three-body system is represented by a point  $(z_1, z_2) \in \mathbb{C}^2$ . However, rotating the entire triangle, that is, both Jacobi vectors  $z_1$  and  $z_2$ , by the same angle  $\theta$  leaves the shape of the triangle unchanged, since the relative positions of the three bodies remain the same and only its overall orientation in the plane changes.

### 3.2.1 Rotational Symmetry

Rotation by the angle  $\theta$  in the complex plane can be represented by the group:

$$S^1 := (S^1, \cdot),$$

with

$$S^1 = \{ e^{i\theta} = \cos \theta + i \sin \theta \mid \theta \in [0, 2\pi) \}.$$

Its elements are complex unit modulus numbers, and the group operation is complex scalar multiplication.

The rotation group acts on the configuration space  $S^1 \curvearrowright \mathbb{C}^2$  via the map

$$\rho : S^1 \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2, \quad (e^{i\theta}, (z_1, z_2)) \longmapsto (e^{i\theta} z_1, e^{i\theta} z_2).$$

Below, I include a short proof written by myself showing that  $\rho$  is indeed a well-defined group action.

**Lemma 3.4** (The rotational action is a group action). *Let  $\rho : S^1 \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$  be the map defined above. Then  $\rho$  is a left group action of the rotational group  $S^1$  on the configuration space  $\mathbb{C}^2$ .*

*Proof.* Let  $1 = e^{i \cdot 0} \in S^1$  be the neutral element and  $(z_1, z_2) \in \mathbb{C}^2$ , with  $z_1, z_2 \in \mathbb{C}$ . Then

$$\rho(1, (z_1, z_2)) = (1 \cdot z_1, 1 \cdot z_2) = (z_1, z_2),$$

so the identity axiom is satisfied.

For the compatibility axiom, let  $e^{i\theta_1}, e^{i\theta_2} \in S^1$ . Then

$$\begin{aligned} \rho(e^{i\theta_1} e^{i\theta_2}, (z_1, z_2)) &= (e^{i\theta_1} e^{i\theta_2} z_1, e^{i\theta_1} e^{i\theta_2} z_2) \\ &= \rho(e^{i\theta_1}, (e^{i\theta_2} z_1, e^{i\theta_2} z_2)) = \rho(e^{i\theta_1}, \rho(e^{i\theta_2}, (z_1, z_2))). \end{aligned}$$

This proves the claim. □

**Remark 3.4.** *The verification for a right group action is completely analogous.*

The rotational action is not free on all of  $\mathbb{C}^2$ , since the point  $(0, 0)$  is fixed by every rotation. However, on the collision-free configuration space  $\mathbb{C}^2 \setminus \{(0, 0)\}$  the action becomes free: if

$$(e^{i\theta}, (z_1, z_2)) = (z_1, z_2) \quad \text{with } (z_1, z_2) \neq (0, 0),$$

then the equalities  $e^{i\theta} z_1 = z_1$ ,  $e^{i\theta} z_2 = z_2$  imply that at least one  $z_i \neq 0$ , and therefore  $e^{i\theta} = 1$ . For this reason, from now on we restrict the configuration space from  $\mathbb{C}^2$  to the collision-free subspace  $\mathbb{C}^2 \setminus \{(0, 0)\}$ .

### 3.2.2 Hopf Coordinates

Our goal is again to rewrite the system in a form that separates the degrees of freedom: those describing the shape of the configuration and those corresponding to its overall rotation in the plane. To achieve this, we introduce the Hopf coordinates [5].

First, we identify the quantities that remain invariant under the  $S^1$ -action on  $\mathbb{C}^2$ , which are  $|z_1|^2$ ,  $|z_2|^2$ ,  $\Re(z_1 \bar{z}_2)$ ,  $\Im(z_1 \bar{z}_2)$ .

**Lemma 3.5** (Rotational invariance of the quadratic quantities). *The quantities*

$$|z_1|^2, \quad |z_2|^2, \quad \Re(z_1 \bar{z}_2), \quad \Im(z_1 \bar{z}_2)$$

are invariant under rotations, i.e. for any  $\theta \in [0, 2\pi)$  the action  $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$  does not change their values.

Next, I present the proof written by myself.

*Proof.* We fix  $\theta \in [0, 2\pi)$ . Since  $e^{i\theta} = \cos \theta + i \sin \theta$  has unit modulus,

$$|e^{i\theta}| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1,$$

and the complex modulus is multiplicative, we obtain

$$|e^{i\theta} z_j|^2 = |e^{i\theta}|^2 |z_j|^2 = |z_j|^2, \quad j = 1, 2.$$

Moreover,

$$(e^{i\theta} z_1) \overline{(e^{i\theta} z_2)} = e^{i\theta} z_1 e^{-i\theta} \bar{z}_2 = z_1 \bar{z}_2,$$

so both  $\Re(z_1 \bar{z}_2)$  and  $\Im(z_1 \bar{z}_2)$  are unchanged by the rotation. Hence the four stated quantities are invariant under rotations.  $\square$

The four rotational invariants identified above can be combined into a new coordinate system on  $\mathbb{C}^2$ , the *Hopf coordinates*. These coordinates encode exactly the information that remains unchanged under the  $S^1$ -action, and therefore describe the shape of the configuration independently of its overall orientation.

The Hopf coordinates are defined as

$$\mathbf{w} = (w_1, w_2, w_3) = \mathcal{H}(z_1, z_2),$$

$$w_1 = \frac{1}{2}(|z_1|^2 - |z_2|^2), \quad w_2 + iw_3 = z_1 \bar{z}_2.$$

The first coordinate  $w_1$  is half the difference of the squared lengths of the two Jacobi vectors. The remaining two coordinates  $w_2$  and  $w_3$  are the real and imaginary parts of the complex number  $z_1 \bar{z}_2$ . Together,  $(w_1, w_2, w_3)$  determine completely the *shape* formed by the three bodies, while ignoring the global orientation in the plane.

The fourth coordinate, which corresponds to the overall rotation angle  $\theta$  of the configuration can be encoded by the off-diagonal entry

$$I_{12} := -z_{11} z_{12} - z_{21} z_{22}$$

of the moment-of-inertia tensor, which describes the orientation of the system in the plane.

Thus, we obtain a map

$$\begin{aligned} \mathcal{H} : \mathbb{C}^2 \setminus \{(0, 0)\} &\longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 2\pi), \\ (z_1, z_2) &\longmapsto (w_1, w_2, w_3, \theta), \end{aligned}$$

with  $z_1, z_2 \in \mathbb{C}$ ,  $(z_1, z_2) \neq (0, 0)$ ,  $w_1, w_2, w_3 \in \mathbb{R}$  and  $\theta \in [0, 2\pi)$ , which separates the configuration into the rotationally invariant part  $(w_1, w_2, w_3)$  and the rotation  $\theta$  of the system.

### 3.2.3 Moment of Inertia

Having described the rotational symmetry of the system and introduced the Hopf coordinates, we now examine how the moment of inertia is expressed in this new coordinate system. Recall that in Jacobi coordinates the moment of inertia takes the form  $I = |z_1|^2 + |z_2|^2$ , which measures the overall size of the triangle formed by the three bodies. Since the Hopf vector satisfies

$$\|w\|^2 = w_1^2 + w_2^2 + w_3^2 = \frac{1}{4}I^2,$$

we obtain  $\|w\| = \frac{1}{2}I$ . In the following lemma we express the moment of inertia directly in terms of the Hopf coordinates [5].

**Lemma 3.6.** *Let  $z_1, z_2 \in \mathbb{C}$  and let  $(w_1, w_2, w_3)$  be the Hopf coordinates defined above. Then the moment of inertia*

$$I := |z_1|^2 + |z_2|^2$$

*can equivalently be expressed as*

$$I = \sqrt{4(w_1^2 + w_2^2 + w_3^2)}.$$

Below, I present the proof written by myself.

*Proof.* We compute

$$\|w\|^2 := w_1^2 + w_2^2 + w_3^2 = \frac{1}{4}I^2.$$

Hence,

$$I = \sqrt{4(w_1^2 + w_2^2 + w_3^2)}.$$

From the definition of the Hopf coordinates we obtain

$$w_1 = \frac{1}{2}(|z_1|^2 - |z_2|^2) \implies w_1^2 = \frac{1}{4}(|z_1|^2 - |z_2|^2)^2$$

$$w_2 + iw_3 = z_1 \bar{z}_2 \implies w_2^2 + w_3^2 = |w_2 + iw_3|^2 = |z_1 \bar{z}_2|^2 = |z_1|^2 |z_2|^2.$$

Substituting these expressions into the formula for  $I$  yields

$$\begin{aligned}
I &= \sqrt{4\left(\frac{1}{4}(|z_1|^2 - |z_2|^2)^2 + |z_1|^2|z_2|^2\right)} \\
&= \sqrt{(|z_1|^2 - |z_2|^2)^2 + 4|z_1|^2|z_2|^2} \\
&= \sqrt{|z_1|^4 - 2|z_1|^2|z_2|^2 + |z_2|^4 + 4|z_1|^2|z_2|^2} \\
&= \sqrt{|z_1|^4 + 2|z_1|^2|z_2|^2 + |z_2|^4} \\
&= \sqrt{(|z_1|^2 + |z_2|^2)^2} \\
&= |z_1|^2 + |z_2|^2.
\end{aligned}$$

□

### 3.2.4 Forming the Quotient $\mathcal{C}/S^1$

We now construct the quotient space, which identifies all configurations that differ only by a global rotation of the entire system.

After passing to Hopf coordinates, every configuration is represented by a quadruple  $(w_1, w_2, w_3, \theta)$ , where  $(w_1, w_2, w_3)$  are rotationally invariant and  $\theta$  encodes the absolute orientation of the system in the plane. Under a rotation  $e^{i\varphi} \in S^1$ , these coordinates transform as

$$(w_1, w_2, w_3, \theta) \longmapsto (w_1, w_2, w_3, \theta + \varphi).$$

Thus each rotation orbit has the form

$$\mathcal{O}_{(w_1, w_2, w_3, \theta)} = \{(w_1, w_2, w_3, \theta + \varphi) : \varphi \in [0, 2\pi)\}.$$

Since the action of  $S^1$  on the collision-free configuration space  $\mathcal{C} := \mathbb{C}^2 \setminus \{(0, 0)\}$  is free, no two distinct rotations produce the same point in the orbit. Moreover, given any point written in Hopf coordinates as  $(w_1, w_2, w_3, \theta)$ , we can always eliminate its rotational component by applying the rotation  $\varphi = -\theta$ , which shifts the angle coordinate to

$$\theta + \varphi = \theta + (-\theta) = 0.$$

In other words, fixing the angle to 0 provides a canonical representative of each rotation orbit.

To remove the rotational degree of freedom, we introduce the equivalence relation

$$(z_1, z_2) \sim (u_1, u_2) \iff \exists e^{i\theta} \in S^1 \text{ such that } (z_1, z_2) = e^{i\theta}(u_1, u_2).$$

Each equivalence class consists of all configurations that differ only by a common rotation of the plane. Since only the coordinate  $\theta$  changes under this rotation, two points are equivalent precisely when they have the same triple  $(w_1, w_2, w_3)$ . Thus each class admits a unique representative, which we choose by setting  $\theta = 0$ .

This leads to the projection

$$\eta : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \quad (w_1, w_2, w_3, \theta) \longmapsto (w_1, w_2, w_3).$$

Its image is naturally isomorphic to  $\mathbb{R}^3$ .

### 3.2.5 Hopf bundle

The projection

$$\pi : \mathcal{C} \longrightarrow \mathcal{C}/S^1$$

defines a fiber bundle with fiber  $S^1$ . Since the action of  $S^1$  on  $\mathcal{C}$  is free, each fiber of  $\pi$  is exactly one rotation orbit,

$$\pi^{-1}([(z_1, z_2)]) = \{ (e^{i\theta} z_1, e^{i\theta} z_2) : \theta \in [0, 2\pi) \},$$

which is homeomorphic to  $S^1$ .

In Hopf coordinates each point of  $\mathcal{C}$  is written as  $(w_1, w_2, w_3, \theta)$ , and the projection extracts the rotationally invariant part,

$$(w_1, w_2, w_3, \theta) \longmapsto (w_1, w_2, w_3).$$

Over an open set  $U \subset \mathcal{C}/S^1$  this decomposition is expressed by the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times S^1 \\ \pi \downarrow & \text{proj}_1 \swarrow & \\ U & & \end{array}$$

where  $\text{proj}_1$  denotes the projection onto the first factor.

Although the classical Hopf fibration  $S^3 \rightarrow S^2$  appears when the configuration space is restricted to a sphere of fixed radius in  $\mathbb{C}^2$ , in our setting we do not impose such a restriction yet. The nonzero configuration space decomposes as

$$\mathcal{C} \cong S^3 \times \mathbb{R}_{>0},$$

where the  $S^1$ -action affects only the  $S^3$ -factor. Taking the quotient therefore gives

$$\mathcal{C}/S^1 \cong (S^3/S^1) \times \mathbb{R}_{>0} \cong S^2 \times \mathbb{R}_{>0}.$$

Thus, the rotational reduction behaves like the usual Hopf fibration on each sphere of fixed radius, with the radial coordinate left free. In particular, this quotient forms a fiber bundle with total space  $S^3 \times \mathbb{R}_{>0}$ , base space  $S^2 \times \mathbb{R}_{>0}$ , and fiber  $S^1$ .

Combining the translational and rotational reductions described above, we see that shape space is obtained by quotienting the full configuration space by the action of the planar Euclidean group

$$\text{SE}(2) = \mathbb{R}^2 \rtimes S^1.$$

Translations give rise to an  $\mathbb{R}^2$ -bundle over  $Q/\mathbb{R}^2$ , while rotations produce the  $S^1$ -bundle described by the Hopf map. Together, these two fibrations show that the configuration space  $Q$  can be viewed as a bundle whose fibers represent changes in position and orientation, whereas the base records only the shape of the triangle formed by the three bodies.

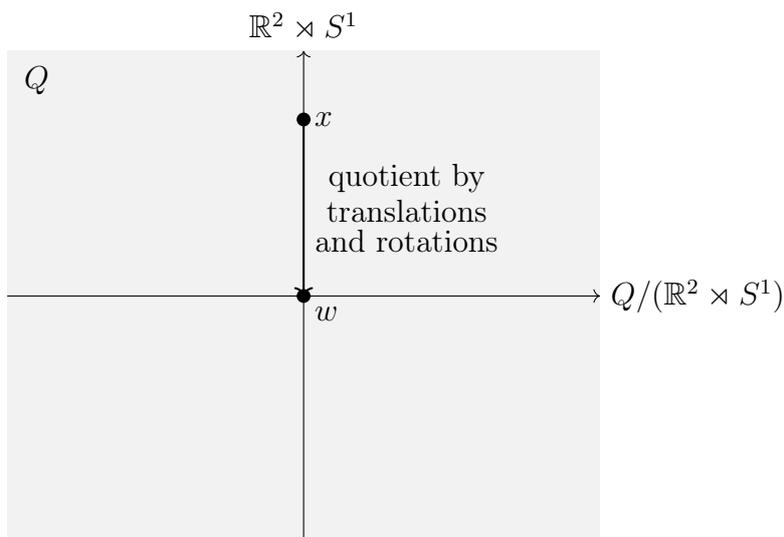


Figure 3.2: Schematic view of the quotient map  $Q \rightarrow Q/(\mathbb{R}^2 \rtimes S^1)$ .

Source: Created by me

In this representation, the horizontal axis encodes the quotient space. Moving vertically corresponds to applying translations or rotations to the same triangle. Thus each vertical line consists of all configurations that share the same shape (and size) but differ only by their position or orientation in the plane.

## 4 Shape Space

Starting from  $Q = \mathbb{R}^6$ , we successively remove the degrees of freedom associated with rigid motions of the plane. On the level of spaces, this produces the chain of maps

$$Q \longrightarrow Q/\mathbb{R}^2 \longrightarrow Q/(\mathbb{R}^2 \rtimes S^1).$$

The final space obtained in this process is the *shape space*: a space that captures only the intrinsic geometry of the configuration, its *shape*.

In this chapter we explain how shape space is constructed and why it plays a central role in the analysis of the Newtonian three-body problem. We also introduce the unit level set of the moment of inertia, the *shape sphere*, and show how its geometry provides a natural framework for encoding and interpreting three-body motions.

### 4.1 Defining Shape Space

In order to formalize the notion of “having the same shape”, we begin by introducing the concept of *oriented congruence*.

**Definition 4.1** (Oriented congruence). *Two configurations of three bodies in the plane are said to be oriented congruent (or to have the same shape) if one can be obtained from the other by a rigid motion of the plane, that is, by a combination of a translation and a rotation. Equivalently, two configurations share the same shape precisely when they lie in the same orbit of the action of the special Euclidean group  $SE(2)$  on the configuration space.*

**Definition 4.2** (Shape space). *The shape space, denoted  $\Sigma$ , is the space whose points are the oriented congruence classes of planar three-body configurations. Formally, it is the quotient*

$$\Sigma = Q/SE(2),$$

where  $Q = \mathbb{R}^6$  is the configuration space.

Geometrically, shape space can be visualized as a *cone*. Once we are in the reduced space, each configuration of the three-body system is represented by its Hopf vector

$$w = (w_1, w_2, w_3) \in \mathbb{R}^3,$$

whose *direction* encodes the shape of the triangle, while its *length* describes its overall size. Since the Hopf coordinates satisfy

$$\|w\| = \frac{1}{2} I, \quad I = |z_1|^2 + |z_2|^2,$$

the norm  $\|w\|$  measures the moment of inertia and therefore the scale of the configuration.

The remaining symmetry of the reduced problem is the uniform scaling of the triangle. A change of size is represented by a factor  $\lambda \in \mathbb{R}_{>0}$ . Using

$$I = 2\sqrt{w_1^2 + w_2^2 + w_3^2} \implies \lambda I = \lambda 2\sqrt{w_1^2 + w_2^2 + w_3^2} = \sqrt{\lambda^2 4(w_1^2 + w_2^2 + w_3^2)},$$

we see that multiplying the moment of inertia by  $\lambda$  requires multiplying each Hopf coordinate by  $\lambda^2$ . Thus the scaling action on the Hopf vector is

$$w \mapsto \lambda^2 w.$$

Consequently, all vectors of the form

$$\{\lambda^2 w : \lambda > 0\}$$

represent triangles with the same shape but different sizes. Each such set is a ray through the origin in  $\mathbb{R}^3$ , and the collection of all these rays forms a cone whose apex corresponds to triple collision [4]. The *shape sphere* will next be obtained by fixing the moment of inertia to  $I = 1$ , that is, by intersecting this cone with the sphere  $\|w\| = \frac{1}{2}$ .

## 4.2 The Shape Sphere

To remove this final degree of freedom, we fix the moment of inertia to the unit value  $I = 1$ . Given any Hopf vector  $w \neq 0$ , we can always rescale it to satisfy  $\|w'\| = \frac{1}{2}$ : indeed, choosing  $\lambda > 0$  such that

$$\|\lambda^2 w\| = \lambda^2 \|w\| = \frac{1}{2} \implies \lambda = \frac{1}{\sqrt{2\|w\|}},$$

produces the normalized vector

$$w' = \lambda^2 w = \frac{1}{2\|w\|} w.$$

Thus every scaling class  $\{\lambda^2 w : \lambda > 0\}$  intersects the sphere of radius  $\frac{1}{2}$  in exactly one point, giving a unique representative of each similarity class of triangles.

**Definition 4.3** (Shape sphere). *The shape sphere is the unit level set of the moment of inertia,*

$$S_{1/2}^2 := \{ w \in \mathbb{R}^3 : \|w\| = \frac{1}{2} \}.$$

*Each point of  $S_{1/2}^2$  represents an oriented similarity class of planar triangles, that is, a triangle modulo translations, rotations, and scalings.*

## 4.3 Spherical Coordinates

To separate the information related to size and shape of the Hopf vector, it is convenient to introduce spherical coordinates on  $\mathbb{R}^3$ . We first normalize  $w$  by its length:

$$\frac{(w_1, w_2, w_3)}{\|w\|} = (\cos \psi \cos \chi, \sin \psi \cos \chi, \sin \chi),$$

which defines the angular coordinates  $(\psi, \chi)$  on the unit sphere. Here  $\psi$  determines the direction in the equatorial plane, while  $\chi$  measures the signed distance from this plane.

These coordinates differ slightly from the standard spherical coordinates  $(\theta, \phi)$ , where

$$x = \sin \theta \cos \phi, \quad y = \sin \theta \sin \phi, \quad z = \cos \theta,$$

with  $\theta \in [0, \pi]$  and  $\phi \in [0, 2\pi)$ . Since in our setting it is more natural to measure angles from the equator rather than from the north pole, we introduce

$$\chi = \frac{\pi}{2} - \theta, \quad \text{so that} \quad \chi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Moreover, the equatorial angle remains unchanged, so we set

$$\psi := \phi.$$

A short computation then gives

$$\sin \theta = \cos \chi, \quad \cos \theta = \sin \chi,$$

and the coordinate change becomes

$$x = \cos \chi \cos \psi, \quad y = \cos \chi \sin \psi, \quad z = \sin \chi,$$

which matches the parametrization of the normalized Hopf vector.

The pair  $(\chi, \psi)$  therefore describes the *shape* of the triangle, while the radial coordinate  $\|w\|$  encodes its *size*. This yields spherical coordinates  $(r, \chi, \psi)$  for the Hopf vector, with

$r = \|w\|$ . On the shape sphere, where  $\|w\| = \frac{1}{2}$ , these coordinates reduce to

$$(r, \chi, \psi) = \left(\frac{1}{2}, \chi, \psi\right).$$

## 4.4 Geometry of the Shape Sphere

In this section we explain how the shape sphere provides a geometric representation of different types of planar three-body configurations.

### Lagrange configurations

Equilateral triangles play a distinguished role. There are two oriented similarity classes: a “right-handed” and a “left-handed” one. These appear as the *north* and *south* poles of the shape sphere. They represent the two Lagrange configurations, usually denoted by  $L_+$  and  $L_-$ . In spherical coordinates  $(r, \chi, \psi)$ , they correspond to the points

$$(r, \chi, \psi) = \left(\frac{1}{2}, \frac{\pi}{2}, \psi\right) \quad \text{and} \quad (r, \chi, \psi) = \left(\frac{1}{2}, -\frac{\pi}{2}, \psi\right),$$

respectively, where the value of  $\psi$  is irrelevant since all longitudes meet at the poles.

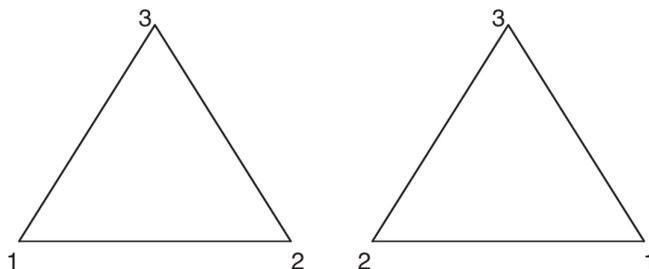


Figure 4.1: Lagrange configurations  
Source: Montgomery, *Infinitely Many Syzygies* (2002)

### Syzygies

A configuration is called a *syzygy* when the three bodies lie on a straight line. All collinear triangles form a two-dimensional set in shape space, and its intersection with the shape sphere is a great circle. This circle is the *equator* of the shape sphere, and each point on it corresponds to a collinear configuration.

Among the collinear configurations, there are three distinguished points on the equator corresponding to the *Euler configurations*, denoted by  $E_1, E_2, E_3$ . These represent collinear central configurations, in which the mutual gravitational forces are aligned with the configuration itself.

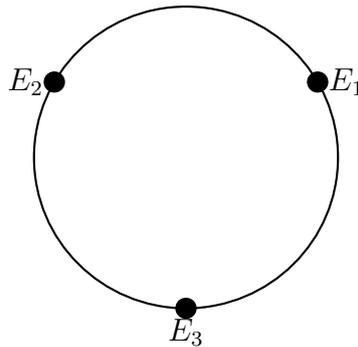


Figure 4.2: Euler configurations on the equator of the shape sphere.  
Source: Created by me

Along any three-body motion that is not collinear for all time and suffers no collisions, we can record the successive syzygies as a symbolic sequence in the symbols 1, 2, 3, called the *syzygy sequence*. If the motion is periodic in shape space, then the corresponding syzygy sequence is periodic as well. This symbolic viewpoint is particularly useful in the study of zero-angular-momentum solutions, where syzygies appear frequently. In fact, Montgomery proved in [5] that every bounded solution of the Newtonian three-body problem with zero angular momentum and no triple collision must experience infinitely many syzygies.

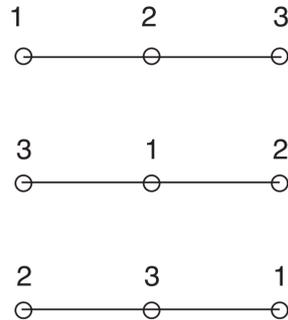


Figure 4.3: The three syzygy types  
Source: Montgomery, *Infinitely Many Syzygies* (2002)

## Binary collisions

Binary collisions occur when exactly two of the bodies approach each other while the third remains at a finite distance. There are three such possibilities, and each determines a ray in shape space issuing from the triple collision. Their intersections with the shape sphere are three distinguished points lying on the equator, equally spaced by symmetry. We denote them by  $B_{12}$ ,  $B_{23}$ ,  $B_{31}$ .

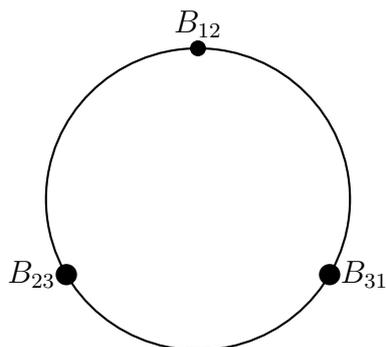


Figure 4.4: Three binary collision configurations on the equator of the shape sphere.  
Source: Created by me

All these geometric features can be visualized in the following figure:

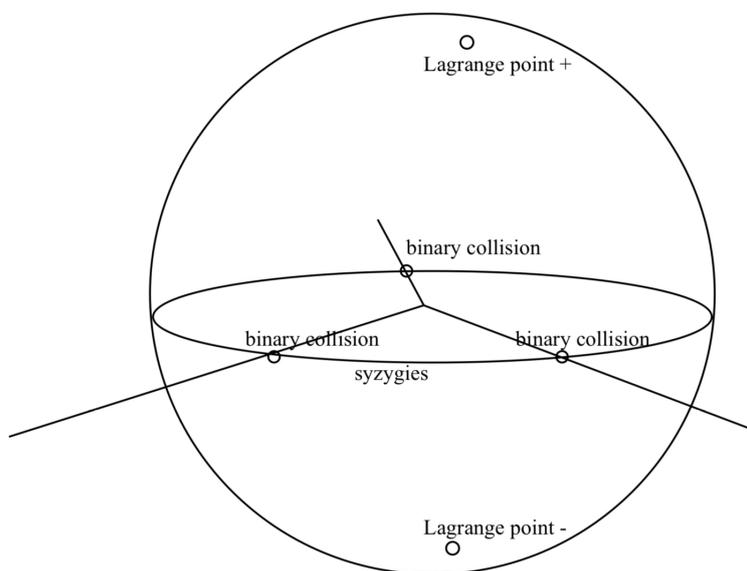


Figure 4.5: Shape Sphere  
Source: Montgomery, *Infinitely Many Syzygies* (2002)

## 4.5 Interpretation of Three-Body Motions in Shape Space

Having described the geometry of shape space and the shape sphere, we now explain how actual solutions of the Newtonian three-body problem are interpreted within this framework.

A key role is played by the *total energy* of the system. The relation between energy and the global behavior of solutions is captured by the *Lagrange–Jacobi identity*, which governs the evolution of the moment of inertia  $I$  and thus of the overall size of the configuration. This identity provides a natural criterion for distinguishing between different types of motion.

Recall that the moment of inertia  $I$  provides a measure of the overall size of the configuration. Its evolution is governed by the *Lagrange-Jacobi identity*,

$$\frac{d^2 I}{dt^2} = 2T - U,$$

where  $T$  denotes the kinetic energy and  $U$  the Newtonian potential energy. This identity shows that the evolution of the size of the configuration is directly linked to the balance between kinetic and potential energy. In particular, this implies that the qualitative behavior of  $I(t)$  is controlled by the total energy of the system. This leads to a natural distinction between bounded motions and motions for which the size of the configuration grows without bound.

### Solutions as fixed points

The simplest motions in shape space correspond to configurations whose shape remains constant in time. Such motions arise from special initial configurations that are represented by fixed points on the shape sphere.

In the Lagrange configurations, the three bodies form an equilateral triangle. If the initial velocities vanish, the system evolves purely under gravitational attraction and the three masses collide in a triple collision. Throughout the motion the shape remains equilateral, so the trajectory in shape space reduces to a single fixed point, namely one of the Lagrange points.

The Euler configurations provide another class of fixed points. In this case, the three bodies are collinear. If the initial velocities vanish, the system evolves under pure gravitational attraction and the three masses collide in a triple collision. Since the configuration remains collinear throughout the motion, this solution is represented by a single point on the shape sphere, namely one of the Euler points.

Both Lagrange and Euler configurations correspond to motions with negative total energy, which leads to a contraction of the overall size of the system.

### Periodic solutions and syzygy sequences

In the three-body problem, many periodic solutions have been found. In shape space, such solutions are represented by closed curves on the shape sphere, reflecting the fact that the shape of the configuration repeats after a finite time.

These periodic motions necessarily belong to the negative-energy case. Indeed, only when the total energy is negative can the size of the configuration remain bounded and allow for recurrent, periodic behavior.

Along any periodic three-body motion that is not collinear for all time, the configuration necessarily passes through syzygies. Therefore, such a motion can be described by a

syzygy sequence. For example, for a periodic solution such as the figure-eight orbit, one of the corresponding syzygy sequences takes the form

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \dots$$

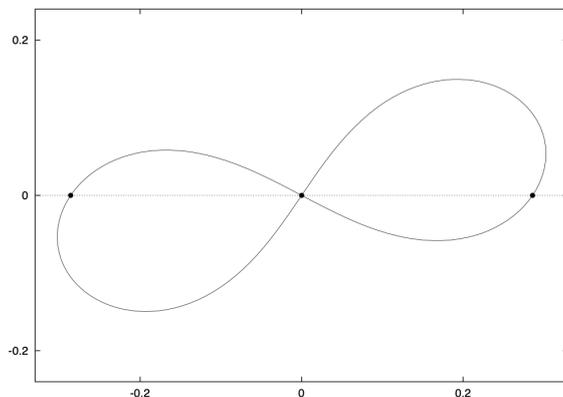


Figure 4.6: Figure Eight at a syzygy

Source: Simó, *Dynamical properties of the figure eight solution of the three-body problem* (2002)

## Non-periodic motions and binary collisions

Generic solutions of the three-body problem are typically non-periodic. Their behavior depends, among others, on the sign of the total energy  $E$ , which determines whether the size of the configuration remains bounded or not.

In the negative-energy case  $E < 0$ , the motion is bounded. Nevertheless, the repeated gravitational interactions between the bodies lead to irregular and highly complex trajectories.

In the positive-energy case  $E > 0$ , the motion is no longer bounded. One possible outcome is that two of the bodies form a bound Keplerian pair, while the third escapes to infinity. In this scenario, the moment of inertia grows without bound. On the shape sphere, this behavior corresponds to trajectories that spiral toward one of the binary collision points  $B_{12}$ ,  $B_{23}$ ,  $B_{31}$ .

This behavior is illustrated in the following figure, where the spiraling trajectories reflect the near-binary interactions occurring during the motion.

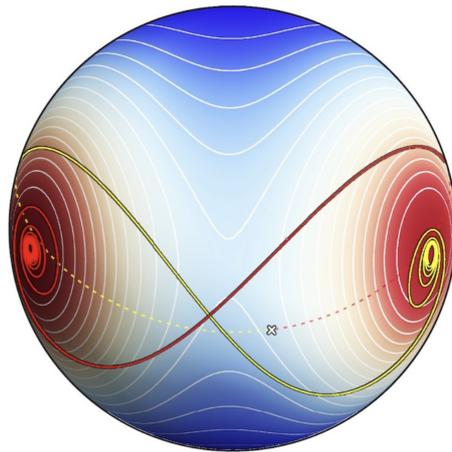


Figure 4.7: Non-periodic three-body trajectories spiralling toward binary collision points on the shape sphere

Source: Barbour, *A Gravitational Origin of the Arrows of Time* (2014)

## 5 Conclusion

In this thesis we have studied the planar Newtonian three-body problem from a geometric point of view, focusing on symmetry reduction. Although the equations of motion are given by explicit expressions, the wide range of possible dynamical behaviours makes a complete analytical description extremely difficult. This motivates the use of alternative frameworks that help reveal the qualitative structure of the problem.

Starting from the full configuration space, we identified the fundamental symmetries of the system and used them to remove redundant degrees of freedom. Translations and rotations were eliminated through Jacobi and Hopf coordinates, leading to a reduced description in terms of shape space. This reduction separates the intrinsic geometry of the configuration from its absolute position and orientation.

The resulting shape space admits a simple and elegant geometric representation. After fixing the moment of inertia, each planar configuration up to translation, rotation, and scaling is represented by a single point on the shape sphere. Within this framework, important configurations such as equilateral triangles, collinear states, and binary collisions appear as distinguished points or curves, providing a clear geometric interpretation of different types of motion.

Viewed on the shape sphere, three-body motions can be interpreted as trajectories whose qualitative features reflect the underlying dynamics. Periodic solutions correspond to closed curves, Lagrange and Euler configurations to fixed points, while more complicated non-periodic motions may appear as spiralling paths around binary collision points. This geometric perspective offers a compact and intuitive way to organize and interpret the dynamics of the three-body problem.

Beyond the specific constructions presented here, the shape-space approach suggests several natural directions for further study. We may investigate the reduced equations of motion directly on shape space, or use the geometric framework to better understand other existing classifications of three-body solutions. More generally, the idea of symmetry reduction and shape space extend beyond the three-body problem and provide a powerful conceptual tool for the study of many-body systems.

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Marta J. Díaz Martín